Linear-Quadratic Control

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Linear dynamical system

$$x_{t+1} = Ax_t + Bu_t, \quad t = 1, 2, \ldots$$

- n-vector x_t is *state* at time t
- m-vector u_t is *input* at time t
- $n \times n$ matrix A is *dynamics matrix*
- $n \times m$ matrix B is *input matrix*
- sequence $x_1, x_2, \ldots$ is called *state trajectory*

Simulation

- given x_1, $u_1, u_2, \ldots$ find $x_2, x_3, \ldots$
- can be done by recursion: for $t = 1, 2, \ldots$,

$$x_{t+1} = Ax_t + Bu_t$$

Vehicle example

consider a vehicle moving in a plane:

- sample position and velocity at times $\tau = 0, h, 2h, \ldots$
- 2-vectors p_t and v_t are position and velocity at time ht
- 2-vector u_t gives applied force on the vehicle time ht
- friction force is $-\eta v_t$
- vehicle has mass m
- for small h,

$$m\frac{v_{t+1} - v_t}{h} \approx -\eta v_t + u_t, \qquad \frac{p_{t+1} - p_t}{h} \approx v_t$$

- we use approximate state update

$$v_{t+1} = (1 - h\eta/m)v_t + (h/m)u_t, \qquad p_{t+1} = p_t + hv_t$$

- vehicle state is 4-vector $x_t = (p_t, v_t)$
- dynamics recursion is

$$x_{t+1} = Ax_t + Bu_t,$$

where

$$A = \begin{bmatrix} 1 & 0 & h & 0 \\ 0 & 1 & 0 & h \\ 0 & 0 & 1 - h\eta/m & 0 \\ 0 & 0 & 0 & 1 - h\eta/m \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ h/m & 0 \\ 0 & h/m \end{bmatrix}$$

Control

- x_1 is given
- choose $u_1, u_2, \ldots, u_{T-1}$ to achieve some goals, *e.g.*,
  - terminal state should have some fixed value: $x_T = x^{\mathrm{des}}$
  - $u_1, u_2, \ldots, u_{T-1}$ should be small, say measured as

    $$\|u_1\|^2 + \cdots + \|u_{T-1}\|^2$$

    (sometimes called 'energy')
- many control problems are linearly constrained least-squares problems

Minimum-energy state transfer

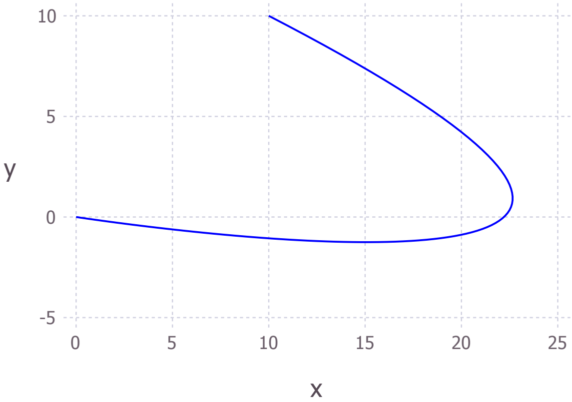
- given initial state x_1 and desired final state x^{des}
- choose $u_1, \ldots, u_{T-1}$ to minimize 'energy'

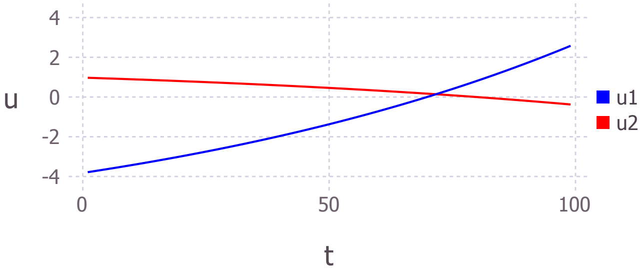
$$\begin{array}{ll} \text{minimize} & \|u_1\|^2 + \cdots + \|u_{T-1}\|^2 \\ \text{subject to} & x_{t+1} = Ax_t + Bu_t, \quad t = 1, \ldots, T-1 \\ & x_T = x^{\mathrm{des}} \end{array}$$

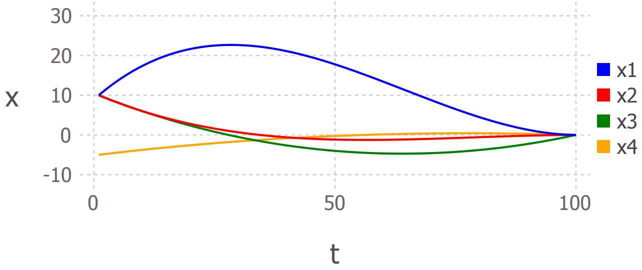
  variables are $x_2, \ldots, x_T, u_1, \ldots, u_{T-1}$
- roughly speaking: find minimum energy inputs that steer the state to given target state over T periods

State transfer example

vehicle model with $T = 100,\ x_1 = (10, 10, 10, -5),\ x^{\rm des} = 0$




u
4
2
0
-2
-4
0
50
100
t
u1
u2


x
30
20
10
0
-10
0
50
100
t
x1
x2
x3
x4

Output tracking

- $y_t = Cx_t$ is output (*e.g.*, position)
- y_t should follow a desired trajectory, *i.e.*, sum square *tracking error*

$$\|y_2 - y_2^{\text{des}}\|^2 + \cdots + \|y_T - y_T^{\text{des}}\|^2$$

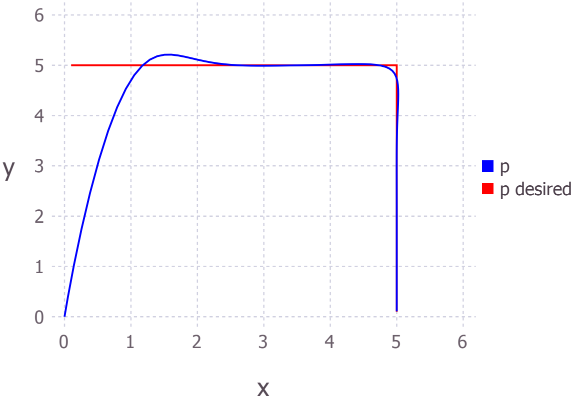
  should be small
- the output tracking problem is

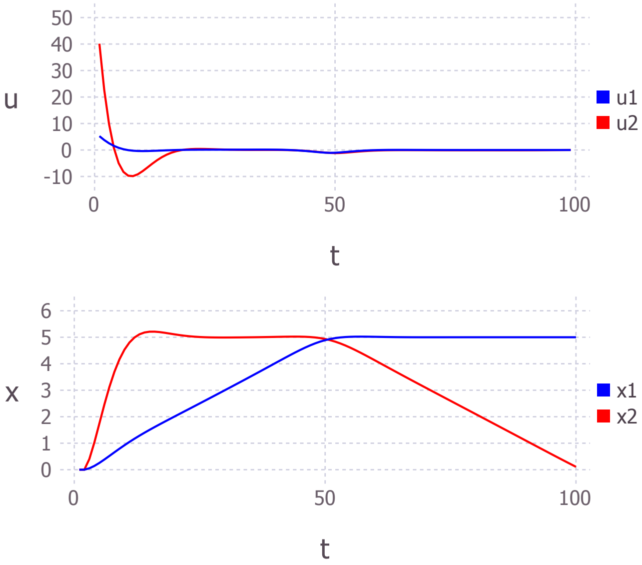
$$\begin{array}{ll} \text{minimize} & \sum_{t=2}^{T} \|y_t - y_t^{\text{des}}\|^2 + \rho \sum_{t=1}^{T-1} \|u_t\|^2 \\ \text{subject to} & x_{t+1} = Ax_t + Bu_t, \quad t = 1, \ldots, T-1 \\ & y_t = Cx_t, \quad t = 1, \ldots, T-1 \end{array}$$

  variables are $x_2, \ldots, x_T, u_1, \ldots, u_{T-1}, y_2, \ldots, y_T$
- parameter $\rho > 0$ trades off control 'energy' and tracking error

Output tracking example

vehicle model with $T = 100,\ \rho = 0.1,\ x_1 = 0,\ y_t = p_t$ (position tracking)




50
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50
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t
u1
u2
6
5
4
3
2
1
0
x
0
50
100
t
x1
x2

Waypoints

- using output, can specify *waypoints*
- specify output (position) $w^{(k)}$ at time t_k at K total places

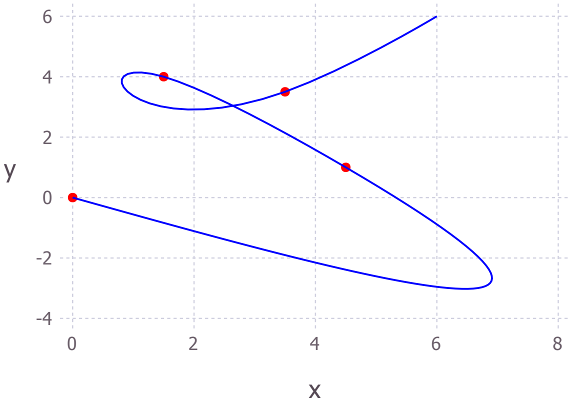
$$\begin{array}{ll} \text{minimize} & \|u_1\|^2 + \cdots + \|u_{T-1}\|^2 \\ \text{subject to} & x_{t+1} = Ax_t + Bu_t, \quad t = 1, \ldots, T-1 \\ & Cx_{t_k} = w^{(k)}, \quad k = 1, \ldots, K \end{array}$$

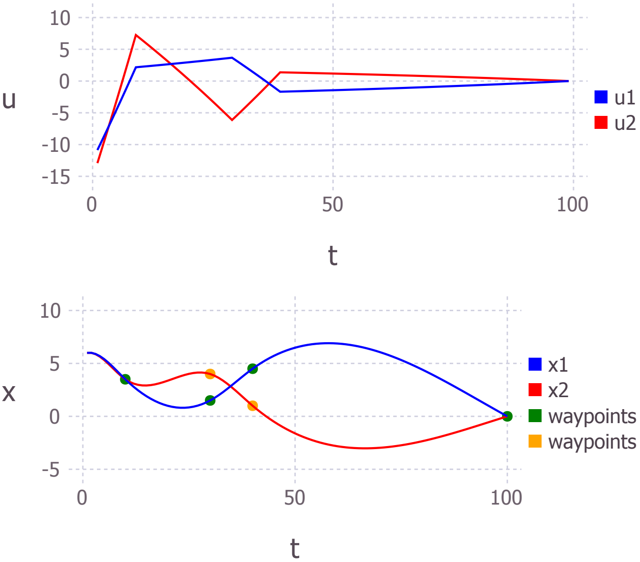
variables are $x_2, \ldots, x_T, u_1, \ldots, u_{T-1}$

Waypoints example

- vehicle model
- $T = 100,\ x_1 = (10, 10, 20, 0),\ x^{\text{des}} = 0$
- $K = 4,\ t_1 = 10,\ t_2 = 30,\ t_3 = 40,\ t_4 = 80$
- $w^{(1)} = \begin{bmatrix} 3.5 \\ 3.5 \end{bmatrix},\ w^{(2)} = \begin{bmatrix} 1.5 \\ 4 \end{bmatrix},\ w^{(3)} = \begin{bmatrix} 4.5 \\ 1 \end{bmatrix},\ w^{(4)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Waypoints example




10
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-10
-15
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50
100
t
u1
u2
10
5
0
-5
x
0
50
100
t
x1
x2
waypoints
waypoints

Rendezvous

- we control two vehicles with dynamics

$$x_{t+1} = Ax_t + Bu_t, \quad z_{t+1} = Az_t + Bv_t$$

- final relative state constraint $x_T = z_T$
- formulate as state transfer problem:

$$\begin{array}{ll} \text{minimize} & \sum_{t=1}^{T-1} (\|u_t\|^2 + \|v_t\|^2) \\ \text{subject to} & x_{t+1} = Ax_t + Bu_t, \quad t = 1, \ldots, T-1, \\ & z_{t+1} = Az_t + Bv_t, \quad t = 1, \ldots, T-1, \\ & x_T = z_T \end{array}$$

variables are $x_2, \ldots, x_T, u_1, \ldots, u_{T-1}$, $z_2, \ldots, z_T, v_1, \ldots, v_{T-1}$

Rendezvous example

$x_1 = (0, 0, 0, -5)$, $z_1 = (10, 10, 5, 0)$