Mathematical Identities

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In the following, the rectangular system is characterized by coordinates (x, y, z) and unit vectors $\begin{bmatrix} \boldsymbol{x}_0 & \boldsymbol{y}_0 & \boldsymbol{z}_0 \end{bmatrix}$. The cylindrical system is characterized by coordinates (ρ, φ, z) and unit vectors $\begin{bmatrix} \boldsymbol{\rho}_0 & \boldsymbol{\varphi}_0 & \boldsymbol{z}_0 \end{bmatrix}$. The spherical system is characterized by coordinates (r, θ, φ) and unit vectors $\begin{bmatrix} \boldsymbol{r}_0 & \boldsymbol{\theta}_0 & \boldsymbol{\varphi}_0 \end{bmatrix}$.

0.1 Point Transformations

Rectangular - Cylindrical

$$\begin{aligned} x &= \rho \cos\varphi \\ y &= \rho \sin\varphi \\ z &= z \end{aligned} \tag{1}$$

Cylindrical - Rectangular

$$\begin{aligned} \rho &= \sqrt{x^2 + y^2} \\ \tan\varphi &= \frac{y}{x} \\ z &= z \end{aligned} \tag{2}$$

Rectangular - Spherical

$$\begin{aligned} x &= r \cos\varphi \sin\theta \\ y &= r \sin\varphi \sin\theta \\ z &= r \cos\theta \end{aligned} \tag{3}$$

Spherical - Rectangular

$$\begin{aligned} r &= \sqrt{x^2 + y^2 + z^2} \\ \tan\theta &= \frac{\sqrt{x^2 + y^2}}{z} \\ \tan\varphi &= \frac{y}{x} \end{aligned} \tag{4}$$

Cylindrical - Spherical

$$\begin{aligned} \rho &= r \sin\theta \\ z &= r \cos\theta \\ \varphi &= \varphi \end{aligned} \tag{5}$$

Spherical - Cylindrical

$$\begin{aligned} r &= \sqrt{\rho^2 + z^2} \\ \tan\theta &= \frac{\rho}{z} \\ \varphi &= \varphi \end{aligned} \tag{6}$$

0.2 Vector Transformations

Rectangular - Cylindrical

$$\begin{aligned} \boldsymbol{x}_0 &= \boldsymbol{\rho}_0 \cos\varphi - \boldsymbol{\varphi}_0 \sin\varphi \\ \boldsymbol{y}_0 &= \boldsymbol{\rho}_0 \sin\varphi + \boldsymbol{\varphi}_0 \cos\varphi \\ \boldsymbol{z}_0 &= \boldsymbol{z}_0 \end{aligned} \tag{7}$$

Cylindrical - Rectangular

$$\begin{aligned} \boldsymbol{\rho}_0 &= \boldsymbol{x}_0 \cos\varphi + \boldsymbol{y}_0 \sin\varphi = \frac{\boldsymbol{x}_0 x + \boldsymbol{y}_0 y}{\sqrt{x^2+y^2}} \\ \boldsymbol{\varphi}_0 &= -\boldsymbol{x}_0 \sin\varphi + \boldsymbol{y}_0 \cos\varphi = \frac{-\boldsymbol{x}_0 y + \boldsymbol{y}_0 x}{\sqrt{x^2+y^2}} \\ \boldsymbol{z}_0 &= \boldsymbol{z}_0 \end{aligned} \tag{8}$$

Rectangular - Spherical

$$\begin{aligned} \boldsymbol{x}_0 &= \boldsymbol{r}_0 \sin\theta\cos\varphi + \boldsymbol{\theta}_0 \cos\theta\cos\varphi - \boldsymbol{\varphi}_0 \sin\varphi \\ \boldsymbol{y}_0 &= \boldsymbol{r}_0 \sin\theta\sin\varphi + \boldsymbol{\theta}_0 \cos\theta\sin\varphi + \boldsymbol{\varphi}_0 \cos\varphi \\ \boldsymbol{z}_0 &= \boldsymbol{r}_0 \cos\theta - \boldsymbol{\theta}_0 \sin\theta \end{aligned} \tag{9}$$

Spherical - Rectangular

$$\begin{aligned} \boldsymbol{r}_0 &= \boldsymbol{x}_0 \sin\theta\cos\varphi + \boldsymbol{y}_0 \sin\theta\sin\varphi + \boldsymbol{z}_0\cos\theta = \frac{\boldsymbol{x}_0 x + \boldsymbol{y}_0 y + \boldsymbol{z}_0 z}{\sqrt{x^2+y^2+z^2}} \\ \boldsymbol{\theta}_0 &= \boldsymbol{x}_0 \cos\theta\cos\varphi + \boldsymbol{y}_0 \cos\theta\sin\varphi - \boldsymbol{z}_0\sin\theta = \frac{\boldsymbol{x}_0 zx + \boldsymbol{y}_0 zy - \boldsymbol{z}_0\left(x^2+y^2\right)}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}} \\ \boldsymbol{\varphi}_0 &= -\boldsymbol{x}_0 \sin\varphi + \boldsymbol{y}_0 \cos\varphi = \frac{-\boldsymbol{x}_0 y + \boldsymbol{y}_0 x}{\sqrt{x^2+y^2}} \end{aligned} \tag{10}$$

Cylindrical - Spherical

$$\begin{aligned} \boldsymbol{\rho}_0 &= \boldsymbol{r}_0 \sin\theta + \boldsymbol{\theta}_0 \cos\theta \\ \boldsymbol{\varphi}_0 &= \boldsymbol{\varphi}_0 \\ \boldsymbol{z}_0 &= \boldsymbol{r}_0 \cos\theta - \boldsymbol{\theta}_0 \sin\theta \end{aligned} \tag{11}$$

Spherical - Cylindrical

$$\begin{aligned} \boldsymbol{r}_0 &= \boldsymbol{\rho}_0 \sin\theta + \boldsymbol{z}_0 \cos\theta = \frac{\boldsymbol{\rho}_0 \rho + \boldsymbol{z}_0 z}{\sqrt{\rho^2+z^2}} \\ \boldsymbol{\theta}_0 &= \boldsymbol{\rho}_0 \cos\theta - \boldsymbol{z}_0 \sin\theta = \frac{\boldsymbol{\rho}_0 z - \boldsymbol{z}_0 \rho}{\sqrt{\rho^2+z^2}} \\ \boldsymbol{\varphi}_0 &= \boldsymbol{\varphi}_0 \end{aligned} \tag{12}$$

0.3 Differential Operators

Rectangular coordinate system

$$\begin{aligned} \nabla f &= \boldsymbol{x}_0 \frac{\partial f}{\partial x} + \boldsymbol{y}_0 \frac{\partial f}{\partial y} + \boldsymbol{z}_0 \frac{\partial f}{\partial z} \\ \nabla\cdot\boldsymbol{F} &= \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \\ \nabla\times\boldsymbol{F} &= \boldsymbol{x}_0\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \boldsymbol{y}_0\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \boldsymbol{z}_0\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) \\ \nabla^2 f &= \nabla\cdot(\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \\ \nabla^2\boldsymbol{F} &= \nabla\left(\nabla\cdot\boldsymbol{F}\right) - \nabla\times\nabla\times\boldsymbol{F} = \boldsymbol{x}_0\nabla^2 F_x + \boldsymbol{y}_0\nabla^2 F_y + \boldsymbol{z}_0\nabla^2 F_z \end{aligned} \tag{13}$$

Cylindrical coordinate system

$$
\begin{aligned}
&\nabla f = \boldsymbol{\rho}_0 \frac{\partial f}{\partial \rho} + \boldsymbol{\varphi}_0 \frac{1}{\rho} \frac{\partial f}{\partial \varphi} + \boldsymbol{z}_0 \frac{\partial f}{\partial z} \\
&\nabla \cdot \boldsymbol{F} = \frac{1}{\rho} \frac{\partial \left(\rho F_\rho \right)}{\partial \rho} + \frac{1}{\rho} \frac{\partial F_\varphi}{\partial \varphi} + \frac{\partial F_z}{\partial z} \\
&\nabla \times \boldsymbol{F} = \boldsymbol{\rho}_0 \left(\frac{1}{\rho} \frac{\partial F_z}{\partial \varphi} - \frac{\partial F_\varphi}{\partial z} \right) + \boldsymbol{\varphi}_0 \left(\frac{\partial F_\rho}{\partial z} - \frac{\partial F_z}{\partial \rho} \right) + \boldsymbol{z}_0 \left(\frac{1}{\rho} \frac{\partial \left(\rho F_\varphi \right)}{\partial \rho} - \frac{1}{\rho} \frac{\partial F_\rho}{\partial \varphi} \right) \\
&\nabla^2 f = \nabla \cdot (\nabla f) = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2} \\
&\nabla^2 \boldsymbol{F} = \nabla \left(\nabla \cdot \boldsymbol{F} \right) - \nabla \times \nabla \times \boldsymbol{F} = \\
&\quad \boldsymbol{\rho}_0 \left(\frac{\partial^2 F_\rho}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial F_\rho}{\partial \rho} - \frac{F_\rho}{\rho^2} + \frac{1}{\rho^2} \frac{\partial^2 F_\rho}{\partial \varphi^2} - \frac{2}{\rho^2} \frac{\partial F_\varphi}{\partial \varphi} + \frac{\partial^2 F_\rho}{\partial z^2} \right) + \\
&\quad \boldsymbol{\varphi}_0 \left(\frac{\partial^2 F_\varphi}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial F_\varphi}{\partial \rho} - \frac{F_\varphi}{\rho^2} + \frac{1}{\rho^2} \frac{\partial^2 F_\varphi}{\partial \varphi^2} + \frac{2}{\rho^2} \frac{\partial F_\rho}{\partial \varphi} + \frac{\partial^2 F_\varphi}{\partial z^2} \right) + \\
&\quad \boldsymbol{z}_0 \left(\frac{\partial^2 F_z}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial F_z}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 F_z}{\partial \varphi^2} + \frac{\partial^2 F_z}{\partial z^2} \right)
\end{aligned} \tag{14}
$$

Spherical coordinate system

$$\nabla f=\boldsymbol{r}_0\frac{\partial f}{\partial r}+\boldsymbol{\theta}_0\frac{1}{r}\frac{\partial f}{\partial \theta}+\boldsymbol{\varphi}_0\frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}$$

$$\nabla\cdot\boldsymbol{F}=\frac{1}{r^2}\frac{\partial\left(r^2F_r\right)}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(F_\theta\sin\theta\right)+\frac{1}{r\sin\theta}\frac{\partial F_\varphi}{\partial\varphi}$$

$$\nabla\times\boldsymbol{F}=\frac{\boldsymbol{r}_0}{r\sin\theta}\left(\frac{\partial}{\partial\theta}\left(F_\varphi\sin\theta\right)-\frac{\partial F_\theta}{\partial\varphi}\right)+\frac{\boldsymbol{\theta}_0}{r}\left(\frac{1}{\sin\theta}\frac{\partial F_r}{\partial\varphi}-\frac{\partial\left(rF_\varphi\right)}{\partial r}\right)+$$

$$\frac{\boldsymbol{\varphi}_0}{r}\left(\frac{\partial\left(rF_\theta\right)}{\partial r}-\frac{\partial F_r}{\partial\theta}\right)$$

$$\nabla^2 f=\nabla\cdot(\nabla f)=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}$$

$$\begin{aligned}
&\nabla^2\boldsymbol{F}=\nabla\left(\nabla\cdot\boldsymbol{F}\right)-\nabla\times\nabla\times\boldsymbol{F}=\\
&\boldsymbol{r}_0\left(\begin{array}{l}\dfrac{\partial^2F_r}{\partial r^2}+\dfrac{2}{r}\dfrac{\partial F_r}{\partial r}-\dfrac{2}{r^2}F_r+\dfrac{1}{r^2}\dfrac{\partial^2F_r}{\partial\theta^2}+\dfrac{\cot\theta}{r^2}\dfrac{\partial F_r}{\partial\theta}+\dfrac{1}{r^2\sin^2\theta}\dfrac{\partial^2F_r}{\partial\varphi^2}\\ -\dfrac{2}{r^2}\dfrac{\partial F_\theta}{\partial\theta}-\dfrac{2\cot\theta}{r^2}F_\theta-\dfrac{2}{r^2\sin\theta}\dfrac{\partial F_\varphi}{\partial\varphi}\end{array}\right)+\\
&\boldsymbol{\theta}_0\left(\begin{array}{l}\dfrac{\partial^2F_\theta}{\partial r^2}+\dfrac{2}{r}\dfrac{\partial F_\theta}{\partial r}-\dfrac{1}{r^2\sin^2\theta}F_\theta+\dfrac{1}{r^2}\dfrac{\partial^2F_\theta}{\partial\theta^2}+\dfrac{\cot\theta}{r^2}\dfrac{\partial F_\theta}{\partial\theta}\\ +\dfrac{1}{r^2\sin^2\theta}\dfrac{\partial^2F_\theta}{\partial\varphi^2}+\dfrac{2}{r^2}\dfrac{\partial F_r}{\partial\theta}-\dfrac{2\cot\theta}{r^2\sin\theta}\dfrac{\partial F_\varphi}{\partial\varphi}\end{array}\right)+\\
&\boldsymbol{\varphi}_0\left(\begin{array}{l}\dfrac{\partial^2F_\varphi}{\partial r^2}+\dfrac{2}{r}\dfrac{\partial F_\varphi}{\partial r}-\dfrac{1}{r^2\sin^2\theta}F_\varphi+\dfrac{1}{r^2}\dfrac{\partial^2F_\varphi}{\partial\theta^2}+\dfrac{\cot\theta}{r^2}\dfrac{\partial F_\varphi}{\partial\theta}\\ +\dfrac{1}{r^2\sin^2\theta}\dfrac{\partial^2F_\varphi}{\partial\varphi^2}+\dfrac{2}{r^2\sin\theta}\dfrac{\partial F_r}{\partial\varphi}+\dfrac{2\cot\theta}{r^2\sin\theta}\dfrac{\partial F_\theta}{\partial\varphi}\end{array}\right)
\end{aligned}\tag{15}$$

0.4 Differential Identities

$$
\begin{aligned}
\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) &= \boldsymbol{b} \cdot (\boldsymbol{c} \times \boldsymbol{a}) = \boldsymbol{c} \cdot (\boldsymbol{a} \times \boldsymbol{b}) \\
\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) &= (\boldsymbol{a} \cdot \boldsymbol{c})\, \boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\, \boldsymbol{c} \\
(\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{c} \times \boldsymbol{d}) &= (\boldsymbol{a} \cdot \boldsymbol{c})\,(\boldsymbol{b} \cdot \boldsymbol{d}) - (\boldsymbol{a} \cdot \boldsymbol{d})\,(\boldsymbol{b} \cdot \boldsymbol{c}) \\
|\boldsymbol{a} \times \boldsymbol{b}|^2 &= |\boldsymbol{a}|^2\, |\boldsymbol{b}|^2 - |\boldsymbol{a} \cdot \boldsymbol{b}^*|^2 \\
\nabla \times (\nabla f) &= 0 \\
\nabla \cdot (\nabla \times \boldsymbol{F}) &= 0 \\
\nabla\,(f \cdot g) &= g \nabla f + f \nabla g \\
\nabla \cdot (\boldsymbol{F} f) &= \boldsymbol{F} \cdot (\nabla f) + f\,(\nabla \cdot \boldsymbol{F}) \\
\nabla \times (\boldsymbol{F} f) &= (\nabla f) \times \boldsymbol{F} + f\,(\nabla \times \boldsymbol{F}) \\
\nabla \left(\frac{f}{g}\right) &= \frac{g \nabla f - f \nabla g}{g^2} \\
\nabla \cdot \left(\frac{\boldsymbol{F}}{f}\right) &= \frac{f\,(\nabla \cdot \boldsymbol{F}) - \boldsymbol{F} \cdot (\nabla f)}{f^2} \\
\nabla \times \left(\frac{\boldsymbol{F}}{f}\right) &= \frac{f\,(\nabla \times \boldsymbol{F}) - (\nabla f) \times \boldsymbol{F}}{f^2} \\
\nabla\,(\boldsymbol{F} \cdot \boldsymbol{G}) &= (\boldsymbol{F} \cdot \nabla)\, \boldsymbol{G} + (\boldsymbol{G} \cdot \nabla)\, \boldsymbol{F} + \boldsymbol{F} \times (\nabla \times \boldsymbol{G}) + \boldsymbol{G} \times (\nabla \times \boldsymbol{F}) \\
\nabla \cdot (\boldsymbol{F} \times \boldsymbol{G}) &= \boldsymbol{G} \cdot (\nabla \times \boldsymbol{F}) - \boldsymbol{F} \cdot (\nabla \times \boldsymbol{G}) \\
\nabla \times (\boldsymbol{F} \times \boldsymbol{G}) &= \boldsymbol{F}\,(\nabla \cdot \boldsymbol{G}) - \boldsymbol{G}\,(\nabla \cdot \boldsymbol{F}) + (\boldsymbol{G} \cdot \nabla)\, \boldsymbol{F} - (\boldsymbol{F} \cdot \nabla)\, \boldsymbol{G} \\
\nabla\,(f \circ g) &= (f' \circ g)\, \nabla g \\
\nabla \cdot (\boldsymbol{F} \circ g) &= \big(\boldsymbol{F}' \circ g\big) \cdot \nabla g \\
\nabla \times (\boldsymbol{F} \circ g) &= -\big(\boldsymbol{F}' \circ g\big) \times \nabla g
\end{aligned}
\tag{16}
$$

0.5 Integration Identities

$$\int\limits_V (\nabla \cdot \boldsymbol{F})\,\mathrm{d}V = \oint\limits_S \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S}$$

$$\int\limits_V (\nabla f)\,\mathrm{d}V = \oint\limits_S f\,\mathrm{d}\boldsymbol{S}$$

$$\int\limits_V (\nabla \times \boldsymbol{F})\,\mathrm{d}V = -\oint\limits_S \boldsymbol{F} \times \mathrm{d}\boldsymbol{S}$$

$$\int\limits_V \left[f\left(\nabla^2 g\right)\right]\mathrm{d}V = \oint\limits_S f\,(\nabla g) \cdot \mathrm{d}\boldsymbol{S} - \int\limits_V \left[(\nabla f) \cdot (\nabla g)\right]\mathrm{d}V$$

$$\int\limits_V \left[\boldsymbol{F} \cdot (\nabla \times \nabla \times \boldsymbol{G})\right]\mathrm{d}V = -\oint\limits_S \left[\boldsymbol{F} \times (\nabla \times \boldsymbol{G})\right] \cdot \mathrm{d}\boldsymbol{S} + \int\limits_V \left[(\nabla \times \boldsymbol{F}) \cdot (\nabla \times \boldsymbol{G})\right]\mathrm{d}V$$

$$\int\limits_V \left[f\left(\nabla^2 g\right) - g\left(\nabla^2 f\right)\right]\mathrm{d}V = \oint\limits_S \left[f\,(\nabla g) - g\,(\nabla f)\right] \cdot \mathrm{d}\boldsymbol{S}$$

$$\int\limits_V \left[\boldsymbol{G} \cdot (\nabla \times \nabla \times \boldsymbol{F}) - \boldsymbol{F} \cdot (\nabla \times \nabla \times \boldsymbol{G})\right]\mathrm{d}V = \oint\limits_S \left[\boldsymbol{F} \times (\nabla \times \boldsymbol{G}) - \boldsymbol{G} \times (\nabla \times \boldsymbol{F})\right] \cdot \mathrm{d}\boldsymbol{S}$$

$$\int\limits_S (\nabla \times \boldsymbol{F}) \cdot \mathrm{d}\boldsymbol{S} = \oint\limits_l \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{l}$$

$$\int\limits_S (\nabla f) \times \mathrm{d}\boldsymbol{S} = -\oint\limits_l f\,\mathrm{d}\boldsymbol{l} \tag{17}$$

0.6 Helmholtz's decomposition

Any vector field $\boldsymbol{F}$ can be decomposed into its irrotational and solenoidal part as $\boldsymbol{F} = -\nabla\phi + \nabla \times \boldsymbol{G}$, where

$$\begin{aligned}
\phi\,(\boldsymbol{r}) &= \frac{1}{4\pi}\int\limits_{V'} \frac{\nabla' \cdot \boldsymbol{F}\,(\boldsymbol{r'})}{|\boldsymbol{r} - \boldsymbol{r'}|}\mathrm{d}V' - \frac{1}{4\pi}\oint\limits_{S'} \frac{\boldsymbol{F}\,(\boldsymbol{r'})}{|\boldsymbol{r} - \boldsymbol{r'}|} \cdot \mathrm{d}\boldsymbol{S}' \\
\boldsymbol{G}\,(\boldsymbol{r}) &= \frac{1}{4\pi}\int\limits_{V'} \frac{\nabla' \times \boldsymbol{F}\,(\boldsymbol{r'})}{|\boldsymbol{r} - \boldsymbol{r'}|}\mathrm{d}V' + \frac{1}{4\pi}\oint\limits_{S'} \frac{\boldsymbol{F}\,(\boldsymbol{r'})}{|\boldsymbol{r} - \boldsymbol{r'}|} \times \mathrm{d}\boldsymbol{S}'
\end{aligned} \tag{18}$$

0.7 3D Fourier's Transformation

Definition

$$\mathrm{FT}\left\{f\left(\boldsymbol{r}\right)\right\} = \tilde{f}\left(\boldsymbol{k}\right) = \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} f\left(\boldsymbol{r}\right)\mathrm{e}^{-\mathrm{j}\boldsymbol{k}\cdot\boldsymbol{r}}\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$\mathrm{FT}^{-1}\left\{\tilde{f}\left(\boldsymbol{k}\right)\right\} = f\left(\boldsymbol{r}\right) = \frac{1}{\left(2\pi\right)^{3}}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \tilde{f}\left(\boldsymbol{k}\right)\mathrm{e}^{\mathrm{j}\boldsymbol{k}\cdot\boldsymbol{r}}\mathrm{d}k_x\mathrm{d}k_y\mathrm{d}k_z \tag{19}$$

Convolution and its Fourier's transformation

$$f\left(\boldsymbol{r}\right) * g\left(\boldsymbol{r}\right) = \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} f\left(\boldsymbol{r}-\boldsymbol{r'}\right)g\left(\boldsymbol{r'}\right)\mathrm{d}x'\mathrm{d}y'\mathrm{d}z'$$

$$\mathrm{FT}\left\{f\left(\boldsymbol{r}\right) * g\left(\boldsymbol{r}\right)\right\} = \tilde{f}\left(\boldsymbol{k}\right)\tilde{g}\left(\boldsymbol{k}\right) \tag{20}$$

Differential operators and their Fourier's transformation

$$\mathrm{FT}\left\{\nabla\cdot\boldsymbol{F}\left(\boldsymbol{r}\right)\right\} = \mathrm{j}\boldsymbol{k}\cdot\tilde{\boldsymbol{F}}\left(\boldsymbol{k}\right)$$

$$\mathrm{FT}\left\{\nabla\times\boldsymbol{F}\left(\boldsymbol{r}\right)\right\} = \mathrm{j}\boldsymbol{k}\times\tilde{\boldsymbol{F}}\left(\boldsymbol{k}\right) \tag{21}$$

0.8 Trigonometric identities

$$\sin^2\left(\alpha\right) = \frac{1-\cos\left(2\alpha\right)}{2}$$

$$\cos^2\left(\alpha\right) = \frac{1+\cos\left(2\alpha\right)}{2}$$

$$\sin\left(\alpha+\beta\right) = \sin\left(\alpha\right)\cos\left(\beta\right) + \cos\left(\alpha\right)\sin\left(\beta\right)$$

$$\sin\left(\alpha-\beta\right) = \sin\left(\alpha\right)\cos\left(\beta\right) - \cos\left(\alpha\right)\sin\left(\beta\right)$$

$$\cos\left(\alpha+\beta\right) = \cos\left(\alpha\right)\cos\left(\beta\right) - \sin\left(\alpha\right)\sin\left(\beta\right)$$

$$\cos\left(\alpha-\beta\right) = \cos\left(\alpha\right)\cos\left(\beta\right) + \sin\left(\alpha\right)\sin\left(\beta\right) \tag{22}$$

$$\sin\left(\alpha\right) + \sin\left(\beta\right) = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\sin\left(\alpha\right) - \sin\left(\beta\right) = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

$$\cos\left(\alpha\right) + \cos\left(\beta\right) = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\cos\left(\alpha\right) - \cos\left(\beta\right) = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

0.9 Identities involving separation vector

The separation vector is defined as $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r'}$ with its magnitude denoted as $R = |\boldsymbol{R}|$ and direction $\boldsymbol{R}_0 = \dfrac{\boldsymbol{R}}{R}$.

$$
\begin{aligned}
\nabla R &= \boldsymbol{R}_0 \\
\nabla R^n &= nR^{n-1}\nabla R = nR^{n-1}\boldsymbol{R}_0 \\
\nabla \cdot \boldsymbol{R} &= 3 \\
\nabla \times \boldsymbol{R} &= 0 \\
\nabla \cdot \boldsymbol{R}_0 &= \frac{2}{R} \\
\nabla \left(\mathrm{e}^{-\mathrm{j}kR}\right) &= -\mathrm{j}k\mathrm{e}^{-\mathrm{j}kR}\boldsymbol{R}_0 \\
\nabla^2 \left(\frac{1}{R}\right) &= -4\pi\delta\left(\boldsymbol{R}\right) \\
\left(\nabla^2 + k^2\right)\frac{\mathrm{e}^{-\mathrm{j}kR}}{R} &= -4\pi\delta\left(\boldsymbol{R}\right) \\
\frac{\partial^2}{\partial r_i \partial r_j}\frac{1}{R} &= \frac{3R_iR_j - R^2\delta_{ij}}{R^5},\ R \neq 0
\end{aligned}
\tag{23}
$$

0.10 Useful integrals

$$\begin{aligned}
\int \frac{\mathrm{d}x}{x^2+a^2} &= \frac{1}{a}\arctan\left(\frac{x}{a}\right) \\
\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} &= \arcsin(x) \\
\int \frac{\mathrm{d}x}{\sqrt{x^2-1}} &= \ln\left(x+\sqrt{x^2-1}\right) \\
\int \frac{\mathrm{d}x}{\sqrt{x^2+a^2}} &= \ln\left(x+\sqrt{x^2+a^2}\right) \\
\int \frac{\mathrm{d}x}{\left(x^2+a^2\right)^{3/2}} &= \frac{x}{a^2\sqrt{x^2+a^2}} \\
\int x^n \ln\left(\frac{a}{x}\right)\mathrm{d}x &= \frac{x^{n+1}}{(n+1)^2}+\frac{x^{n+1}}{n+1}\ln\left(\frac{a}{x}\right) \\
\int x\mathrm{e}^{ax}\mathrm{d}x &= \frac{ax-1}{a^2}\mathrm{e}^{ax} \\
\int x^2\mathrm{e}^{ax}\mathrm{d}x &= \frac{a^2x^2-2ax+2}{a^3}\mathrm{e}^{ax} \\
\int \sin^3(x)\,\mathrm{d}x &= -\cos(x)+\frac{\cos^3(x)}{3} \\
\int \cos^3(x)\,\mathrm{d}x &= \sin(x)-\frac{\sin^3(x)}{3} \\
\int \frac{\mathrm{d}x}{\cos(x)} &= \ln\left(\frac{1+\sin(x)}{\cos(x)}\right) \\
\int \frac{\mathrm{d}x}{\sin(x)} &= \ln\left(\frac{1-\cos(x)}{\sin(x)}\right) \\
\int \frac{\cos(x)\,\mathrm{d}x}{\left(1-a^2\cos^2(x)\right)^{3/2}} &= \frac{\sin(x)}{\left(1-a^2\right)\sqrt{1-a^2\cos^2(x)}} \\
\int \frac{\sin(x)\,\mathrm{d}x}{\left(1-a^2\sin^2(x)\right)^{3/2}} &= \frac{-\cos(x)}{\left(1-a^2\right)\sqrt{1-a^2\sin^2(x)}}
\end{aligned} \tag{24}$$