Electromagnetic Field Theory

Week 14

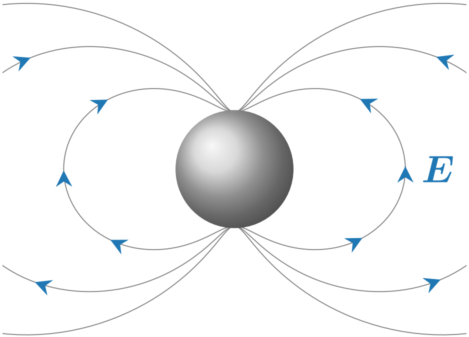
Miloslav Čapek

Department of Electromagnetic Field
Czech Technical University in Prague
Czech Republic
em@fel.cvut.cz

January 7, 2026
Winter semester 2025/26

Outline

1. Displacement Current
2. Maxwell's Equations
3. Poynting's Theorem
4. Wave Equation
5. Continuity Equation
6. Time-Harmonic Domain



Need For Displacement Current

Maxwell's equations before Maxwell

$$\begin{aligned} \nabla \cdot \boldsymbol{E} &= \frac{\rho}{\varepsilon_0}, \\ \nabla \cdot \boldsymbol{B} &= 0, \\ \nabla \times \boldsymbol{E} &= -\frac{\partial \boldsymbol{B}}{\partial t}, \\ \nabla \times \boldsymbol{B} &= \mu_0 \boldsymbol{J}. \end{aligned}$$

Displacement Current

Complete Ampère's law:

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0 \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}$$

with

$$\boldsymbol{J}_{\mathrm{d}} = \frac{\partial \boldsymbol{D}}{\partial t}$$

being the displacement current.

Maxwell's Equations

Complete Maxwell's equations in differential and integral form:

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \boldsymbol{B} = 0$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \frac{1}{c_0^2}\frac{\partial \boldsymbol{E}}{\partial t}$$

$$\iint_{S'} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \frac{1}{\varepsilon_0} \iiint_{V'} \rho \, \mathrm{d}V'$$

$$\oiint_{S'} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = 0$$

$$\oint_{l'} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l}' = -\frac{\mathrm{d}}{\mathrm{d}t} \iint \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S}$$

$$\oint_{l'} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l}' = \mu_0 \iint_{S'} \boldsymbol{J} \cdot \mathrm{d}\boldsymbol{S} + \frac{1}{c_0^2}\frac{\mathrm{d}}{\mathrm{d}t} \iint_{S'} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S}$$

Material constitutive relations:

$$\boldsymbol{D} = \varepsilon_0 \varepsilon_{\mathrm{r}} \boldsymbol{E}$$

$$\boldsymbol{B} = \mu_0 \mu_{\mathrm{r}} \boldsymbol{H}$$

Lorentz force:

$$\boldsymbol{F} = q\left(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}\right)$$

Poynting's Theorem (Derivation)

Taking Ampère's law multiplied by $\boldsymbol{E}$

$$\boldsymbol{E} \cdot (\nabla \times \boldsymbol{H}) = \boldsymbol{E} \cdot \boldsymbol{J} + \left(\boldsymbol{D} \cdot \frac{\partial \boldsymbol{E}}{\partial t} \right)$$

and applying vector identity

$$\nabla \cdot (\boldsymbol{E} \times \boldsymbol{H}) = \boldsymbol{H} \cdot (\nabla \times \boldsymbol{E}) - \boldsymbol{E} \cdot (\nabla \times \boldsymbol{H})$$

gives

$$\boldsymbol{H} \cdot (\nabla \times \boldsymbol{E}) - \nabla \cdot (\boldsymbol{E} \times \boldsymbol{H}) = \boldsymbol{E} \cdot \boldsymbol{J} + \boldsymbol{E} \cdot \frac{\partial \boldsymbol{D}}{\partial t}.$$

By substituting Faraday's law of induction, we get

$$\boldsymbol{J} \cdot \boldsymbol{E} + \boldsymbol{E} \cdot \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{H} \cdot \frac{\partial \boldsymbol{B}}{\partial t} + \nabla \cdot (\boldsymbol{E} \times \boldsymbol{H}) = 0.$$

Poynting's Theorem (Differential Form)

Considering non-dispersive media, we can use $\boldsymbol{D} = \varepsilon \boldsymbol{E}$ and $\boldsymbol{B} = \mu \boldsymbol{H}$, so

$$\frac{\partial}{\partial t}\left(\boldsymbol{E} \cdot \varepsilon \boldsymbol{E}\right) = \frac{\partial \boldsymbol{E}}{\partial t} \cdot \varepsilon \boldsymbol{E} + \boldsymbol{E} \cdot \frac{\partial \varepsilon \boldsymbol{E}}{\partial t} = 2\varepsilon \boldsymbol{E} \cdot \frac{\partial \boldsymbol{E}}{\partial t}$$

and analogously for the magnetic field, leading to

$$\boldsymbol{J} \cdot \boldsymbol{E} + \frac{\partial u}{\partial t} + \nabla \cdot \boldsymbol{S} = 0,$$

where

$$\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}$$

is the Poynting vector in $[\mathrm{W/m^2}]$ and

$$u = \frac{1}{2}\left(\boldsymbol{E} \cdot \boldsymbol{D} + \boldsymbol{H} \cdot \boldsymbol{B}\right) = \frac{\varepsilon}{2}|\boldsymbol{E}|^2 + \frac{1}{2\mu}|\boldsymbol{B}|^2$$

is the total electromagnetic energy density (even for time-dependent fields).

Poynting's Theorem (Integral Form)

The Poynting's theorem states that the electromagnetic energy is conserved

$$-\oiint_S \boldsymbol{S} \cdot \mathrm{d}\boldsymbol{S} = \frac{\partial W}{\partial t} + \iiint_V \boldsymbol{E} \cdot \boldsymbol{J} \, \mathrm{d}V$$

instantaneous power absorbed in volume V

$$\iiint_V \boldsymbol{E} \cdot \boldsymbol{J} \, \mathrm{d}V$$

electromagnetic energy in volume V

$$W = \frac{1}{2} \iiint (\boldsymbol{D} \cdot \boldsymbol{E} + \boldsymbol{B} \cdot \boldsymbol{H}) \, \mathrm{d}V$$

instantaneous power escaping closed surface $S \equiv \partial V$

$$-\oiint_S \boldsymbol{S} \cdot \mathrm{d}\boldsymbol{S}$$

Wave Equation

Consider free space (vacuum):

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \qquad \nabla \times \boldsymbol{B} = \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t}$$

$$\nabla \cdot \boldsymbol{E} = 0 \qquad \nabla \cdot \boldsymbol{B} = 0$$

Take the curl of Faraday's law

$$\nabla \times (\nabla \times \boldsymbol{E}) = -\frac{\partial}{\partial t}\nabla \times \boldsymbol{B}$$

and the identity

$$\nabla \times (\nabla \times \boldsymbol{E}) = \nabla (\nabla \cdot \boldsymbol{E}) - \nabla^2 \boldsymbol{E},$$

so

$$\nabla^2 \boldsymbol{E} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} = 0$$

and analogously for magnetic field $\boldsymbol{B}$.

Continuity Equation

We start with Ampère's law and take the divergence of both sides

$$\nabla \cdot (\nabla \times \boldsymbol{H}) = \nabla \cdot \boldsymbol{J} + \nabla \cdot \left(\frac{\partial \boldsymbol{D}}{\partial t} \right).$$

The left-hand side is identically zero, and we get

$$\nabla \cdot \boldsymbol{J} = -\frac{\partial}{\partial t} (\nabla \cdot \boldsymbol{D}).$$

By substituting Gauss's law

$$\nabla \cdot \boldsymbol{D} = \rho,$$

where ρ represents free charge density, we get

$$\nabla \cdot \boldsymbol{J} = -\frac{\partial \rho}{\partial t}.$$

Phasors

- Let us consider a time-harmonic domain:

$$\boldsymbol{a}(x, y, z, t) = \boldsymbol{a}(\boldsymbol{r}, t) = \boldsymbol{A}(\boldsymbol{r}) \sin(\omega t + \varphi) = \operatorname{Im}\{\boldsymbol{A}(\boldsymbol{r}) \mathrm{e}^{\mathrm{j}\varphi} \mathrm{e}^{\mathrm{j}\omega t}\}$$

Useful rules to deal with:

$$\frac{\partial \boldsymbol{A}(\boldsymbol{r}, t)}{\partial t} \longrightarrow \mathrm{j}\omega \mathbf{A}(\boldsymbol{r}, \omega)$$

$$\int \boldsymbol{A}(\boldsymbol{r}, t) \, \mathrm{d}t \longrightarrow \frac{1}{\mathrm{j}\omega} \mathbf{A}(\boldsymbol{r}, \omega)$$

Maxwell's Equations in Time-Harmonic Domain

$$\begin{aligned}
\nabla \cdot \mathbf{D} &= \rho, \\
\nabla \cdot \mathbf{B} &= 0, \\
\nabla \times \mathbf{E} &= -\mathrm{j}\omega \mathbf{B}, \\
\nabla \times \mathbf{H} &= \mathbf{J} + \mathrm{j}\omega \mathbf{D},
\end{aligned}$$

$$\begin{aligned}
\oiint_S \mathbf{D} \cdot \mathrm{d}\boldsymbol{S} &= \iiint_V \rho \, \mathrm{d}V, \\
\oiint_S \mathbf{B} \cdot \mathrm{d}\boldsymbol{S} &= 0, \\
\oint_l \mathbf{E} \cdot \mathrm{d}\boldsymbol{l} &= -\mathrm{j}\omega \iint_S \mathbf{B} \cdot \mathrm{d}\boldsymbol{S}, \\
\oint_l \mathbf{H} \cdot \mathrm{d}\boldsymbol{l} &= \iint_S \boldsymbol{J} \cdot \mathrm{d}\boldsymbol{S} + \mathrm{j}\omega \iint_S \mathbf{D} \cdot \mathrm{d}\boldsymbol{S}.
\end{aligned}$$

RMS (Effective Values)

Root mean square value (inspired by the definition $x_{\mathrm{RMS}} = \sqrt{(x_1^2 + x_2 + \dots)/n}$):

$$f_{\mathrm{RMS}} = \sqrt{\frac{1}{T_2 - T_1} \int\limits_{T_1}^{T_2} |f(t)|^2 \, \mathrm{d}t} = \sqrt{\frac{1}{T_2 - T_1} \int\limits_{T_1}^{T_2} |I_0 \sin(\omega t)|^2 \, \mathrm{d}t}$$

evaluated as

$$I_{\mathrm{RMS}} = I_0 \sqrt{\frac{1}{T_2 - T_1} \int\limits_{T_1}^{T_2} \sin^2(\omega t) \, \mathrm{d}t} = I_0 \sqrt{\frac{1}{T_2 - T_1} \int\limits_{T_1}^{T_2} \frac{1 - \cos(2\omega t)}{2} \, \mathrm{d}t}$$

$$= I_0 \sqrt{\frac{1}{T_2 - T_1} \left[\frac{t}{2} - \frac{\sin(2\omega t)}{4\omega} \right]_{T_1}^{T_2}} = \frac{I_0}{\sqrt{2}}.$$

Wave Equation in Time-Harmonic Domain

Take the wave equation

$$\nabla^2 \boldsymbol{E} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} = 0$$

and use $\mu\varepsilon = 1/c^2$ and $\partial^2/\partial t^2 \to -\omega^2$ to get

$$\nabla^2 \mathbf{E} + k^2 \mathbf{E} = 0,$$

where the wavenumber $k = \omega\sqrt{\mu\varepsilon} = \omega/c$.

- The situation is considerably more complicated for an arbitrary environment!

Free-Space Propagation

We simplify the discussion to

$$\mathbf{E}(z) = E(z)\hat{\boldsymbol{x}} = E_x(z).$$

Solving a differential equation

$$\frac{\mathrm{d}^2 E_x(z)}{dz^2} + k^2 E_x(z) = 0$$

gives

$$E_x(z) = E_x(0)\mathrm{e}^{-\mathrm{j}kz} + E_x(0)\mathrm{e}^{\mathrm{j}kz} = E_x^+(z) + E_x^-(z).$$

- Two general solutions (waves).
- One is always non-physical (the amplitude is non-increasing when traveling from its sources).

Questions?

BAB17EMP
em@fel.cvut.cz

January 7, 2026
Winter semester 2025/26

This document has been created as a part of BAB17EMP course.