GVG: Cylindrical panorama

1 Cylindrical coordiante system

Consider an orthonormal coordinate system $(A, \underbrace{(\vec{a}_1, \vec{a}_2, \vec{a}_3)}_{\alpha})$ and a cylinder defined by the set of points

$$\mathcal{C}_{(A,\alpha)} \stackrel{\text{def}}{=} \{e \mid e_{(A,\alpha)} = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix}^\top, \ e_1^2 + e_2^2 = 1\}$$

as is depicted in Figure 1.

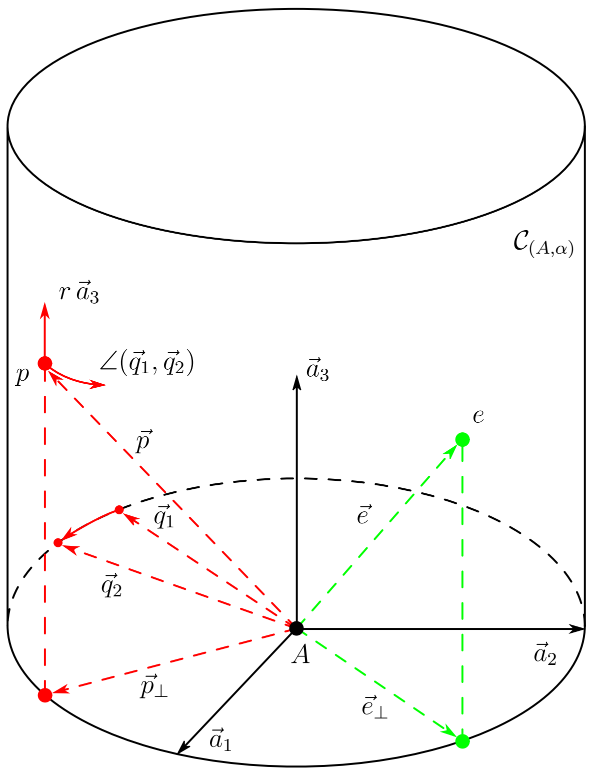

Figure 1: The cylinder $\mathcal{C}_{(A,\alpha)}$ and its coordinate system

The cylindrical coordinate system $\left(p, \underbrace{\left(\angle(\vec{q}_1, \vec{q}_2), r\right)}_{\psi}\right)$ of the cylinder $\mathcal{C}_{(A,\alpha)}$ consists of 3 elements:

1. The origin $p \in \mathcal{C}_{(A,\alpha)}$,
2. The angular resolution $\angle(\vec{q}_1, \vec{q}_2)$ defined by some $\vec{q}_1, \vec{q}_2 \perp \vec{a}_3$, $\vec{q}_1 \nparallel \vec{q}_2$,
3. The vertical resolution $r \in \mathbb{R}\backslash\{0\}$ in α units.

In order to define the coordinates of a point $e \in \mathcal{C}_{(A,\alpha)}$ in a cylindrical coordinate system (p, ψ), we express $\vec{e}$ and $\vec{p}$ in α

$$\vec{e}_\alpha = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}, \quad \vec{p}_\alpha = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix},$$

project $\vec{e}$ and $\vec{p}$ onto the plane spanned by $\vec{a}_1$ and $\vec{a}_2$ as

$$\vec{e}_\perp = e_1\vec{a}_1 + e_2\vec{a}_2, \quad \vec{p}_\perp = p_1\vec{a}_1 + p_2\vec{a}_2$$

and define

$$e_{(p,\psi)} \stackrel{\text{def}}{=} \begin{bmatrix} \frac{\angle(\vec{p}_\perp, \vec{e}_\perp)}{\angle(\vec{q}_1, \vec{q}_2)} \\ \frac{e_3 - p_3}{r} \end{bmatrix} \tag{1}$$

If we denote

$$\mathcal{C} = \left\{ e_{(A,\alpha)} \mid e \in \mathcal{C}_{(A,\alpha)} \right\} \subset \mathbb{R}^3$$

then, depending on how we define the angle function $\angle(\cdot,\cdot)$, the function

$$\varphi \colon \mathcal{C} \to \mathbb{R}^2$$
$$e_{(A,\alpha)} \mapsto e_{(p,\psi)}$$

will have discontinuities at different lines on the cylinder. There are two common choices for the angle to make:

(a) $\angle(\cdot,\cdot) \in [0, 2\pi)$

(b) $\angle(\cdot,\cdot) \in (-\pi, \pi]$

(There is also a choice in the direction, which, however, does not influence the discontinuities). The vertical line across which the tearing occurs in case (b) is shown in Figure 2 in blue.

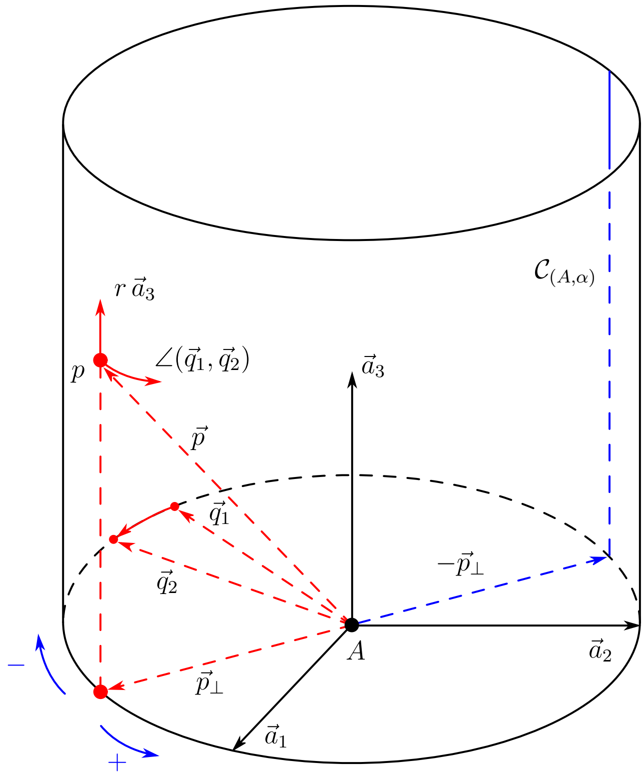

Figure 2: The angle function $\angle(\cdot,\cdot) \in (-\pi, \pi]$ causes the tearing of the cylinder along the blue vertical line.

In case (a), the tearing would occur along the vertical line that passes through p.

We will see later that, for the construction of the cylindrical image, it is important to choose an appropriate definition of the angle function $\angle(\cdot,\cdot)$ to avoid tearing the cylindrical image somewhere in the middle.

2 Projection to cylinder

If we have a general point x in space (not necessarily in $\mathcal{C}_{(A,\alpha)}$), we can project it along the ray that joins A and x denoted by $\vec{x}$ to $e \in \mathcal{C}_{(A,\alpha)}$. We are looking for λ such that

$$\vec{e} = \lambda \vec{x}$$

$$\vec{e}_\alpha = \lambda \vec{x}_\alpha$$

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Since $e \in \mathcal{C}_{(A,\alpha)}$, then we have $e_1^2 + e_2^2 = 1$, and hence

$$1 = e_1^2 + e_2^2 = \lambda^2 x_1^2 + \lambda^2 x_2^2 \iff \lambda^2 = \frac{1}{x_1^2 + x_2^2} \iff \lambda = \pm \frac{1}{\sqrt{x_1^2 + x_2^2}}$$

Having 2 values for λ corresponds to the fact that the ray defined by $\vec{x}$ intersects the cylinder $\mathcal{C}_{(A,\alpha)}$ at two different points represented by vectors

$$\vec{e}_1 = \frac{1}{\sqrt{x_1^2 + x_2^2}} \vec{x} \tag{2}$$

$$\vec{e}_2 = -\frac{1}{\sqrt{x_1^2 + x_2^2}} \vec{x} = -\vec{e}_1 \tag{3}$$

3 Constructing panorama

3.1 Cylindrical image surface

Having a projective camera with a cartesian camera coordinate system (C, γ), we first define the cylinder $\mathcal{C}_{(C,\gamma)}$ and its coordinate system. The cylinder is defined by the set of points

$$\mathcal{C}_{(C,\gamma)} = \{e \mid e_{(C,\gamma)} = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix}^\top,\ e_1^2 + e_3^2 = 1\}.$$

Notice that, unlike in the previous sections, we define the cylinder here a bit differently: its axis follows $\vec{c}_2$ (not $\vec{c}_3$). Hence, formulas (1), (2), and (3) must be changed appropriately.

The center p of the coordinate system of $\mathcal{C}_{(C,\gamma)}$ is defined to be the principal point of the camera. The angular resolution is defined to be $\angle(\vec{c}_3, \vec{c}_3 + \vec{b}_1)$, since we would like to achieve approximately the same horizontal resolution in the cylindrical image as in the perspective image itself. As for vertical resolution, we would like to remove the (possible) affine distortion in the perspective image caused by the (possible) non-orthonormality of $(\vec{b}_1, \vec{b}_2)$. To achieve this, we define the vertical resolution r to be the length of $\vec{b}_1$ in units of γ, that is, $r = \frac{\|\vec{b}_1\|}{f}$.

Before looking at the angular resolution we define the angle function $\angle(\cdot,\cdot)$. Since we defined p to be the principal point and the angular resolution to be $\angle(\vec{c}_3, \vec{c}_3 + \vec{b}_1)$, it will be sufficient for us to define $\angle(\vec{c}_3, \vec{v})$ for $\vec{v} \in \langle \vec{c}_1, \vec{c}_3 \rangle$, since this is all we need to evaluate Equation (1). If

$$\vec{v}_\gamma = \begin{bmatrix} v_1 \\ 0 \\ v_3 \end{bmatrix}$$

then we define

$$\angle(\vec{c}_3, \vec{v}) \overset{\text{def}}{=} \operatorname{atan2}(v_1, v_3) \in (-\pi, \pi]$$

The geometry of such a definition is visualized in Figure 3 in magenta color. The tearing in the cylindrical coordinates (i.e., when we unwrap the cylinder and visualize its points in the plane using cylindrical coordinates) happens along the line ℓ_1, and as a consequence when we project the image to the cylinder it will not be teared when visualized in the cylindrical coordinates, because ℓ_1 is behind the camera. If we used another common definition of the angle $\angle(\vec{c}_3, \vec{v}) \in [0, 2\pi)$, we would tear the cylindrical image along the line ℓ_2, because ℓ_2 is in front of the camera.

By looking at the angular resolution of the cylindrical coordinate system, we see that

$$\vec{c}_3 \perp \vec{b}_1 \Rightarrow \angle(\vec{c}_3, \vec{c}_3 + \vec{b}_1) = \arctan\left(\frac{\left\|\vec{b}_1\right\|}{\|\vec{c}_3\|}\right)$$

which, since $\|\vec{c}_3\| \gg \left\|\vec{b}_1\right\|$, can be written approximately as

$$\angle(\vec{c}_3, \vec{c}_3 + \vec{b}_1) \approx \frac{\left\|\vec{b}_1\right\|}{\|\vec{c}_3\|} = \frac{\left\|\vec{b}_1\right\|}{f} = \frac{1}{k_{11}}$$

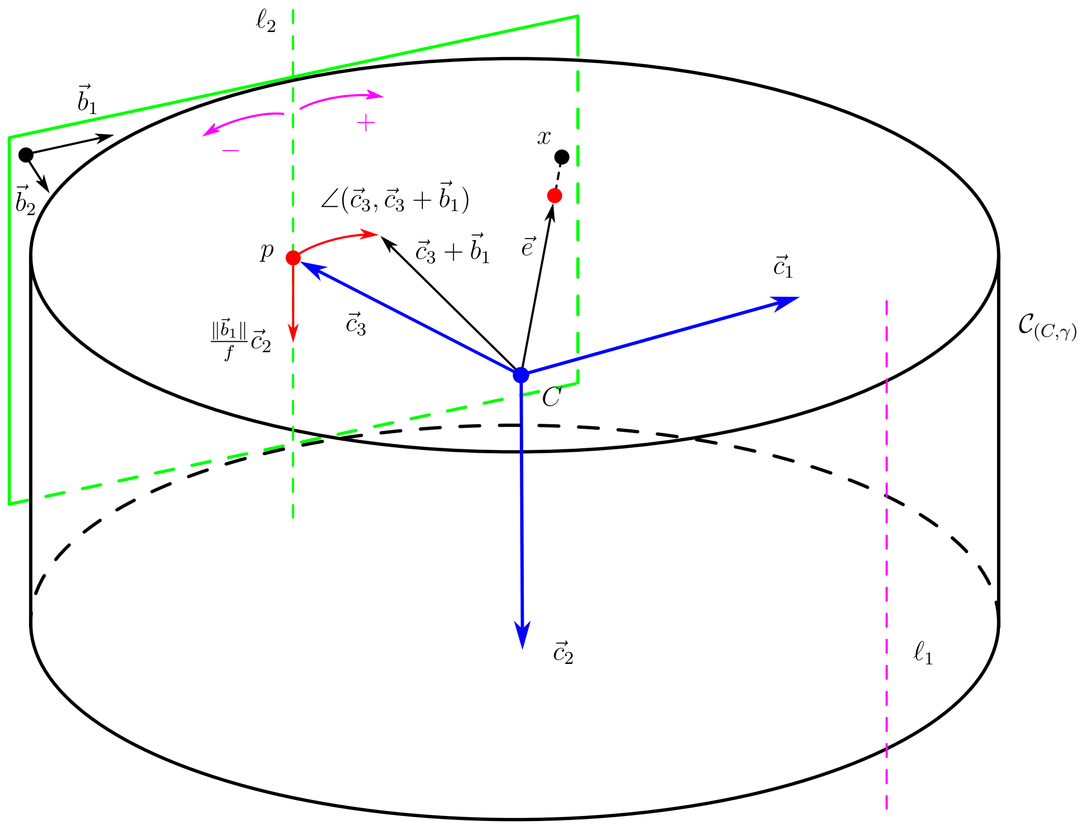

Figure 3: The cylindrical image surface $\mathcal{C}_{(C,\gamma)}$ and its coordinate system

The vertical resolution is

$$r = \frac{\left\|\vec{b}_1\right\|}{f} = \frac{1}{k_{11}}$$

and the coordinates $p_{(C,\gamma)}$ of the center of the cylindrical coordinate system in the camera cartesian coordinate system are

$$p_{(C,\gamma)} = \vec{c}_{3_\gamma} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Taking a general point x which is represented by vector $\vec{x}$ in (C, γ), we project x along $\vec{x}$ to the cylinder and get a point $e \in \mathcal{C}_{(C,\gamma)}$ represented by vector $\vec{e}$. To obtain its coordinates in γ, we calculate

$$\vec{x}_\gamma = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow \vec{e}_\gamma = \frac{1}{\sqrt{x_1^2 + x_3^2}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \vec{e}_\perp \sim x_1\vec{c}_1 + x_3\vec{c}_3$$

where the scale λ according to Section 2 was chosen to be positive to obtain the projected point in front of the camera. As for the angle between $\vec{c}_3$ and $\vec{e}_\perp$ we have

$$\angle(\vec{c}_3, \vec{e}_\perp) = \operatorname{atan2}(x_1, x_3)$$

The coordinates of e in the cylindrical coordinate system defined above can be now obtained as

$$e_{(p,\psi)} = \begin{bmatrix} \frac{\angle(\vec{c}_3, \vec{e}_\perp)}{\angle(\vec{c}_3, \vec{c}_3 + \vec{b}_1)} \\ \frac{\vec{e}_{\gamma,2} - \vec{c}_{3\gamma,2}}{r} \end{bmatrix} = k_{11} \cdot \begin{bmatrix} \operatorname{atan2}(x_1, x_3) \\ \frac{x_2}{\sqrt{x_1^2 + x_3^2}} \end{bmatrix}$$

3.2 Gluing images on the cylinder

Suppose that we took n images with n cameras that all have the same projection center C. Let's order the cameras in a way that the cylinder will be defined in the coordinate system of the first camera. Let us take the j-th camera and show how to express the projections of its image points to $\mathcal{C}_{(C,\gamma_1)}$ in the coordinate system of $\mathcal{C}_{(C,\gamma_1)}$.

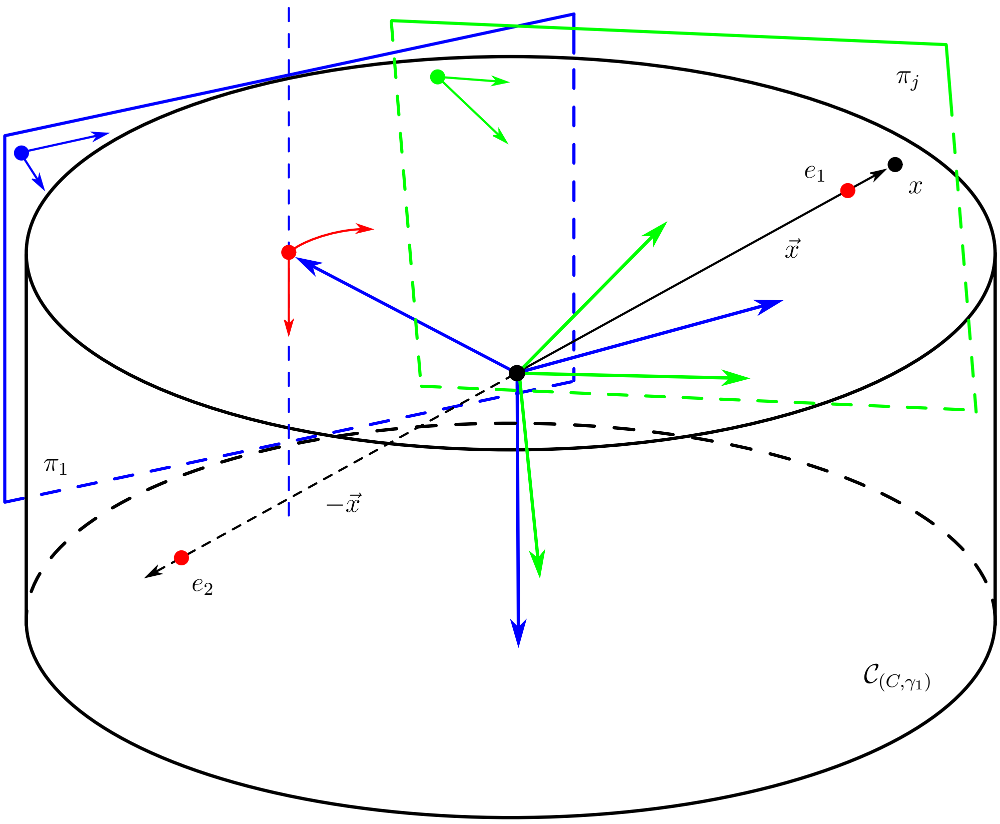

Figure 4: Projecting the image points in π_j onto the cylinder $\mathcal{C}_{(C,\gamma_1)}$

Since $C_1 = C_j = C$, then according to [1, Section 8.1], there is a homography $\mathtt{H}_j$ that transforms the image points from the j-th to the 1st camera (see [1, Equation (8.4)]). Furthermore, we have

$$\mathtt{H}_j = \mathtt{T}_{\beta_j \to \beta_1} = \mathtt{T}_{\gamma_1 \to \beta_1} \mathtt{T}_{\gamma_j \to \gamma_1} \mathtt{T}_{\beta_j \to \gamma_j} = \mathtt{K}_1 \, s\mathtt{R} \, \mathtt{K}_j^{-1}$$

where $\mathtt{K}_1, \mathtt{K}_j$ are the camera calibration matrices of the 1st and j-th cameras, and $s\mathtt{R}$ (scaled rotation) is the transition matrix from γ_j to γ_1. Notice that $s > 0$, since γ_1 and γ_j are both right-handed (and thus a transition matrix between them must have positive determinant). (In case when the images are made by the same camera, we have $\mathtt{K}_1 = \mathtt{K}_j$ and $s = 1$). This gives a constraint on $\mathtt{H}_j$:

$$\operatorname{sgn}(\det \mathtt{H}_j) = \operatorname{sgn}\left(s^3 \det \mathtt{K}_1 \det \mathtt{R} \frac{1}{\det \mathtt{K}_j}\right) = \operatorname{sgn}\left(\frac{\det \mathtt{K}_1}{\det \mathtt{K}_j}\right)$$

If we have recovered only a multiple $\mathtt{G}_j = \tau \mathtt{H}_j$, then we can obtain a multiple of the transition matrix from γ_j to γ_1:

$$\tau \mathtt{T}_{\gamma_j \to \gamma_1} = \tau s \mathtt{R} = \tau \mathtt{K}_1^{-1} \mathtt{H}_j \mathtt{K}_j = \mathtt{K}_1^{-1} \mathtt{G}_j \mathtt{K}_j$$

In order to project a general point $x \in \pi_j$ to the cylinder $\mathcal{C}_{(C,\gamma_1)}$, we express a τ-multiple of vector $\vec{x}$ (that joins C and x) in γ_1 as

$$(\tau \vec{x})_{\gamma_1} = \tau \vec{x}_{\gamma_1} = \tau \mathtt{T}_{\gamma_j \to \gamma_1} \vec{x}_{\gamma_j} = \mathtt{K}_1^{-1} \mathtt{G}_j \mathtt{K}_j \vec{x}_{\gamma_j} = \mathtt{K}_1^{-1} \mathtt{G}_j \vec{x}_{\beta_j}$$

Our aim is to obtain the projection of x onto the cylinder $\mathcal{C}_{(C,\gamma_1)}$ that will be in front of the j-th camera. As shown in Figure 4, we are interested in e_1, and not in e_2. For this, we need to apply the projection of $\tau \vec{x}$ for $\tau > 0$ according to Equation (2). All that is left is to obtain a positive multiple of $\mathtt{H}_j$ from $\mathtt{G}_j$. This can be done by considering

$$\operatorname{sgn}(\det \mathtt{G}_j) \cdot \mathtt{G}_j = \operatorname{sgn}(\tau^3 \det \mathtt{H}_j) \cdot \tau \mathtt{H}_j = \operatorname{sgn}(\tau)^3 \cdot \operatorname{sgn}(\det \mathtt{H}_j) \cdot \tau \mathtt{H}_j = \operatorname{sgn}(\tau) \cdot \tau \cdot \mathtt{H}_j = |\tau| \cdot \mathtt{H}_j,$$

where we have used that

$$\operatorname{sgn}(\tau)^3 = \operatorname{sgn}(\tau), \quad \operatorname{sgn}(\det \mathtt{H}_j) = \operatorname{sgn}\left(\frac{\det \mathtt{K}_1}{\det \mathtt{K}_j}\right) = 1, \quad \operatorname{sgn}(\tau) \cdot \tau = |\tau|.$$

Remark. *Beware that not considering a positive multiple of* $\mathtt{H}_j = \mathtt{T}_{\beta_j \to \beta_1}$ *will cause mirroring effects in the panorama.*

References

[1] Tomas Pajdla, *Elements of geometry for computer vision*, `https://cw.fel.cvut.cz/wiki/_media/courses/gvg/pajdla-gvg-lecture-2021.pdf`.