Electromagnetic Field Theory 1 *(fundamental relations and definitions)*

Lukas Jelinek

Ver. 2024/11/30



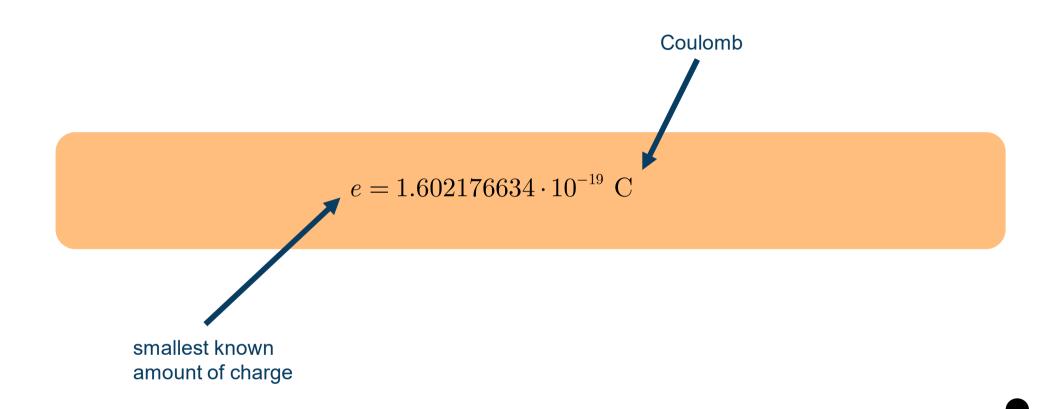
Fundamental Question of Classical Electrodynamics

A specified distribution of elementary charges is in a state of arbitrary (but known) motion. At certain time we pick one of them and ask what is the force acting on it.

Rather difficult question – will not be fully answered



Elementary Charge



As far as we know, all charges in nature have values $\pm Ne, N \in \mathbb{Z}$





Charge conservation

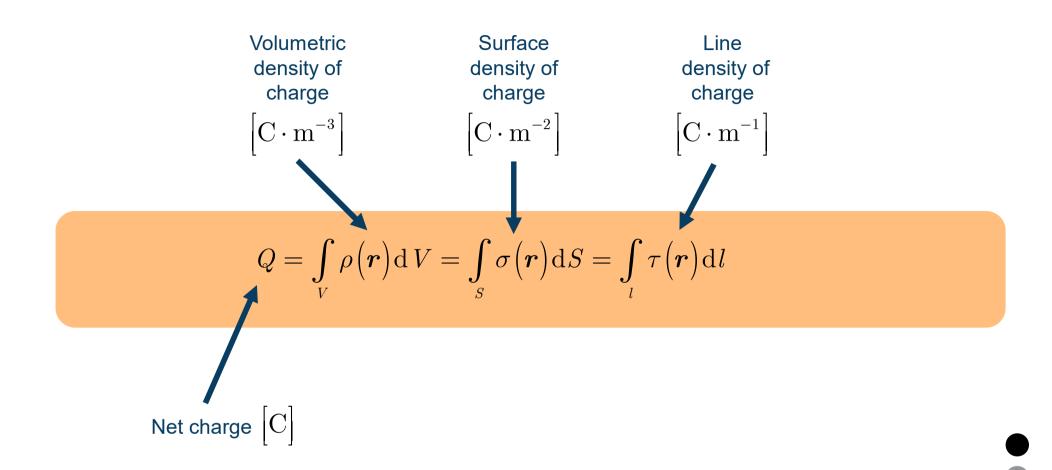
Amount of charge is conserved in every frame (even non-inertial).

Neutrality of atoms has been verified to 20 digits





Continuous approximation of charge distribution



Continuous approximation allows for using powerful mathematics





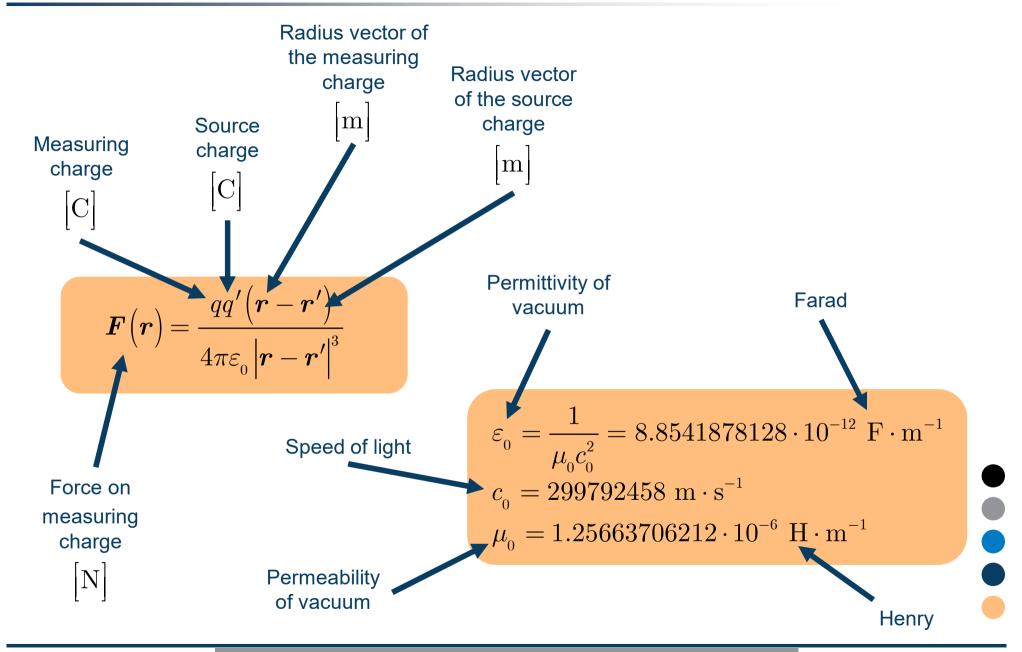
Fundamental Question of Electrostatics

There exist a specified distribution of static elementary charges. We pick one of them and ask what is the force acting on it.

This will be answered in full details



Coulomb('s) Law





Coulomb('s) Law + Superposition Principle

$$oldsymbol{F}ig(oldsymbol{r}ig) = rac{q}{4\piarepsilon_0} \sum_n rac{q_n'ig(oldsymbol{r}-oldsymbol{r}_n'ig)}{ig|oldsymbol{r}-oldsymbol{r}_n'ig|^3}$$

Entire electrostatics can be deduced from this formula





Electric Field

$$oldsymbol{F}ig(oldsymbol{r}ig)=qoldsymbol{E}ig(oldsymbol{r}ig)$$

$$oldsymbol{E}(oldsymbol{r}) = rac{1}{4\piarepsilon_0} \sum_n rac{q_n' \left(oldsymbol{r} - oldsymbol{r}_n'
ight)}{\left|oldsymbol{r} - oldsymbol{r}_n'
ight|^3}$$

Intensity of electric field

$$\left[\mathbf{V} \cdot \mathbf{m}^{-1} \right]$$

Force is represented by field – entity generated by charges and permeating the space





Continuous Distribution of Charge

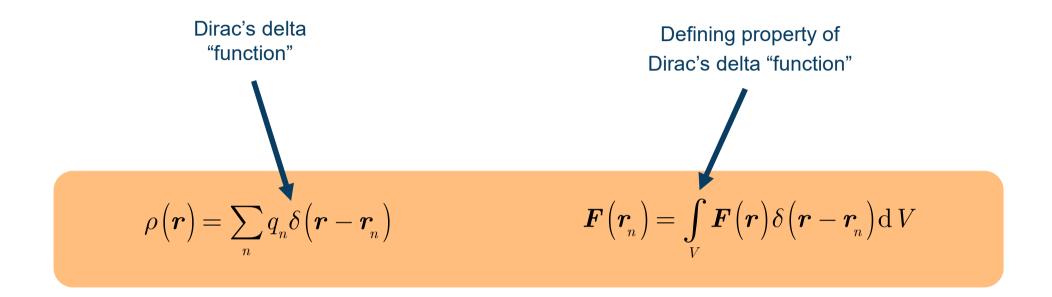
$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{n} \frac{q'_n(\boldsymbol{r} - \boldsymbol{r}'_n)}{\left|\boldsymbol{r} - \boldsymbol{r}'_n\right|^3} \qquad \boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0} \int_{V'} \frac{\rho(\boldsymbol{r}')(\boldsymbol{r} - \boldsymbol{r}')}{\left|\boldsymbol{r} - \boldsymbol{r}'\right|^3} dV'$$

Continuous description of charge allows for using powerful mathematics



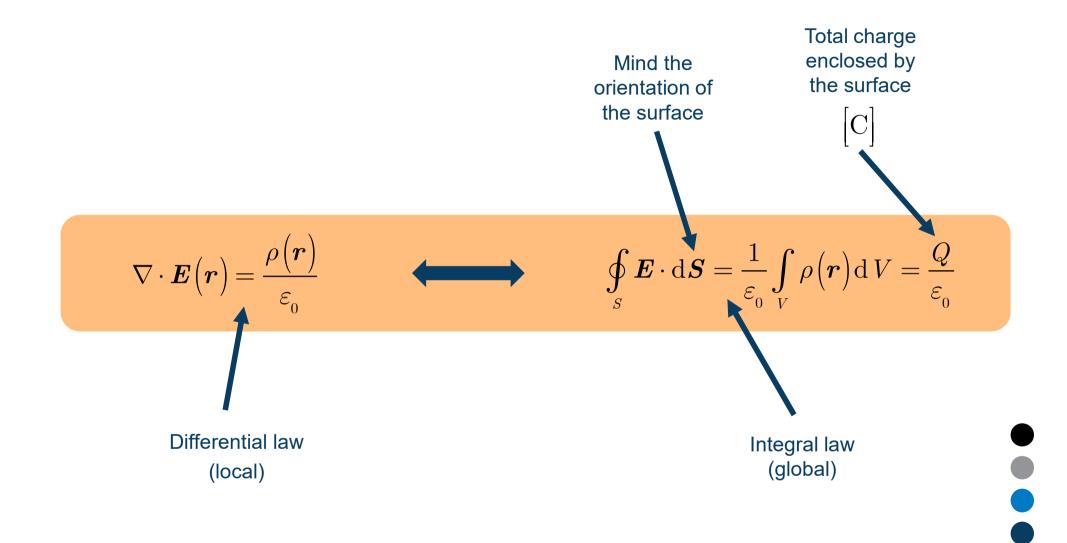


Continuous Description of a Point Charge



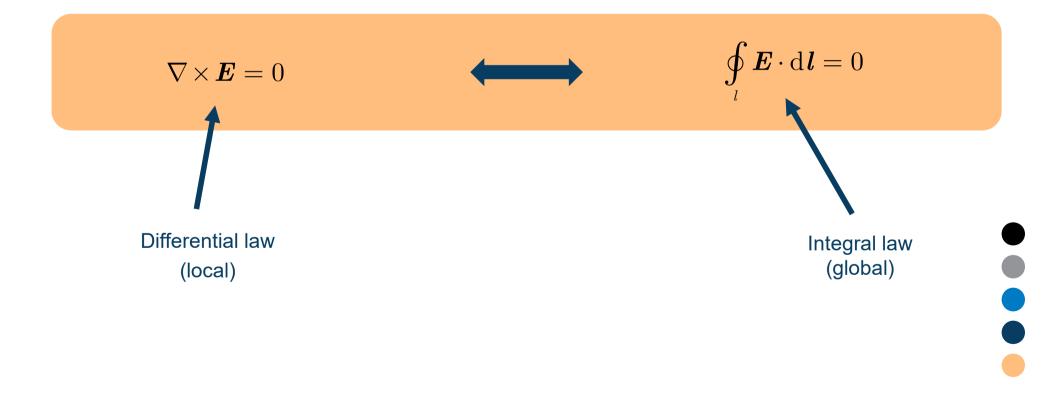


Gauss(') Law





Rotation of Electric Field





Various Views on Electrostatics

Integral laws of electrostatics



Differential laws of electrostatics



Coulomb's law



$$\oint_{S} \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{Q}{\varepsilon_{0}}$$

$$\oint_{\mathbf{r}} \mathbf{E} \cdot \mathrm{d} \mathbf{l} = 0$$

$$abla \cdot oldsymbol{E}ig(oldsymbol{r}ig) = rac{
ho\left(oldsymbol{r}
ight)}{arepsilon_0}$$

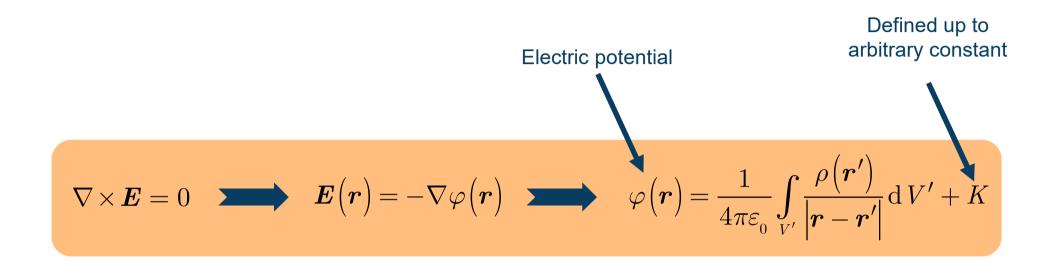
$$\nabla \times \boldsymbol{E} = 0$$

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0} \int_{V'} \frac{\rho(\boldsymbol{r}')(\boldsymbol{r} - \boldsymbol{r}')}{\left|\boldsymbol{r} - \boldsymbol{r}'\right|^3} dV'$$

The physics content is the same, the formalism is different.



Electric potential

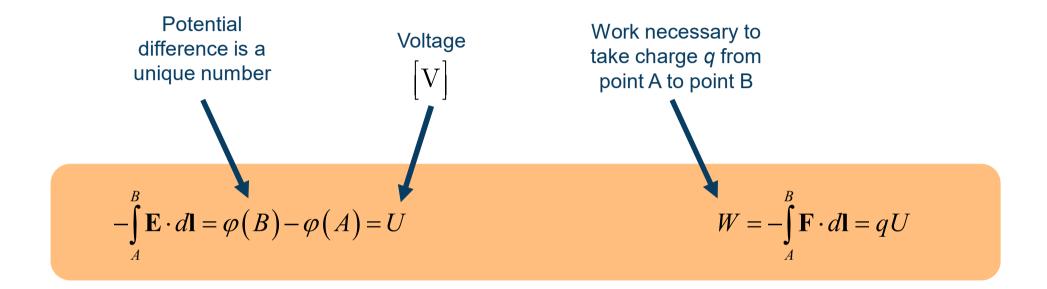


Scalar description of electrostatic field





Voltage



Voltage represents connection of abstract field theory with experiments

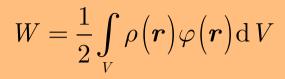




Electrostatic Energy









Energy is carried by charges and fields





$$W = \frac{1}{8\pi\varepsilon_0} \sum_{\stackrel{i,j}{j \neq i}} \frac{q_i q_j}{\left| \boldsymbol{r}_i - \boldsymbol{r}_j \right|}$$

$$W = \frac{1}{8\pi\varepsilon_0} \sum_{\substack{i,j\\j\neq i}} \frac{q_i q_j}{\left| \boldsymbol{r}_i - \boldsymbol{r}_j \right|}$$

$$W = \frac{1}{8\pi\varepsilon_0} \int_{V} \int_{V'} \frac{\rho\left(\boldsymbol{r}\right)\rho\left(\boldsymbol{r}'\right)}{\left| \boldsymbol{r} - \boldsymbol{r}' \right|} dV'dV$$



$$W = rac{1}{2} arepsilon_0 \int_V \left| oldsymbol{E} \left(oldsymbol{r}
ight)
ight|^2 \mathrm{d}\, V$$

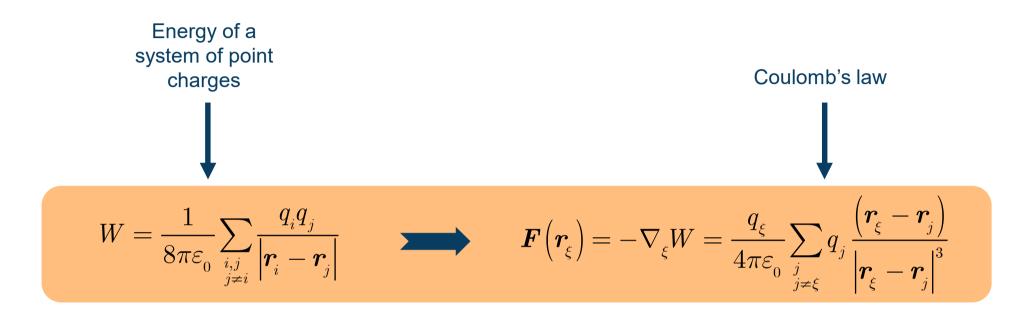


Energy is carried by fields



Be careful with point charges (self-energy)

Electrostatic Energy vs Force



Electrostatic forces are always acting so as to minimize energy of the system





Electric Stress Tensor

Total electric force acting in a volume



Stress tensor



$$\mathbf{F} = \int_{V} \rho(\mathbf{r}) \mathbf{E}(\mathbf{r}) dV = \varepsilon_{0} \oint_{S} \mathbf{T} \cdot d\mathbf{S}$$



$$egin{aligned} & oldsymbol{ar{T}} & = oldsymbol{E} oldsymbol{E} - rac{1}{2} oldsymbol{ar{I}} ig| oldsymbol{E} ig|^2 \end{aligned}$$

All the information on the volumetric Coulomb's force is contained at the boundary



Ideal Conductor – classical description

Ideal conductor contains unlimited amount of free charges which under action of external electric field rearrange so as to annihilate electric field inside the conductor.

In 3D, the free charge always resides on the external bounding surface of the conductor.



Generally, free charges in conductors move so as to minimize the energy



Ideal Conductor – quantum description

In an ideal conductor, wave functions of electrons in outer shells perceive flat potential background. In reaction to an external electric field, these wave functions are slightly modified so as to provide zero average charge density inside the conductor. Due to flat potential background, there is no counter interaction.

Long-range transport of charge does not truly happen in a solid conductor



Boundary Conditions on Ideal Conductor

Inside conductor

• $\boldsymbol{E}(\boldsymbol{r}) = 0 \quad \Leftrightarrow \quad \varphi(\boldsymbol{r}) = \text{const.}$

Just outside conductor

Potential is continuous across the boundary

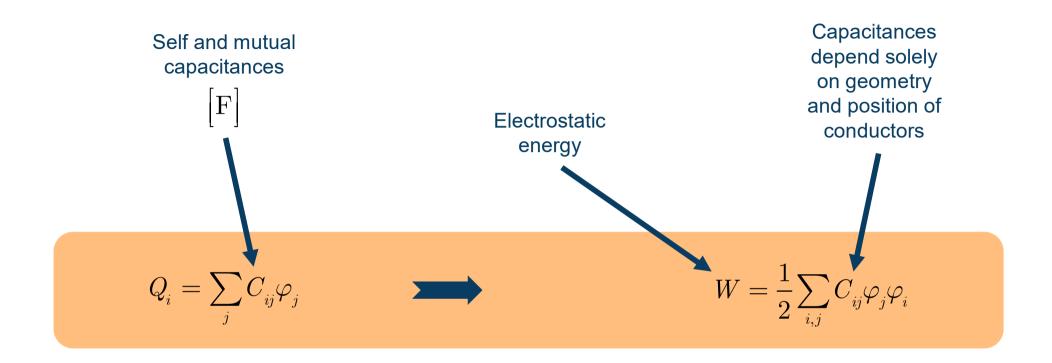
- $n(\mathbf{r}) \times \mathbf{E}(\mathbf{r}) = 0 \Leftrightarrow \varphi(\mathbf{r}) = \text{const.}$ $n(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r}) = \frac{\sigma}{\varepsilon_0} \Leftrightarrow \frac{\partial \varphi(\mathbf{r})}{\partial n} = -\frac{\sigma}{\varepsilon_0}$

Outward normal to the conductor

Normal derivative Surface charge residing on the outer surface of the conductor



Capacitance of a System of N conductors

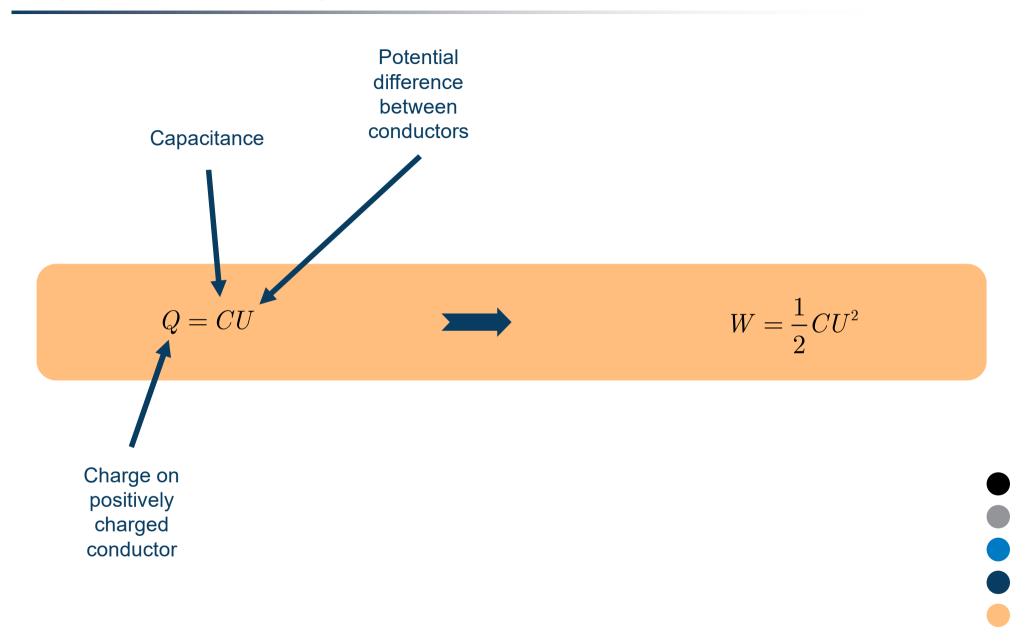


Electrostatic system is fully characterized by capacitances (we know the energy)





Capacitance of a System of two conductors





Poisson('s) equation

$$\Delta arphi \left(oldsymbol{r}
ight) = -rac{
ho \left(oldsymbol{r}
ight)}{arepsilon_0}$$

The solution to Poisson's equation is unique in a given volume once the potential is known on its bounding surface and the charge density is known throughout the volume.



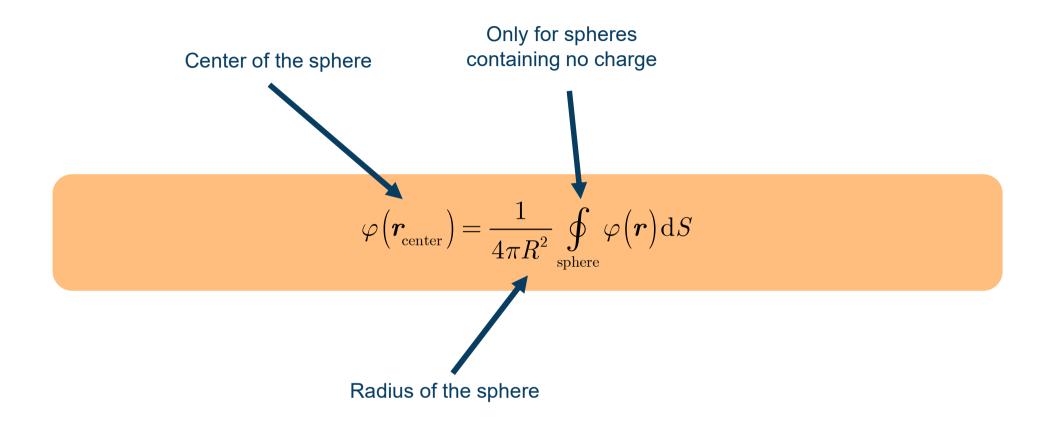
Laplace('s) equation

$$\Delta \varphi(\mathbf{r}) = 0$$

The solution to Laplace's equation is unique in a given volume once the potential is known on its bounding surface.



Mean Value Theorem



The solution to Laplace's equation posses neither maxima nor minima inside the solved volume.



Earnshaw('s) Theorem

Consequence of mean value theorem

A charged particle cannot be held in stable equilibrium by electrostatic forces alone.

Mind that the solution to Laplace's equation posses neither maxima nor minima inside the solved volume. This means that charged particle will always travel towards the boundary.





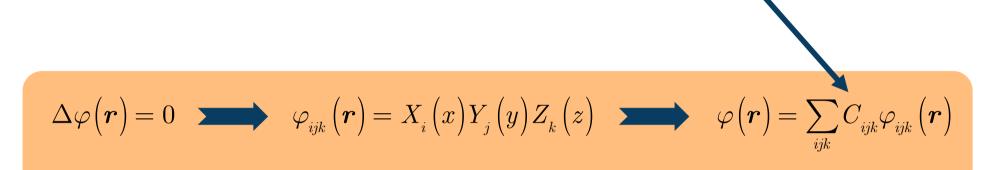
Image Method

When solving field generated by charges in the presence of conductors, it is sometimes possible to remove the conductor and mimic its boundary conditions by adding extra charges to the exterior of the solution volume. The uniqueness theorem claims that this is a correct solution.

Image method always works with planes and spheres.

Separation of Variables

Constants determined by boundary conditions



Semi-analytical method for canonical problems





Finite Differences

$$\varphi\left(x+h,y,z\right)\to\varphi_{(i+1)jk}$$

$$\Delta\varphi\left(\boldsymbol{r}\right)\approx\frac{\varphi_{(i+1)jk}-2\varphi_{ijk}+\varphi_{(i-1)jk}}{h^{2}}+\frac{\varphi_{i(j+1)k}-2\varphi_{ijk}+\varphi_{i(j-1)k}}{h^{2}}+\frac{\varphi_{ij(k+1)}-2\varphi_{ijk}+\varphi_{ij(k-1)}}{h^{2}}$$

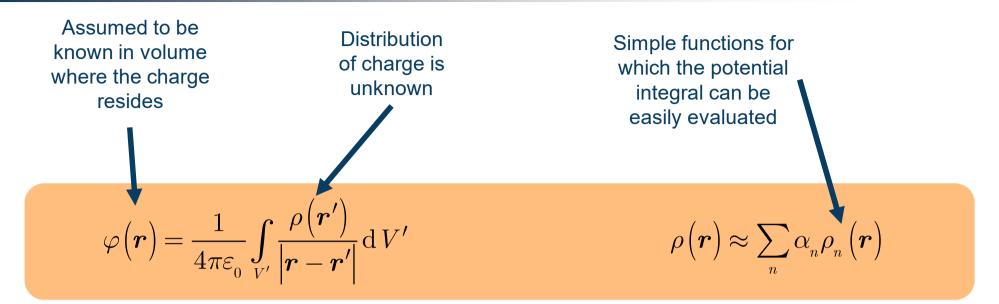
$$\Delta \varphi(\mathbf{r}) = 0 \qquad \qquad \varphi_{ijk} = \frac{\varphi_{(i+1)jk} + \varphi_{(i-1)jk} + \varphi_{i(j+1)k} + \varphi_{i(j-1)k} + \varphi_{ij(k-1)}}{6}$$

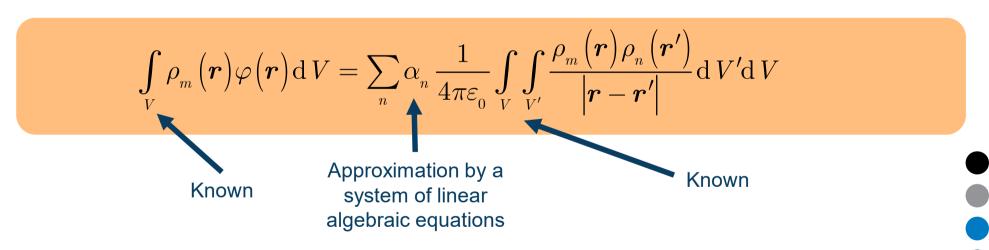
Approximation by a system of linear algebraic equations

Mind the mean value theorem

Powerful numerical method for closed problems

Integral Equation & Method of Moments





Powerful numerical method for open problems





Dielectrics

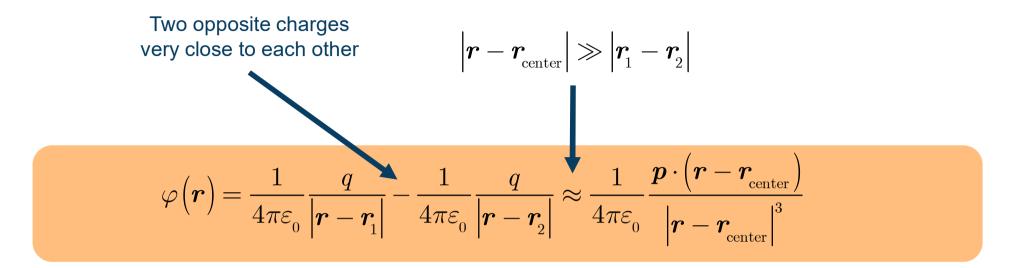
Material in which charges cannot move freely

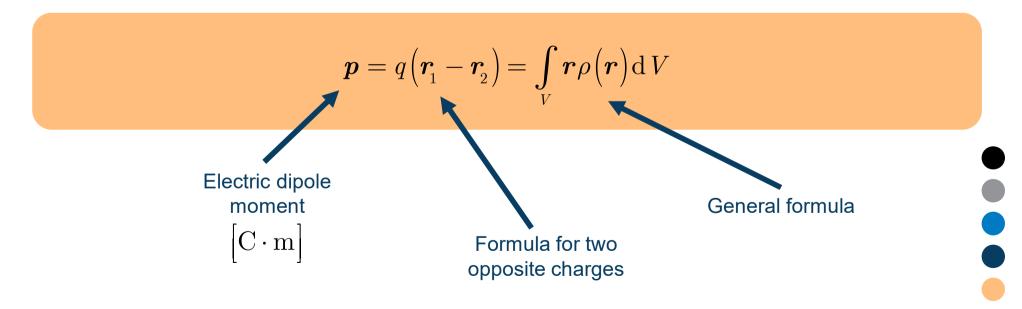
Clusters are electrically neutral

- Charges are forming clusters (atoms, molecules)
- Under influence of electric field the clusters change shape or rotate
- ullet Electric field induces electric dipoles with density $m{P}(m{r})$ $\left[\mathrm{C}\cdot\mathrm{m}^{-2}\right]$

Number of dipoles in unitary volume

Electric Field of a Dipole





Field Produced by Polarized Matter

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int_{V'} \frac{\mathbf{P}(\mathbf{r}') \cdot (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} dV' = \frac{1}{4\pi\varepsilon_0} \oint_{S'} \frac{\mathbf{P}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \cdot d\mathbf{S}' - \frac{1}{4\pi\varepsilon_0} \int_{V'} \frac{\nabla' \cdot \mathbf{P}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV'$$

Only apply at infinitely sharp boundary (unrealistic)

Potential of volumetric charge density

This formula holds very well outside the matter and, curiously, it also well approximates the field inside



Electric Displacement



$$abla \cdot oldsymbol{D}(oldsymbol{r}) =
ho(oldsymbol{r})$$

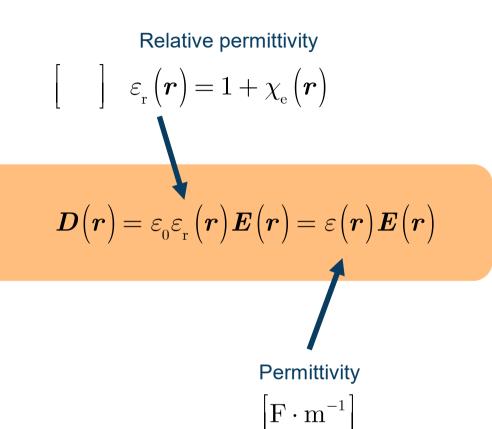
$$oldsymbol{D}ig(oldsymbol{r}ig) = arepsilon_{_{0}}oldsymbol{E}ig(oldsymbol{r}ig) + oldsymbol{P}ig(oldsymbol{r}ig)$$

$$\oint_{S} \mathbf{D} \cdot d\mathbf{S} = \int_{V} \rho(\mathbf{r}) dV = Q$$

Only free charge (compare to divergence of electric field)



Linear Isotropic Dielectrics



 $oldsymbol{P}ig(oldsymbol{r}ig) = arepsilon_0 \chi_{
m e}ig(oldsymbol{r}ig) oldsymbol{E}ig(oldsymbol{r}ig)$

Electric susceptibility

All the complicated structure of matter reduces to a simple scalar quantity



Fields in Presence of Dielectrics 1/2

Analogy with electric field in vacuum can only be used when entire space is homogeneously filled with dielectric.

$$abla imes m{D}ig(m{r}ig) =
abla imes ig[arepsilonig(m{r}ig)m{E}ig(m{r}ig)ig]
eq 0$$
Inequality is due to

boundaries

Analogy with vacuum can only be used when space is homogeneously filled with dielectric

Fields in Presence of Dielectrics 2/2

$$\nabla \times \boldsymbol{E}(\boldsymbol{r}) = 0 \Leftrightarrow \boldsymbol{E}(\boldsymbol{r}) = -\nabla \varphi(\boldsymbol{r}) \qquad \longrightarrow \qquad \nabla \cdot \left[\varepsilon(\boldsymbol{r}) \nabla \varphi(\boldsymbol{r}) \right] = -\rho(\boldsymbol{r})$$

$$\Delta arphi \left(oldsymbol{r}
ight) = -rac{
ho \left(oldsymbol{r}
ight)}{arepsilon}$$

Not a function of coordinates

Poisson's equation holds only when permittivity does not depend on coordinates



Dielectric Boundaries

$$\boldsymbol{n}\left(\boldsymbol{r}\right) \times \left[\boldsymbol{E}_{1}\left(\boldsymbol{r}\right) - \boldsymbol{E}_{2}\left(\boldsymbol{r}\right)\right] = 0 \quad \Leftrightarrow \quad \varphi_{1}\left(\boldsymbol{r}\right) - \varphi_{2}\left(\boldsymbol{r}\right) = 0$$

$$m{n}ig(m{r}ig)\cdotig[arepsilon_1m{E}_1ig(m{r}ig)-arepsilon_2m{E}_2ig(m{r}ig)ig]=\sigmaig(m{r}ig)\quad\Leftrightarrow\quadarepsilon_1rac{\partialarphi_1ig(m{r}ig)}{\partial n}-arepsilon_2rac{\partialarphi_2ig(m{r}ig)}{\partial n}=-\sigmaig(m{r}ig)$$

Normal pointing to region (1)

Both conditions are needed for unique solution



Electrostatic Energy in Dielectrics

$$W = rac{1}{2} arepsilon_0 \int_V \left| oldsymbol{E}(oldsymbol{r})
ight|^2 \mathrm{d}\,V$$



$$W = rac{1}{2} \int_{V} \boldsymbol{E}(\boldsymbol{r}) \cdot \boldsymbol{D}(\boldsymbol{r}) dV$$



Forces on Dielectrics

This only holds when charge is held constant

$$W = \frac{1}{2}CU^2 = \frac{1}{2}\frac{Q^2}{C}$$

$$W = rac{1}{2} \int_{V} \boldsymbol{E}(\boldsymbol{r}) \cdot \boldsymbol{D}(\boldsymbol{r}) dV$$

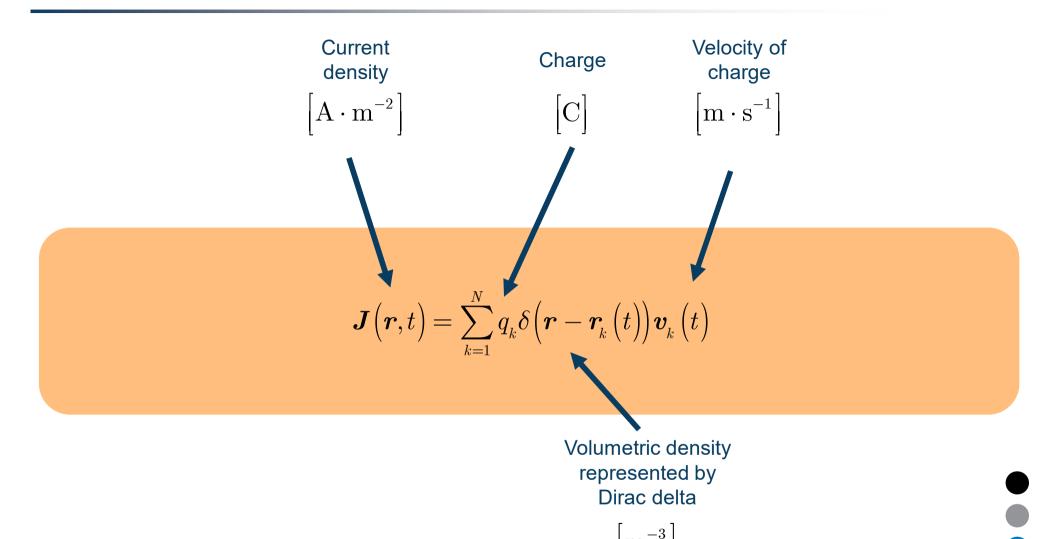


$$oldsymbol{F}ig(oldsymbol{r}_{\!arepsilon}ig) = -
abla_{\!arepsilon}W$$





Electric Current



Charges in motion are represented by current density





Local Charge Conservation

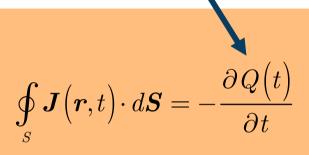
$$abla \cdot oldsymbol{J}ig(oldsymbol{r},tig) = -rac{\partial}{\partial t} \sum_{k=1}^N q_k \deltaig(oldsymbol{r} - oldsymbol{r}_kig(tig)ig) = -rac{\partial
hoig(oldsymbol{r},tig)}{\partial t}$$

Charge is conserved locally at every space-time point



Global Charge Conservation

When charge leaves a given volume, it is always accompanied by a current through the bounding envelope



Charge can neither be created nor destroyed. It can only be displaced.



Stationary Current

When charge enters a volume, another must leave it without any delay

$$\nabla \cdot \boldsymbol{J}(\boldsymbol{r}) = 0$$



$$\oint_{S} \boldsymbol{J}(\boldsymbol{r}) \cdot d\boldsymbol{S} = 0$$

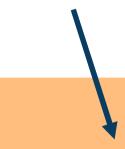
There is no charge accumulation in stationary flow



Ohm('s) Law

Conductivity

$$\left[S \cdot m^{-1} \right]$$



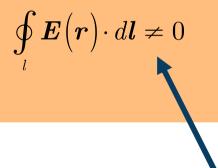
$$oldsymbol{J}ig(oldsymbol{r}ig)=\sigmaig(oldsymbol{r}ig)oldsymbol{E}ig(oldsymbol{r}ig)$$

This simple linear relation holds for enormous interval of electric field strengths

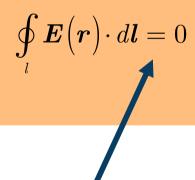


Electromotive Force

Stationary flow of charges cannot be caused by electrostatic field. The motion forces are non-conservative, are called electromotive forces, and are commonly of chemical, magnetic or photoelectric origin.



For curves passing through sources of electromotive force



For curves not crossing sources of electromotive force

Boundary Conditions for Stationary Current

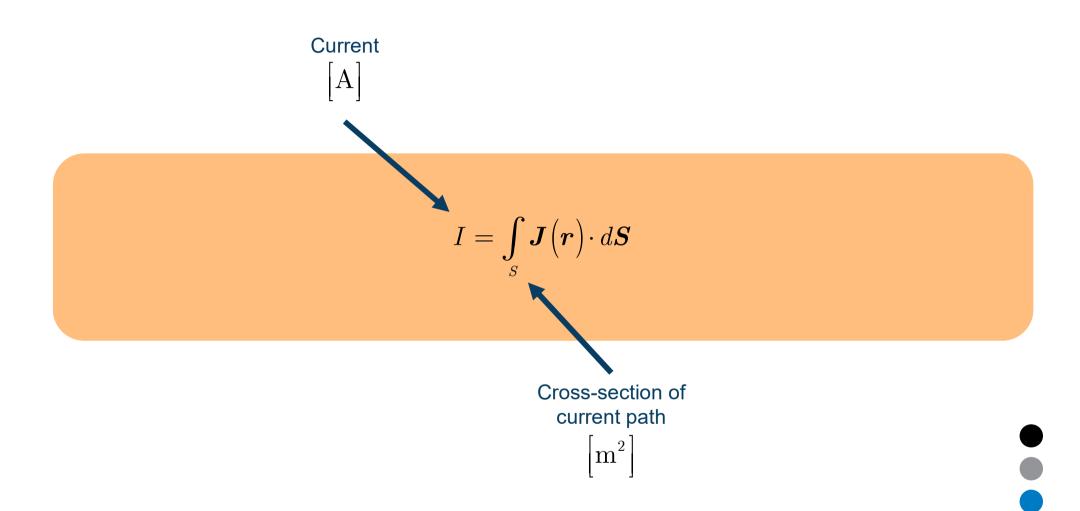
$$\boldsymbol{n}\left(\boldsymbol{r}\right) \times \left[\boldsymbol{E}_{1}\left(\boldsymbol{r}\right) - \boldsymbol{E}_{2}\left(\boldsymbol{r}\right)\right] = 0 \quad \Leftrightarrow \quad \varphi_{1}\left(\boldsymbol{r}\right) - \varphi_{2}\left(\boldsymbol{r}\right) = 0$$

$$m{n}ig(m{r}ig)\cdotig[arepsilon_1m{E}_1ig(m{r}ig)-arepsilon_2m{E}_2ig(m{r}ig)ig]=\sigmaig(m{r}ig)\quad\Leftrightarrow\quadarepsilon_1rac{\partialarphi_1ig(m{r}ig)}{\partial n}-arepsilon_2rac{\partialarphi_2ig(m{r}ig)}{\partial n}=-\sigmaig(m{r}ig)$$

$$m{n} \left(m{r}
ight) \cdot \left[\sigma_1 m{E}_1 \left(m{r}
ight) - \sigma_2 m{E}_2 \left(m{r}
ight)
ight] = 0 \quad \Leftrightarrow \quad \sigma_1 \frac{\partial arphi_1 \left(m{r}
ight)}{\partial n} - \sigma_2 \frac{\partial arphi_2 \left(m{r}
ight)}{\partial n} = 0$$

Charge conservation forces the continuity of current across the boundary

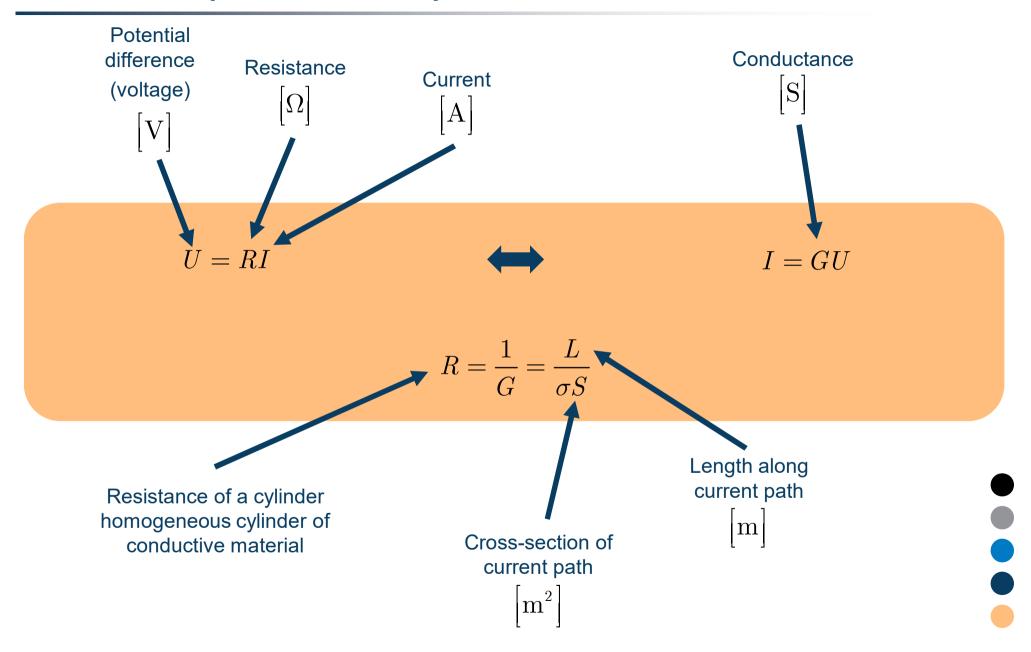
Electric Current



Existence of high contrast in conductivity between conductors and dielectrics allows for well defined current paths.

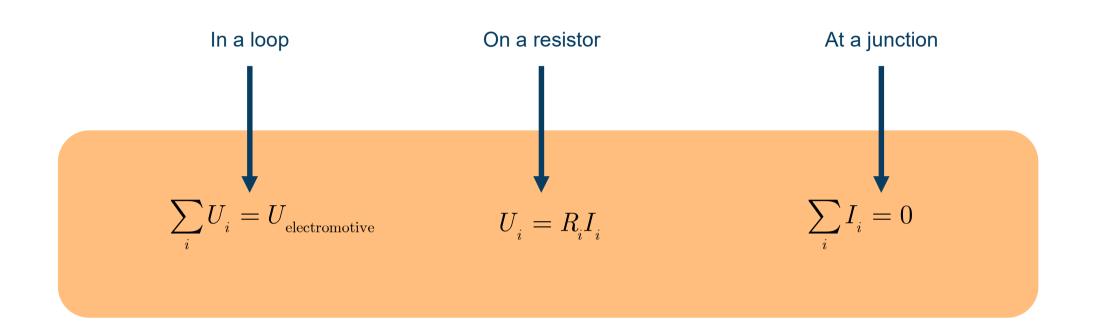


Resistance (Conductance)





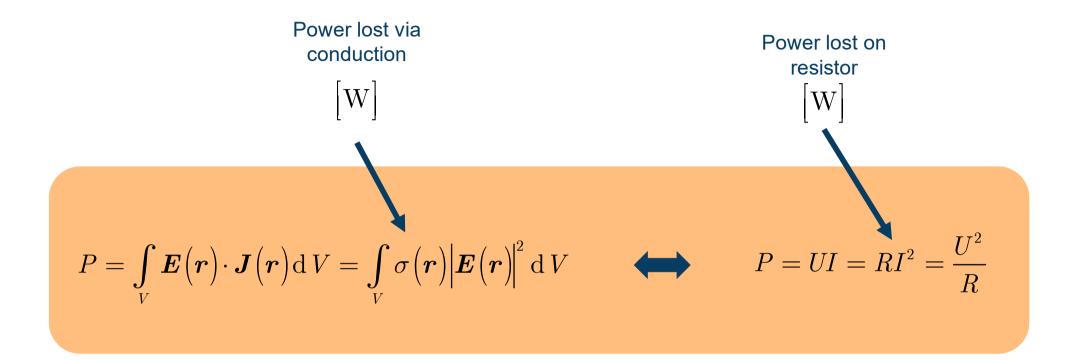
Resistive Circuits and Kirchhoff('s) Laws



Kirchhoff's laws are a consequence of electrostatics and law's of stationary current flow



Joule('s) Heat



Electric field within conducting material produces heat

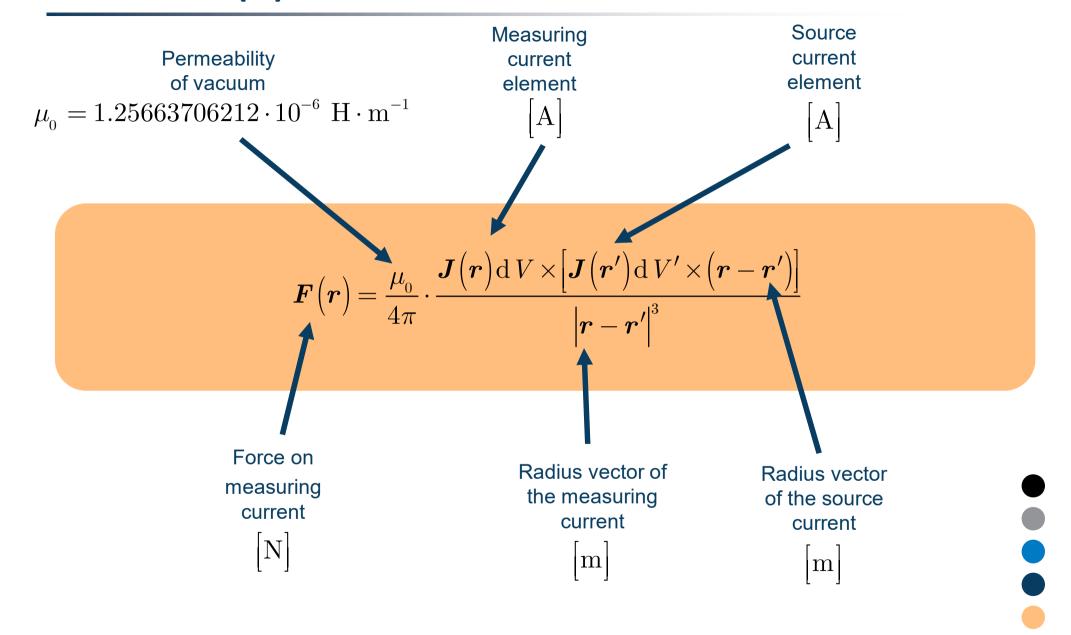


Fundamental Question of Magnetostatics

There exist a specified distribution of stationary current. We pick a differential volume of it and ask what is the force acting on it.



Biot-Savart('s) Law







Biot-Savart('s) Law + Superposition Principle

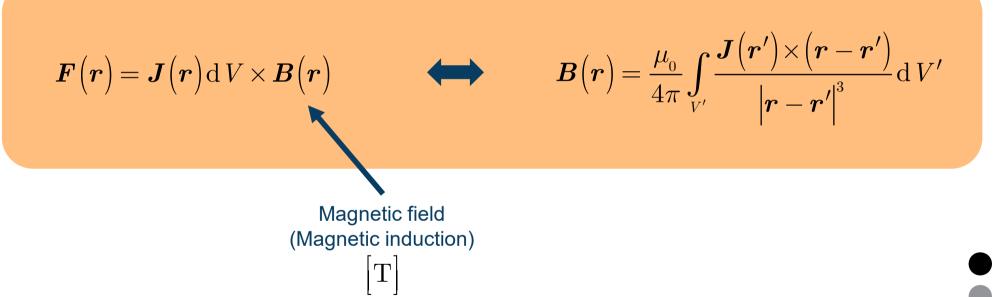
$$\boldsymbol{F}(\boldsymbol{r}) = \boldsymbol{J}(\boldsymbol{r}) dV \times \frac{\mu_0}{4\pi} \int_{V'} \frac{\boldsymbol{J}(\boldsymbol{r}') \times (\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3} dV'$$

Entire magnetostatics can be deduced from this formula





Magnetic Field





Divergence of Magnetic Field

$$\nabla \cdot \boldsymbol{B}(\boldsymbol{r}) = 0$$



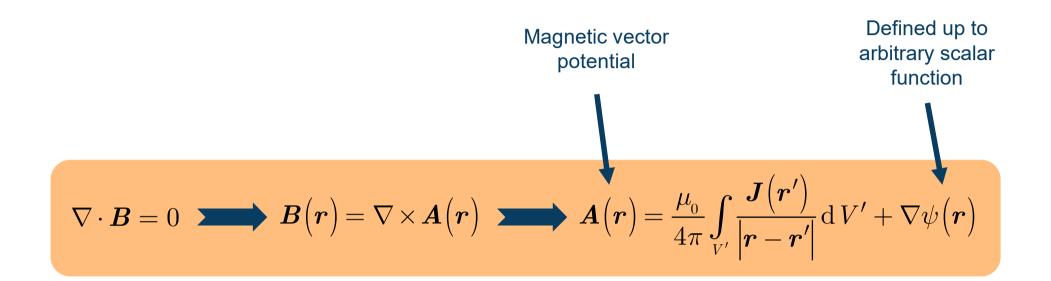
$$\oint_{S} \boldsymbol{B}(\boldsymbol{r}) \cdot d\boldsymbol{S} = 0$$

There are no point sources of magnetostatic field



Curl of Magnetic Field – Ampere('s) Law

Magnetic Vector Potential



Reduced description of magnetostatic field





Poisson('s) equation

$$\Delta oldsymbol{A}ig(oldsymbol{r}ig) = -\mu_0 oldsymbol{J}ig(oldsymbol{r}ig)$$

The solution to Poisson's equation is unique in a given volume once the potential is known on its bounding surface and the current density is known through out the volume.





Boundary Conditions



$$m{n}\left(m{r}
ight) imes \left[m{B}_{\!\scriptscriptstyle 1}\left(m{r}
ight) - m{B}_{\!\scriptscriptstyle 2}\left(m{r}
ight)
ight] = \mu_{\!\scriptscriptstyle 0}m{K}\left(m{r}
ight)$$

$$oldsymbol{n}\left(oldsymbol{r}
ight)\cdot\left[oldsymbol{B}_{\!\scriptscriptstyle 1}\left(oldsymbol{r}
ight)-oldsymbol{B}_{\!\scriptscriptstyle 2}\left(oldsymbol{r}
ight)
ight]=0$$

$$oldsymbol{A}_{\!\scriptscriptstyle 1}\!\left(oldsymbol{r}
ight)\!-oldsymbol{A}_{\!\scriptscriptstyle 2}\!\left(oldsymbol{r}
ight)\!=0$$

Normal pointing to region (1)





Magnetostatic Energy

$$W = \frac{1}{2} \int_{V} \boldsymbol{A}(\boldsymbol{r}) \cdot \boldsymbol{J}(\boldsymbol{r}) dV$$



$$W = rac{1}{2\mu_0} \int_V \left| \boldsymbol{B}(\boldsymbol{r}) \right|^2 dV$$

For now it is just a formula that works – it must be derived with the help of time varying fields



Magnetostatic Energy – Current Circuits

$$M_{ij} = M_{ji} = rac{\mu_0}{4\pi I_i I_j} \int\limits_{V_j} \int\limits_{V_i'} rac{oldsymbol{J}_jig(oldsymbol{r}_jig) \cdot oldsymbol{J}_iig(oldsymbol{r}_i'ig)}{ig|oldsymbol{r}_j - oldsymbol{r}_i'ig|} \operatorname{d} V_i' \operatorname{d} V_j$$

Mutual-Inductance $\left[H \right]$

$$W = \frac{1}{2} \sum_{i=1}^{N} L_{i} I_{i}^{2} + \frac{1}{2} \sum_{i \neq j} M_{ij} I_{i} I_{j}$$

Self-Inductance H

$$L_{i} = rac{\mu_{0}}{4\pi I_{i}^{2}} \int\limits_{V_{i}} \int\limits_{V_{i}^{\prime}} rac{oldsymbol{J}_{i}\left(oldsymbol{r}_{i}
ight) \cdot oldsymbol{J}_{i}\left(oldsymbol{r}_{i}^{\prime}
ight)}{\left|oldsymbol{r}_{i} - oldsymbol{r}_{i}^{\prime}
ight|} \operatorname{d}V_{i}^{\prime} \operatorname{d}V_{i}^{\prime}$$





Mutual Inductance – Thin Current Loop

$$\Phi_{ji} = \int\limits_{S_j} \pmb{B}_i \Big(\pmb{r}_j \Big) \cdot \mathrm{d} \pmb{S}_j$$
 Magnetic flux induced by *i*-th

current through *j*-th current

$$M_{_{ij}}=rac{\Phi_{_{j}}}{I_{_{i}}}$$



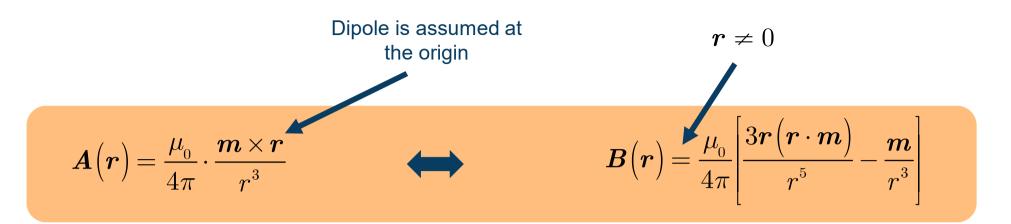


Magnetic Materials

- Material response is due to magnetic dipole moments
- Magnetic moment comes from spin or orbital motion of an electron
- Magnetic field tends to align magnetic moments
- ullet Magnetic field induces magnetic dipoles with density $m{M}m{r}$ $\left[\mathbf{A} \cdot \mathbf{m}^{-1} \right]$

Number of dipoles in unitary volume

Magnetic Field of a Dipole



$$\bm{m} = \frac{1}{2} \int_V \bm{r} \times \bm{J} \left(\bm{r} \right) \mathrm{d}\,V$$
 Magnetic dipole moment
$$\left[\mathbf{A} \cdot \mathbf{m}^2 \right]$$

Magnetic dipole approximates infinitesimally small current loop





Field Produced by Magnetized Matter

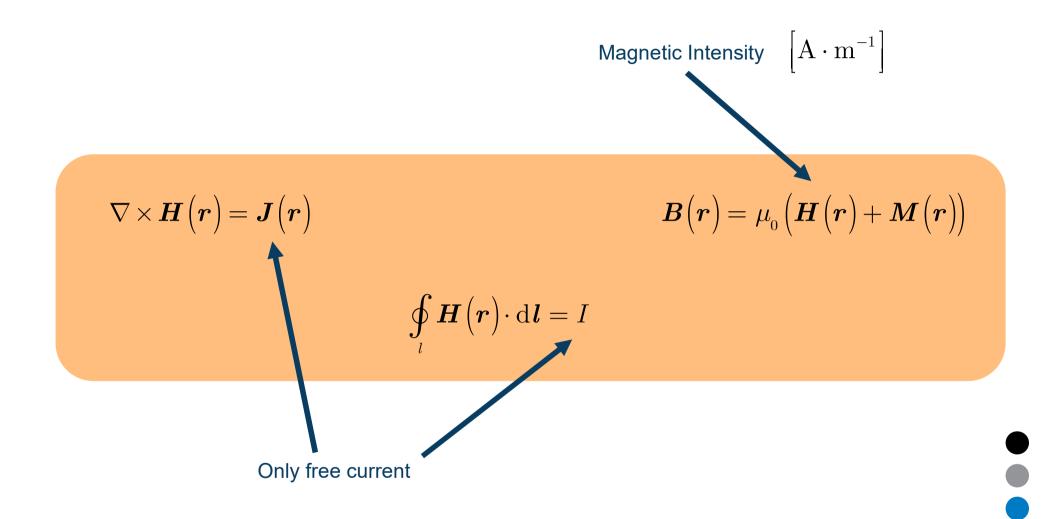
$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int_{V'} \frac{\boldsymbol{M}(\boldsymbol{r}') \times (\boldsymbol{r} - \boldsymbol{r}')}{\left|\boldsymbol{r} - \boldsymbol{r}'\right|^3} dV' = \frac{\mu_0}{4\pi} \oint_{S'} \frac{\boldsymbol{M}(\boldsymbol{r}') \times d\boldsymbol{S}'}{\left|\boldsymbol{r} - \boldsymbol{r}'\right|} + \frac{\mu_0}{4\pi} \int_{V'} \frac{\nabla' \times \boldsymbol{M}(\boldsymbol{r}')}{\left|\boldsymbol{r} - \boldsymbol{r}'\right|} dV'$$

Only applies at infinitely sharp boundary (unrealistic)

Potential of volumetric current density

This formula holds very well outside the matter and, curiously, it also well approximates the field inside

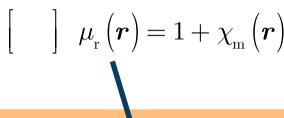
Magnetic Intensity





Linear Isotropic Magnetic Materials

Relative permeability



$$oldsymbol{M}ig(oldsymbol{r}ig) = \chi_{_{\mathrm{m}}}ig(oldsymbol{r}ig)oldsymbol{H}ig(oldsymbol{r}ig)$$

$$oldsymbol{B}ig(oldsymbol{r}ig) = \dot{\mu_{_0}}\dot{\mu_{_{
m r}}}ig(oldsymbol{r}ig)oldsymbol{H}ig(oldsymbol{r}ig) = \muig(oldsymbol{r}ig)oldsymbol{H}ig(oldsymbol{r}ig)$$

Magnetic susceptibility

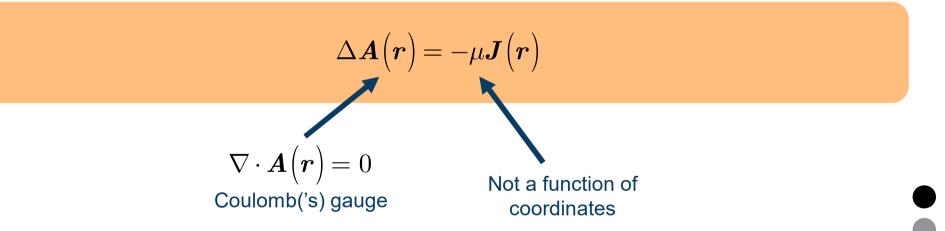
Permeability
$$\left[H \cdot m^{-1} \right]$$

All the complicated structure of matter reduces to a simple scalar quantity



Fields in Presence of Magnetic Material

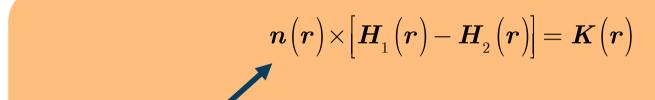
$$\nabla \cdot \boldsymbol{B}(\boldsymbol{r}) = 0 \Leftrightarrow \boldsymbol{B}(\boldsymbol{r}) = \nabla \times \boldsymbol{A}(\boldsymbol{r}) \qquad \longrightarrow \qquad \nabla \times \left[\frac{1}{\mu(\boldsymbol{r})} \nabla \times \boldsymbol{A}(\boldsymbol{r}) \right] = \boldsymbol{J}(\boldsymbol{r})$$



Poisson's equation holds only when permittivity does not depend on coordinates



Magnetic Material Boundaries



$$oldsymbol{n}\left(oldsymbol{r}
ight)\cdot\left[\mu_{_{\!1}}oldsymbol{H}_{_{\!1}}\left(oldsymbol{r}
ight)-\mu_{_{\!2}}oldsymbol{H}_{_{\!2}}\left(oldsymbol{r}
ight)
ight]=0$$

Normal pointing to region (1)

Both conditions are needed for unique solution



Magnetostatic Energy in Magnetic Material

$$W = rac{1}{2\mu_0} \int_V \left| \boldsymbol{B}(\boldsymbol{r}) \right|^2 \mathrm{d}V$$



$$W = \frac{1}{2} \int_{V} \boldsymbol{H}(\boldsymbol{r}) \cdot \boldsymbol{B}(\boldsymbol{r}) dV$$





Magnetic Materials

- Paramagnetic small positive susceptibility (small attraction – linear)
- Diamagnetic small negative susceptibility (small repulsion – linear)
- Ferromagnetic "large positive susceptibility" (large attraction – nonlinear)



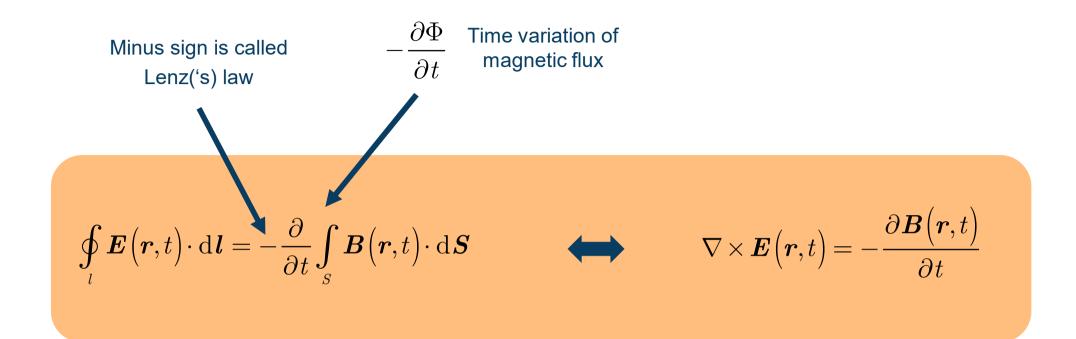
Ferromagnetic Materials

- Spins are ordered within domains
- Magnetization is a non-linear function of field intensity
- Magnetization curve Hysteresis, Remanence
- Susceptibility can only be defined as local approximation
- Above Curie('s) temperature ferromagnetism disappears

Exact calculations are very difficult – use simplified models (soft material, permanent magnet)



Faraday('s) Law



Time variation in magnetic field produces electric field that tries to counter the change in magnetic flux (electromotive force)



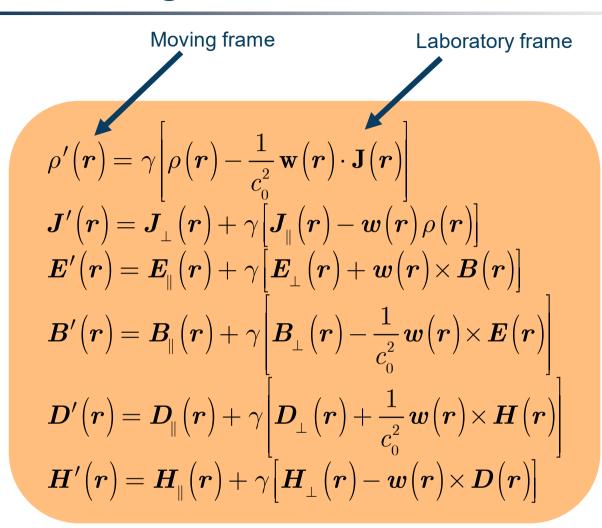
Lenz('s) Law

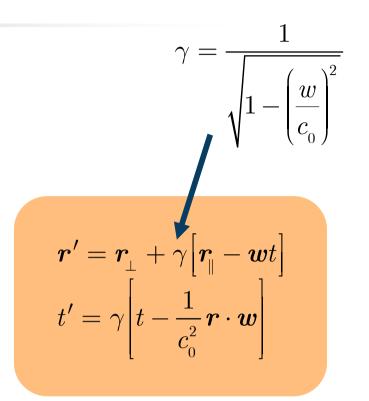
The current created by time variation of magnetic flux is directed so as to oppose the flux creating it.





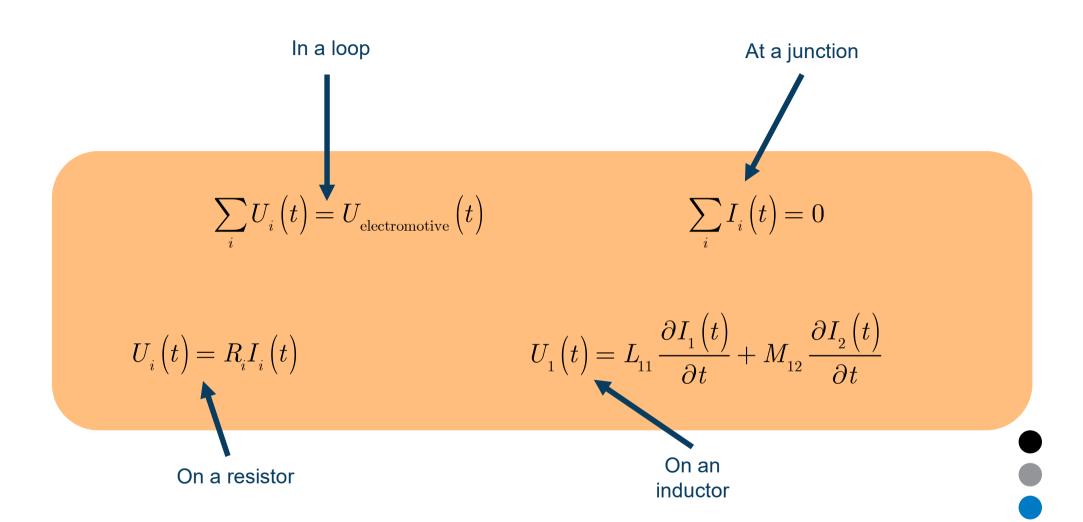
Electromagnetic Field and Motion





Careful, this is only valid for small accelerations (gravity).

Time Varying RL Circuits



Circuit laws are valid as long as time variations are not too fast





Time Varying Potentials

Potential calibration

$$abla \cdot oldsymbol{A}ig(oldsymbol{r},tig) = -\sigma\muarphiig(oldsymbol{r},tig)$$

$$oldsymbol{B}ig(oldsymbol{r},tig) =
abla imes oldsymbol{A}ig(oldsymbol{r},tig) = -
abla arphiig(oldsymbol{r},tig) - rac{\partial oldsymbol{A}ig(oldsymbol{r},tig)}{\partial t}$$

In time varying fields scalar potential becomes redundant





Source and Induced Currents

Those are fixed, not reacting to fields

$$abla imes oldsymbol{H}\left(oldsymbol{r},t
ight) = oldsymbol{J}_{ ext{source}}\left(oldsymbol{r},t
ight) + oldsymbol{J}_{ ext{induced}}\left(oldsymbol{r},t
ight) = oldsymbol{J}_{ ext{source}}\left(oldsymbol{r},t
ight) + \sigma oldsymbol{E}\left(oldsymbol{r},t
ight)$$



Diffusion Equation

$$\Delta \boldsymbol{A} \Big(\boldsymbol{r}, t \Big) - \sigma \mu \frac{\partial \boldsymbol{A} \Big(\boldsymbol{r}, t \Big)}{\partial t} = -\mu \boldsymbol{J}_{\text{source}} \Big(\boldsymbol{r}, t \Big)$$

$$\Delta \boldsymbol{H}\left(\boldsymbol{r},t\right) - \sigma\mu \frac{\partial \boldsymbol{H}\left(\boldsymbol{r},t\right)}{\partial t} = -\nabla \times \boldsymbol{J}_{\text{source}}\left(\boldsymbol{r},t\right)$$

$$\Delta \boldsymbol{E}(\boldsymbol{r},t) - \sigma \mu \frac{\partial \boldsymbol{E}(\boldsymbol{r},t)}{\partial t} = \frac{1}{\varepsilon} \nabla \rho_{\text{source}}(\boldsymbol{r},t) + \mu \frac{\partial \boldsymbol{J}_{\text{source}}(\boldsymbol{r},t)}{\partial t}$$

Material parameters are assumed independent of coordinates





Maxwell('s)-Lorentz('s) Equations

$$abla imes oldsymbol{H} \left(oldsymbol{r}, t
ight) = oldsymbol{J} \left(oldsymbol{r}, t
ight) + rac{\partial oldsymbol{D} \left(oldsymbol{r}, t
ight)}{\partial t}$$
 $abla imes oldsymbol{E} \left(oldsymbol{r}, t
ight) = -rac{\partial oldsymbol{B} \left(oldsymbol{r}, t
ight)}{\partial t}$

$$\nabla \cdot \boldsymbol{B}(\boldsymbol{r},t) = 0$$

$$abla \cdot oldsymbol{B}ig(oldsymbol{r},tig) = 0
abla \cdot oldsymbol{D}ig(oldsymbol{r},tig) =
hoig(oldsymbol{r},tig)$$

Equations of motion for fields

Equation of motion for particles

$$oldsymbol{f}ig(oldsymbol{r},tig) =
hoig(oldsymbol{r},tig)oldsymbol{E}ig(oldsymbol{r},tig) + oldsymbol{J}ig(oldsymbol{r},tig) imesoldsymbol{B}ig(oldsymbol{r},tig)$$

Interaction with materials

$$egin{aligned} oldsymbol{D}ig(oldsymbol{r},tig) &= arepsilon_0 oldsymbol{E}ig(oldsymbol{r},tig) + oldsymbol{P}ig(oldsymbol{r},tig) \ oldsymbol{B}ig(oldsymbol{r},tig) &= \mu_0 ig(oldsymbol{H}ig(oldsymbol{r},tig) + oldsymbol{M}ig(oldsymbol{r},tig) \end{aligned}$$

Absolute majority of things happening around you is described by these equations



Boundary Conditions

$$\boldsymbol{n} \left(\boldsymbol{r} \right) \times \left[\boldsymbol{E}_1 \left(\boldsymbol{r}, t \right) - \boldsymbol{E}_2 \left(\boldsymbol{r}, t \right) \right] = 0$$

$$\boldsymbol{n} \left(\boldsymbol{r} \right) \times \left[\boldsymbol{H}_1 \left(\boldsymbol{r}, t \right) - \boldsymbol{H}_2 \left(\boldsymbol{r}, t \right) \right] = \boldsymbol{K} \left(\boldsymbol{r}, t \right)$$

$$\boldsymbol{n} \left(\boldsymbol{r} \right) \cdot \left[\boldsymbol{B}_1 \left(\boldsymbol{r}, t \right) - \boldsymbol{B}_2 \left(\boldsymbol{r}, t \right) \right] = 0$$
 Normal pointing to region (1)
$$\boldsymbol{n} \left(\boldsymbol{r} \right) \cdot \left[\boldsymbol{D}_1 \left(\boldsymbol{r}, t \right) - \boldsymbol{D}_2 \left(\boldsymbol{r}, t \right) \right] = \sigma \left(\boldsymbol{r}, t \right)$$



Electromagnetic Potentials

Lorentz('s) calibration

$$\nabla \cdot \boldsymbol{A} \big(\boldsymbol{r}, t \big) = -\sigma \mu \varphi \big(\boldsymbol{r}, t \big) - \varepsilon \mu \frac{\partial \varphi \big(\boldsymbol{r}, t \big)}{\partial t}$$

$$oldsymbol{B}ig(oldsymbol{r},tig) =
abla imes oldsymbol{A}ig(oldsymbol{r},tig)$$

$$oldsymbol{B}ig(oldsymbol{r},tig) =
abla imes oldsymbol{A}ig(oldsymbol{r},tig) = -
abla arphiig(oldsymbol{r},tig) - rac{\partial oldsymbol{A}ig(oldsymbol{r},tig)}{\partial t}$$



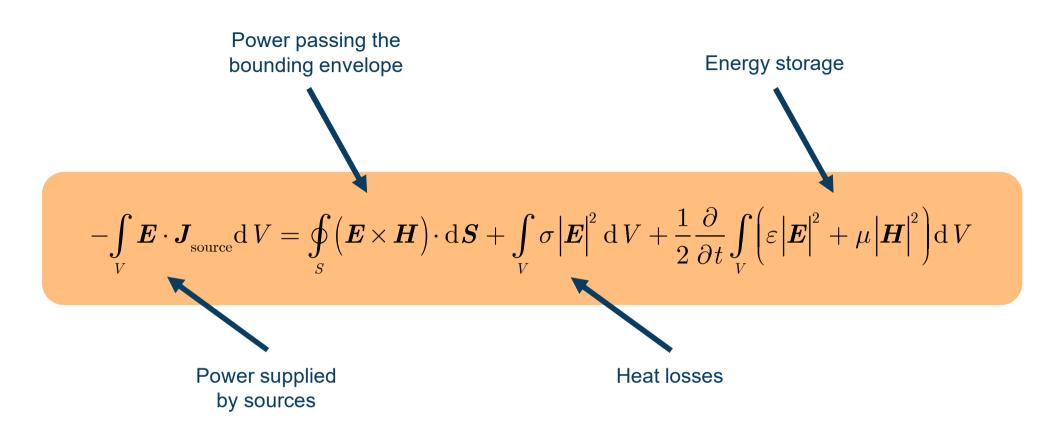
Wave Equation

$$\Delta \boldsymbol{A} (\boldsymbol{r}, t) - \sigma \mu \frac{\partial \boldsymbol{A} (\boldsymbol{r}, t)}{\partial t} - \varepsilon \mu \frac{\partial^2 \boldsymbol{A} (\boldsymbol{r}, t)}{\partial t^2} = -\mu \boldsymbol{J}_{\text{source}} (\boldsymbol{r}, t)$$

Material parameters are assumed independent of coordinates



Poynting('s)-Umov('s) Theorem



Energy balance in an electromagnetic system

Linear Momentum Carried by Fields

Volume integration considerably change the meaning of Poynting('s) vector

$$oldsymbol{p} = rac{1}{c_0^2} \int\limits_V \left(oldsymbol{E} imes oldsymbol{H}
ight) \mathrm{d}\,V$$

This formula is only valid in vacuum. In material media things are more tricky.



Angular Momentum Carried by Fields

$$oldsymbol{L} = rac{1}{c_0^2} \int\limits_V oldsymbol{r} imes \left(oldsymbol{E} imes oldsymbol{H}
ight) \mathrm{d}\,V$$

This formula is only valid in vacuum. In material media things are more tricky.



Frequency Domain

$$m{F}ig(m{r},tig) \in \mathbb{R}$$
 $\hat{m{F}}ig(m{r},\omegaig) \in \mathbb{C}$ $m{F}ig(m{r},\omegaig) = \int_{-\infty}^{\infty} m{F}ig(m{r},tig) e^{\mathrm{j}\omega t} d\omega$ $\hat{m{F}}ig(m{r},\omegaig) = \int_{-\infty}^{\infty} m{F}ig(m{r},tig) e^{-\mathrm{j}\omega t} dt$

$$\frac{\partial \boldsymbol{F} \big(\boldsymbol{r}, t \big)}{\partial t} \leftrightarrow \mathrm{j} \omega \hat{\boldsymbol{F}} \big(\boldsymbol{r}, \omega \big)$$
 Spatial derivatives are untouched
$$\frac{\partial \boldsymbol{F} \big(\boldsymbol{r}, t \big)}{\partial t} \leftrightarrow \frac{\partial \hat{\boldsymbol{F}} \big(\boldsymbol{r}, \omega \big)}{\partial r_{\xi}} \leftrightarrow \frac{\partial \hat{\boldsymbol{F}} \big(\boldsymbol{r}, \omega \big)}{\partial r_{\xi}}$$

Frequency domain helps us to remove explicit time derivatives





Phasors

$$\hat{m{F}}ig(m{r},-\omegaig)=\hat{m{F}}^*ig(m{r},\omegaig)$$

$$m{F}ig(m{r},tig) = rac{1}{\pi}\int\limits_0^\infty \mathrm{Re}ig[\hat{m{F}}ig(m{r},\omegaig)\mathrm{e}^{\mathrm{j}\omega t}ig]d\omega$$

Reduced frequency domain representation



Maxwell('s) Equations – Frequency Domain

$$abla imes \hat{\boldsymbol{H}} \left(\boldsymbol{r}, \omega \right) = \hat{\boldsymbol{J}} \left(\boldsymbol{r}, \omega \right) + \mathrm{j} \omega \varepsilon \hat{\boldsymbol{E}} \left(\boldsymbol{r}, \omega \right)$$

$$abla imes \hat{m{E}}ig(m{r},\omegaig) = -\mathrm{j}\omega\mu\hat{m{H}}ig(m{r},\omegaig)$$

$$\nabla \cdot \hat{\boldsymbol{H}}(\boldsymbol{r},\omega) = 0$$

$$abla \cdot \hat{m{E}} ig(m{r}, \omegaig) = rac{\hat{
ho} ig(m{r}, \omegaig)}{arepsilon}$$

We assume linearity of material relations



Wave Equation – Frequency Domain

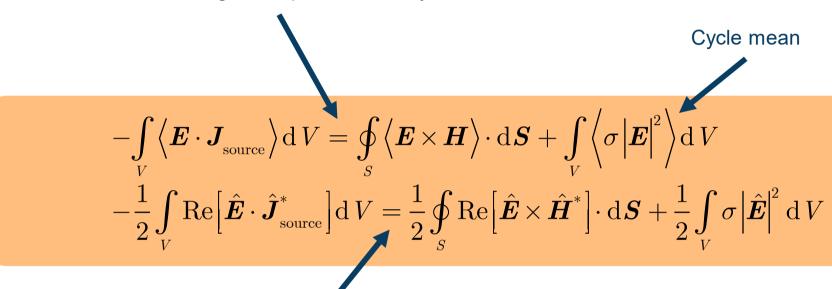
$$\Delta \hat{\boldsymbol{A}}(\boldsymbol{r},\omega) - j\omega\mu(\sigma + j\omega\varepsilon)\hat{\boldsymbol{A}}(\boldsymbol{r},\omega) = -\mu\hat{\boldsymbol{J}}_{\text{source}}(\boldsymbol{r},\omega)$$

Helmholtz('s) equation



Heat Balance in Time-Harmonic Steady State

Valid for general periodic steady state



Valid for time-harmonic steady state





Plane Wave

 $\hat{m{E}}ig(m{r},\omegaig)=m{E}_{_0}ig(\omegaig)\mathrm{e}^{-\mathrm{j}km{n}\cdotm{r}}$ Electric and magnetic fields $\hat{\boldsymbol{H}}(\boldsymbol{r},\omega) = \frac{k}{\omega\mu} [\boldsymbol{n} \times \boldsymbol{E}_{0}(\omega)] e^{-jk\boldsymbol{n}\cdot\boldsymbol{r}}$ are orthogonal to propagation direction $\boldsymbol{n} \cdot \boldsymbol{E}_{\scriptscriptstyle 0} \left(\omega\right) = 0$ $\boldsymbol{n} \cdot \boldsymbol{H}_0(\omega) = 0$ $k^2 = -\mathrm{j}\omega\mu\left(\sigma + \mathrm{j}\omega\varepsilon\right)$ Wave-number

Unitary vector representing the direction of propagation

Electric and magnetic fields are mutually orthogonal

The simplest wave solution of Maxwell('s) equations



Plane Wave Characteristics

$$k = \sqrt{-j\omega\mu(\sigma + j\omega\varepsilon)}$$

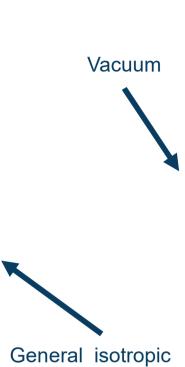
$$\operatorname{Re}[k] > 0; \operatorname{Im}[k] < 0$$

$$\lambda = \frac{2\pi}{\operatorname{Re}[k]}$$

$$v_{\rm f} = \frac{\omega}{{\rm Re}[k]}$$

$$Z = \frac{\omega \mu}{k}$$

$$\delta = -\frac{1}{\mathrm{Im}[k]}$$



material

$$k = \frac{\omega}{c_0}$$

$$\operatorname{Re}[k] > 0; \operatorname{Im}[k] = 0$$

$$\lambda = \frac{c_0}{f}$$

$$v_{\rm f} = c_{\rm o}$$

$$Z = c_{\scriptscriptstyle 0} \mu_{\scriptscriptstyle 0} = \sqrt{\frac{\mu_{\scriptscriptstyle 0}}{\varepsilon_{\scriptscriptstyle 0}}} \approx 377~\Omega$$

$$\delta \to \infty$$





Cycle Mean Power Density of a Plane Wave

Power propagation coincides with phase propagation

$$\left\langle oldsymbol{E}(oldsymbol{r},t) imes oldsymbol{H}(oldsymbol{r},t)
ight
angle = rac{1}{2} rac{\mathrm{Re}ig[kig]}{\omega \mu} ig| oldsymbol{E}_0 \left(\omega
ight) ig|^2 \, \mathrm{e}^{2\,\mathrm{Im}[k]oldsymbol{n}\cdotoldsymbol{r}} oldsymbol{n}$$



Lukas Jelinek

Ver. 2024/11/30

