Electromagnetic Field Theory (BAB17EMP)
Useful Mathematical Identities

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Contents

1 Operations on Radius Vectors

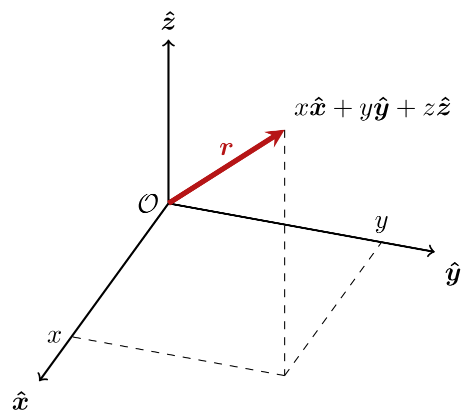

Figure 1: Radius vector $\boldsymbol{r}$.

Observation vector:

$$\boldsymbol{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}} \tag{1}$$

Vector magnitude:

$$r = |\boldsymbol{r}| = \sqrt{x^2 + y^2 + z^2} \tag{2}$$

Unit vector:

$$\hat{\boldsymbol{r}} = \frac{\boldsymbol{r}}{r} \tag{3}$$

Gradient applied to r:

$$\nabla r = \frac{x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}}{\sqrt{x^2 + y^2 + z^2}} = \hat{\boldsymbol{r}} \tag{4}$$

Gradient applied to $1/r$:

$$\nabla \frac{1}{r} = -\frac{x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}}{\left(\sqrt{x^2 + y^2 + z^2}\right)^3} = -\frac{\boldsymbol{r}}{r^3} \tag{5}$$

Divergence applied to $\boldsymbol{r}$:

$$\nabla \cdot \boldsymbol{r} = 3 \tag{6}$$

Curl applied to $\boldsymbol{r}$:

$$\nabla \times \boldsymbol{r} = 0 \tag{7}$$

1.1 Identities Involving Separation Vector

Source vector:

$$\boldsymbol{r}' = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = x'\hat{\boldsymbol{x}} + y'\hat{\boldsymbol{y}} + z'\hat{\boldsymbol{z}} \tag{8}$$

$$\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}' = \begin{bmatrix} x - x' \\ y - y' \\ z - z' \end{bmatrix} = (x - x')\,\hat{\boldsymbol{x}} + (y - y')\,\hat{\boldsymbol{y}} + (z - z')\,\hat{\boldsymbol{z}} \tag{9}$$

$$R = |\boldsymbol{r} - \boldsymbol{r}'| = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2} \tag{10}$$

$$R = |\boldsymbol{r} - \boldsymbol{r}'| = \sqrt{r^2 - 2rr'\cos(\theta) + r'^2} \tag{11}$$

$$\nabla R = \frac{(x - x')\,\hat{\boldsymbol{x}} + (y - y')\,\hat{\boldsymbol{y}} + (z - z')\,\hat{\boldsymbol{z}}}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}} = \frac{\boldsymbol{r} - \boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|} \tag{12}$$

$$\nabla\left(\frac{1}{R}\right) = -\frac{(x - x')\,\hat{\boldsymbol{x}} + (y - y')\,\hat{\boldsymbol{y}} + (z - z')\,\hat{\boldsymbol{z}}}{\left(\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}\right)^3} = -\frac{\boldsymbol{r} - \boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|^3} \tag{13}$$

$$\nabla \cdot \boldsymbol{R} = 3 \tag{14}$$

$$\nabla \times \boldsymbol{R} = 0 \tag{15}$$

2 Trigonometric Identities

$$\sin^2\alpha + \cos^2\alpha = 1 \tag{16}$$

$$\sin^2\alpha = \frac{1 - \cos 2\alpha}{2} \tag{17}$$

$$\cos^2\alpha = \frac{1 + \cos 2\alpha}{2} \tag{18}$$

$$\mathrm{e}^{\mathrm{j}\alpha} = \cos\alpha + \mathrm{j}\sin\alpha \tag{19}$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha \tag{20}$$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha \tag{21}$$

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \tag{22}$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \tag{23}$$

$$\sin\left(\arctan\left(\frac{x}{a}\right)\right) = \frac{x}{\sqrt{x^2 + a^2}} \tag{24}$$

$$\cos\left(\arctan\left(\frac{x}{a}\right)\right) = \frac{a}{\sqrt{x^2 + a^2}} \tag{25}$$

3 Coordinate System Transformations

3.1 Point Transformations

Cylindrical to Cartesian:

$$x = \rho \cos \phi \tag{26}$$
$$y = \rho \sin \phi \tag{27}$$
$$z = z \tag{28}$$

Cartesian to cylindrical:

$$\rho = \sqrt{x^2 + y^2} \tag{29}$$
$$\phi = \arctan \frac{y}{x} \tag{30}$$
$$z = z \tag{31}$$

Spherical to Cartesian:

$$x = r \cos \phi \sin \theta \tag{32}$$
$$y = r \sin \phi \sin \theta \tag{33}$$
$$z = r \cos \theta \tag{34}$$

Cartesian to Spherical:

$$r = \sqrt{x^2 + y^2 + z^2} \tag{35}$$
$$\theta = \arctan \frac{\sqrt{x^2 + y^2}}{z} \tag{36}$$
$$\phi = \arctan \frac{y}{x} \tag{37}$$

3.2 Vector Transformations

Cylindrical to Cartesian:

$$\hat{\boldsymbol{x}} = \hat{\boldsymbol{\rho}} \cos \phi - \hat{\boldsymbol{\phi}} \sin \phi \tag{38}$$
$$\hat{\boldsymbol{y}} = \hat{\boldsymbol{\rho}} \sin \phi + \hat{\boldsymbol{\phi}} \cos \phi \tag{39}$$
$$\hat{\boldsymbol{z}} = \hat{\boldsymbol{z}} \tag{40}$$

Cartesian to Cylindrical:

$$\hat{\boldsymbol{\rho}} = \hat{\boldsymbol{x}} \frac{x}{\sqrt{x^2 + y^2}} + \hat{\boldsymbol{y}} \frac{y}{\sqrt{x^2 + y^2}} \tag{41}$$
$$\hat{\boldsymbol{\phi}} = -\hat{\boldsymbol{x}} \frac{y}{\sqrt{x^2 + y^2}} + \hat{\boldsymbol{y}} \frac{x}{\sqrt{x^2 + y^2}} \tag{42}$$
$$\hat{\boldsymbol{z}} = \hat{\boldsymbol{z}} \tag{43}$$

Spherical to Cartesian:

$$\hat{\boldsymbol{x}} = \hat{\boldsymbol{r}} \sin\theta\cos\phi + \hat{\boldsymbol{\theta}}\cos\theta\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi \tag{44}$$

$$\hat{\boldsymbol{y}} = \hat{\boldsymbol{r}} \sin\theta\sin\phi + \hat{\boldsymbol{\theta}}\cos\theta\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi \tag{45}$$

$$\hat{\boldsymbol{z}} = \hat{\boldsymbol{r}} \cos\theta - \hat{\boldsymbol{\theta}}\sin\theta \tag{46}$$

Cartesian to Spherical:

$$\hat{\boldsymbol{r}} = \hat{\boldsymbol{x}}\frac{x}{\sqrt{x^2+y^2+z^2}} + \hat{\boldsymbol{y}}\frac{y}{\sqrt{x^2+y^2+z^2}} + \hat{\boldsymbol{z}}\frac{z}{\sqrt{x^2+y^2+z^2}} \tag{47}$$

$$\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{x}}\frac{zx}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}} + \hat{\boldsymbol{y}}\frac{zy}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}} - \hat{\boldsymbol{z}}\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}} \tag{48}$$

$$\hat{\boldsymbol{\phi}} = -\hat{\boldsymbol{x}}\frac{y}{\sqrt{x^2+y^2}} + \hat{\boldsymbol{y}}\frac{x}{\sqrt{x^2+y^2}} \tag{49}$$

3.3 Conversion Between Coordinate Systems (Example)

The relation

$$\hat{\boldsymbol{z}} \times \hat{\boldsymbol{r}} = \sin\theta\hat{\boldsymbol{\phi}} \tag{50}$$

holds since the vector

$$\hat{\boldsymbol{\phi}} = -\hat{\boldsymbol{x}}\sin\phi + \hat{\boldsymbol{y}}\cos\phi \tag{51}$$

can be compared with the vector multiplication and the vector multiplication

$$\hat{\boldsymbol{z}} \times \hat{\boldsymbol{r}} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ 0 & 0 & 1 \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{vmatrix} = -\sin\theta\sin\phi\hat{\boldsymbol{x}} + \sin\theta\cos\phi\hat{\boldsymbol{y}} = \sin\theta\hat{\boldsymbol{\Phi}}, \tag{52}$$

where we used

$$\hat{\boldsymbol{r}} = \sin\theta\cos\phi\hat{\boldsymbol{x}} + \sin\theta\sin\phi\hat{\boldsymbol{y}} + \cos\theta\hat{\boldsymbol{z}}. \tag{53}$$

4 Differential Operators

4.1 Differential Elements

$$\begin{aligned} \mathrm{d}\boldsymbol{\ell}_x &= \mathrm{d}x & \mathrm{d}\boldsymbol{\ell}_\rho &= \mathrm{d}\rho & \mathrm{d}\boldsymbol{\ell}_r &= r \\ \mathrm{d}\boldsymbol{\ell}_y &= \mathrm{d}y & \mathrm{d}\boldsymbol{\ell}_\phi &= \rho\,\mathrm{d}\phi & \mathrm{d}\boldsymbol{\ell}_\theta &= r\,\mathrm{d}\theta \\ \mathrm{d}\boldsymbol{\ell}_z &= \mathrm{d}z & \mathrm{d}\boldsymbol{\ell}_z &= \mathrm{d}z & \mathrm{d}\boldsymbol{\ell}_\phi &= r\sin\theta\,\mathrm{d}\phi \end{aligned}$$

4.2 Rectangular Coordinate System

$$\boldsymbol{F} = F_x\hat{\boldsymbol{x}} + F_y\hat{\boldsymbol{y}} + F_z\hat{\boldsymbol{z}} \tag{54}$$

$$\nabla f = \frac{\partial f}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial f}{\partial y}\hat{\boldsymbol{y}} + \frac{\partial f}{\partial z}\hat{\boldsymbol{z}} \tag{55}$$

$$\nabla \cdot \boldsymbol{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \tag{56}$$

$$\nabla \times \boldsymbol{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{\boldsymbol{x}} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{\boldsymbol{y}} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{\boldsymbol{z}} \tag{57}$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \tag{58}$$

$$\nabla^2 \boldsymbol{F} = \nabla^2 F_x\,\hat{\boldsymbol{x}} + \nabla^2 F_y\,\hat{\boldsymbol{y}} + \nabla^2 F_z\,\hat{\boldsymbol{z}} \tag{59}$$

4.3 Polar Coordinate System

$$\boldsymbol{F} = F_\rho\hat{\boldsymbol{\rho}} + F_\phi\hat{\boldsymbol{\phi}} \tag{60}$$

$$\nabla f = \frac{\partial f}{\partial \rho}\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}} \tag{61}$$

$$\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} \tag{62}$$

4.4 Cylindrical Coordinate System

$$\boldsymbol{F} = F_\rho\hat{\boldsymbol{\rho}} + F_\phi\hat{\boldsymbol{\phi}} + F_z\hat{\boldsymbol{z}} \tag{63}$$

$$\nabla f = \frac{\partial f}{\partial \rho}\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial f}{\partial z}\hat{\boldsymbol{z}} \tag{64}$$

$$\nabla \cdot \boldsymbol{F} = \frac{1}{\rho}\frac{\partial\left(\rho F_\rho\right)}{\partial \rho} + \frac{1}{\rho}\frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z} \tag{65}$$

$$\nabla \times \boldsymbol{F} = \left(\frac{1}{\rho}\frac{\partial F_z}{\partial \phi} - \frac{\partial F_\phi}{\partial z}\right)\hat{\boldsymbol{\rho}} + \left(\frac{\partial F_\rho}{\partial z} - \frac{\partial F_z}{\partial \rho}\right)\hat{\boldsymbol{\phi}} + \left(\frac{1}{\rho}\frac{\partial\left(\rho F_\phi\right)}{\partial \rho} - \frac{1}{\rho}\frac{\partial F_\rho}{\partial \phi}\right)\hat{\boldsymbol{z}} \tag{66}$$

$$\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 z}{\partial z^2} \tag{67}$$

4.5 Spherical Coordinate System

$$\boldsymbol{F} = F_r \hat{\boldsymbol{r}} + F_\theta \hat{\boldsymbol{\theta}} + F_\phi \hat{\boldsymbol{\phi}} \tag{68}$$

$$\nabla f = \frac{\partial f}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}} \tag{69}$$

$$\nabla \cdot \boldsymbol{F} = \frac{1}{r^2}\frac{\partial\left(r^2 F_r\right)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial\left(F_\theta \sin\theta\right)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial \phi} \tag{70}$$

$$\nabla \times \boldsymbol{F} = \frac{1}{r\sin\theta}\left(\frac{\partial\left(F_\phi \sin\theta\right)}{\partial \theta} - \frac{\partial F_\theta}{\partial \phi}\right)\hat{\boldsymbol{r}} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial F_r}{\partial \phi} - \frac{\partial\left(rF_\phi\right)}{\partial r}\right)\hat{\boldsymbol{\theta}} + \frac{1}{r}\left(\frac{\partial\left(rF_\theta\right)}{\partial r} - \frac{\partial F_r}{\partial \theta}\right)\hat{\boldsymbol{\phi}} \tag{71}$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2} \tag{72}$$

5 Vector Identities

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 \tag{73}$$

$$|\mathbf{u}| = \sqrt{\mathbf{u} \cdot \mathbf{u}} \tag{74}$$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} u_1 & v_1 & \hat{\boldsymbol{e}}_1 \\ u_2 & v_2 & \hat{\boldsymbol{e}}_2 \\ u_3 & v_3 & \hat{\boldsymbol{e}}_3 \end{vmatrix} = \left(u_2 v_3 - u_3 v_2\right)\hat{\boldsymbol{e}}_1 + \left(u_3 v_1 - u_1 v_3\right)\hat{\boldsymbol{e}}_2 + \left(u_1 v_2 - u_2 v_1\right)\hat{\boldsymbol{e}}_3 \tag{75}$$

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\,\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\,\mathbf{w} \tag{76}$$

6 Differential Identities

$$\nabla \times \nabla f = 0 \tag{77}$$

$$\nabla \cdot \nabla \times \boldsymbol{F} = 0 \tag{78}$$

$$\nabla \times \nabla \times \boldsymbol{F} = \nabla\nabla \cdot \boldsymbol{F} - \nabla^2 \boldsymbol{F} \tag{79}$$

$$\nabla \cdot (f\boldsymbol{F}) = f(\boldsymbol{r})\,\nabla \cdot \boldsymbol{F} + \boldsymbol{F}(\boldsymbol{r}) \cdot \nabla f \tag{80}$$

$$\nabla \times (f\boldsymbol{F}) = f\nabla \times \boldsymbol{F} + \nabla f \times \boldsymbol{F} \tag{81}$$

$$\nabla \cdot (\boldsymbol{F} \times \boldsymbol{G}) = (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{G} - (\nabla \times \boldsymbol{G}) \cdot \boldsymbol{F} \tag{82}$$

7 Integration Identities

Substitution of cylindrical coordinates:

$$\iiint\limits_V f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint\limits_V f(\rho,\phi,z)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}z \tag{83}$$

Substitution of spherical coordinates:

$$\iiint\limits_V f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint\limits_V f(r,\theta,\phi)\,r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \tag{84}$$

Curl theorem:

$$\oint\limits_{\partial S} \boldsymbol{F}(\boldsymbol{r}) \cdot \,\mathrm{d}\boldsymbol{l} = \iint\limits_S \nabla \times \boldsymbol{F} \cdot \,\mathrm{d}\boldsymbol{S} \tag{85}$$

Divergence theorem:

$$\oiint\limits_{\partial V} \boldsymbol{F}(\boldsymbol{r}) \cdot \,\mathrm{d}\boldsymbol{S} = \iiint\limits_V \nabla \cdot \boldsymbol{F}\,\mathrm{d}V \tag{86}$$

$$\oint\limits_{l'} \mathrm{d}\boldsymbol{l}' = 0 \tag{87}$$

$$\oint\limits_{l'} (\hat{\boldsymbol{r}} \cdot \boldsymbol{r}')\,\mathrm{d}\boldsymbol{l} = -\hat{\boldsymbol{r}} \times \iint\limits_{S'} \mathrm{d}\boldsymbol{S}' \tag{88}$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,\mathrm{d}x = \ln\left|x + \sqrt{x^2+a^2}\right| + C \tag{89}$$

$$\int \frac{x}{\sqrt{x^2+a^2}}\,\mathrm{d}x = \sqrt{x^2+a^2} + C \tag{90}$$

$$\int \frac{1}{\left(\sqrt{x^2+a^2}\right)^3}\,\mathrm{d}x = \frac{x}{a^2\sqrt{x^2+a^2}} + C \tag{91}$$

$$\int \frac{x}{\left(\sqrt{x^2+a^2}\right)^3}\,\mathrm{d}x = -\frac{1}{\sqrt{x^2+a^2}} + C \tag{92}$$

8 Fourier Transform

$$\mathscr{F}\left\{\boldsymbol{F}\left(\boldsymbol{r},t\right)\right\} = \hat{\boldsymbol{F}}\left(\boldsymbol{r},\omega\right) \tag{93}$$

$$\mathscr{F}\left\{\frac{\partial \boldsymbol{F}\left(\boldsymbol{r},t\right)}{\partial t}\right\} = \mathrm{j}\omega\hat{\boldsymbol{F}}\left(\boldsymbol{r},\omega\right) \tag{94}$$

$$\mathscr{F}\left\{f\left(\omega\right) * \boldsymbol{F}\left(\boldsymbol{r},t\right)\right\} = \hat{f}\left(\omega\right)\hat{\boldsymbol{F}}\left(\boldsymbol{r},\omega\right) \tag{95}$$

9 Useful Functions

9.1 Dirac Delta Function

$$\int\limits_{-\infty}^{\infty} \delta\left(x\right)\,\mathrm{d}x = 1 \tag{96}$$

$$\int\limits_{V'} f\left(\boldsymbol{r}\right)\delta\left(\boldsymbol{r}-\boldsymbol{r}'\right)\,\mathrm{d}V' = \begin{cases} f\left(\boldsymbol{r}'\right), & \boldsymbol{r}' \in V' \\ 0, & \text{otherwise} \end{cases} \tag{97}$$

10 Series Expansions

10.1 Taylor Series

$$f(a) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots \tag{98}$$

10.2 The Generalized Binomial Theorem

$$(x+y)^r = \sum_{k=0}^{\infty} \binom{r}{k} x^{r-k} y^k \tag{99}$$

for $|x| > |y|$ real numbers and any complex number r.

10.3 Fourier Series

$$f\left(x\right) = \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{\mathrm{j}k_0 x} \tag{100}$$