GVG Lab-09 Solution

Task 1. *Let us have two vanishing points in the image represented by vectors* $\vec{u}_{1\alpha} = [0,0]^\top$ *and* $\vec{u}_{2\alpha} = [2,0]^\top$*, which come from the image of an observed rectangle. Find all values of parameter* a *in the matrix*

$$\mathtt{K} = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

of a camera which captured the image.

Solution: Let us denote by $\vec{x}_{1\beta} = [0,0,1]^\top$ and $\vec{x}_{2\beta} = [2,0,1]^\top$ the two vectors representing given vanishing points in the camera coordinate system (C, β). Since the given vanishing points are images of points at infinity of two perpendicular lines in the world, then $\vec{x}_1 \perp \vec{x}_2$. To express this constraint algebraically we need to pass to the coordinates of $\vec{x}_1$ and $\vec{x}_2$ in some orthogonal basis (e.g. γ):

$$\vec{x}_{1\gamma}^\top \vec{x}_{2\gamma} = 0 \Leftrightarrow \vec{x}_{1\beta}^\top \mathtt{K}^{-\top}\mathtt{K}^{-1}\vec{x}_{2\beta} = 0$$

We compute

$$\mathtt{K}^{-1} = \begin{bmatrix} 1 & 0 & -a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathtt{K}^{-\top} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -a & 0 & 1 \end{bmatrix}, \quad \mathtt{K}^{-\top}\mathtt{K}^{-1} = \begin{bmatrix} 1 & 0 & -a \\ 0 & 1 & 0 \\ -a & 0 & a^2+1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -a \\ 0 & 1 & 0 \\ -a & 0 & a^2+1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = 0$$

$$a^2 - 2a + 1 = 0 \Leftrightarrow a = 1.$$

□

Task 2. *Let the camera be given by the following camera projection matrix*

$$\mathtt{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Let the rectangle in space be defined by the following 4 points:

$$\vec{X}_{1\delta} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{X}_{2\delta} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \vec{X}_{3\delta} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \vec{X}_{4\delta} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Find the horizon of the plane defined by the rectangle.

Solution: In order to draw a picture what is happening in this task we need to take one of the cameras which has the given camera projection matrix.

Remark. *Since the set of camera projection matrices is in bijective correspondence with the set of triples* $(\mathtt{K}, \mathtt{R}, \vec{C}_\delta)$ *and every camera is uniquely defined by a 4-tuple* $(f, \mathtt{K}, \mathtt{R}, \vec{C}_\delta)$*, then it is obvious that for fixed* $\mathtt{K}, \mathtt{R}$ *and* $\vec{C}_\delta$ *the set of cameras*

$$\left\{ (f, \mathtt{K}, \mathtt{R}, \vec{C}_\delta) \;\middle|\; f \in \mathbb{R}^+ \right\}$$

have the same camera projection matrix. Out of those we take the one with $f = 1$ *to create a picture for this task (see Figure 1). Since* $\mathtt{P}_{1:3,1:3} = \mathtt{I}$ *and* $\mathtt{P}_{1:3,4} = \mathbf{0}$ *results into* $\mathtt{K} = \mathtt{R} = \mathtt{I}$ *and* $\vec{C}_\delta = \mathbf{0}$*, then*

$$O = C, \quad \mathtt{T}_{\delta\to\gamma} = \frac{1}{f}\mathtt{R} = \mathtt{I}, \quad \mathtt{T}_{\gamma\to\beta} = \mathtt{K} = \mathtt{I},$$

which means that $(O, \delta) = (C, \gamma) = (C, \beta)$.

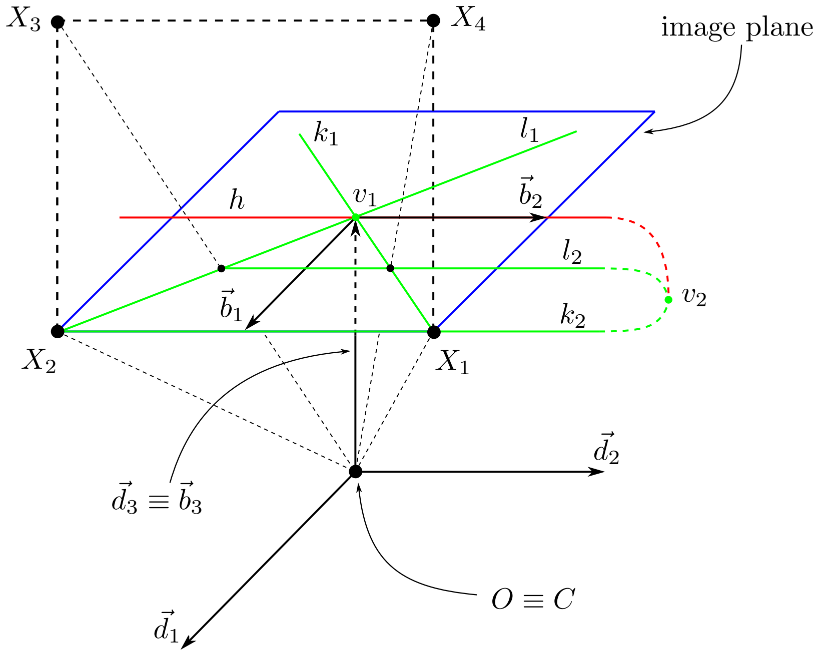

Figure 1: Camera observes the square $\square X_1X_2X_3X_4$

We first project the world points to the camera:

$$\mathbf{x}_1 = \mathtt{P}\begin{bmatrix}\vec{X}_{1\delta}\\1\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix}, \quad \mathbf{x}_2 = \mathtt{P}\begin{bmatrix}\vec{X}_{2\delta}\\1\end{bmatrix} = \begin{bmatrix}1\\-1\\1\end{bmatrix}, \quad \mathbf{x}_3 = \mathtt{P}\begin{bmatrix}\vec{X}_{3\delta}\\1\end{bmatrix} = \begin{bmatrix}1\\-1\\2\end{bmatrix}, \quad \mathbf{x}_4 = \mathtt{P}\begin{bmatrix}\vec{X}_{4\delta}\\1\end{bmatrix} = \begin{bmatrix}1\\1\\2\end{bmatrix}.$$

There is no need to find the image points $\vec{u}_{i\alpha}, i = 1,\ldots,4$ (by dividing $\mathbf{x}_i$ by the last coordinate), since it is easier to work in homogeneous coordinates to work with lines in $\mathbb{P}^2$ and their intersections. In order to find the horizon (the projection of a line at infinity of the plane τ defined by $\square X_1X_2X_3X_4$) it is sufficient to find two vanishing points of two pairs of parallel lines from τ. Those 2 pairs will be defined by $(\overline{X_2X_3}, \overline{X_1X_4})$ and $(\overline{X_1X_2}, \overline{X_3X_4})$.

Two find the representatives of the images k_1 and l_1 of the first pair we apply the cross product rule to the homogeneous representatives of the projected points to the camera:

$$\mathbf{k}_1 = \mathbf{x}_1 \times \mathbf{x}_4 = \begin{bmatrix}1\\1\\1\end{bmatrix} \times \begin{bmatrix}1\\1\\2\end{bmatrix} = \begin{bmatrix}1\\-1\\0\end{bmatrix}, \quad \mathbf{l}_1 = \mathbf{x}_2 \times \mathbf{x}_3 = \begin{bmatrix}1\\-1\\1\end{bmatrix} \times \begin{bmatrix}1\\-1\\2\end{bmatrix} = \begin{bmatrix}-1\\-1\\0\end{bmatrix}$$

The vanishing point associated to the pair of world lines $(\overline{X_2X_3}, \overline{X_1X_4})$ is then the intersection of k_1 and l_1. In homogeneous coordinates we have:

$$\mathbf{v}_1 = \mathbf{k}_1 \times \mathbf{l}_1 = \begin{bmatrix}1\\-1\\0\end{bmatrix} \times \begin{bmatrix}-1\\-1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\-2\end{bmatrix}$$

Since the last coordinate of $\mathbf{v}_1$ is nonzero, the we can pass to the affine coordinates (by dividing by the last coordinate and taking the first two) and see that $v_{1_{(o,\alpha)}} = \begin{bmatrix}0 & 0\end{bmatrix}^\top$.

Similarly, the representatives of the images k_2 and l_2 of the second pair of world lines $(\overline{X_1X_2}, \overline{X_3X_4})$ are:

$$\mathbf{k}_2 = \mathbf{x}_1 \times \mathbf{x}_2 = \begin{bmatrix}1\\1\\1\end{bmatrix} \times \begin{bmatrix}1\\-1\\1\end{bmatrix} = \begin{bmatrix}2\\0\\-2\end{bmatrix}, \quad \mathbf{l}_2 = \mathbf{x}_3 \times \mathbf{x}_4 = \begin{bmatrix}1\\-1\\2\end{bmatrix} \times \begin{bmatrix}1\\1\\2\end{bmatrix} = \begin{bmatrix}-4\\0\\2\end{bmatrix}$$

The vanishing point associated to the pair of world lines $(\overline{X_1X_2}, \overline{X_3X_4})$ is then the intersection of k_2 and l_2 (in $\mathbb{P}^2$). In homogeneous coordinates we have:

$$\mathbf{v}_2 = \mathbf{k}_2 \times \mathbf{l}_2 = \begin{bmatrix}2\\0\\-2\end{bmatrix} \times \begin{bmatrix}-4\\0\\2\end{bmatrix} = \begin{bmatrix}0\\4\\0\end{bmatrix}$$

Since the last coordinate of $\mathbf{v}_2$ is zero, then v_2 is a point at infinity of $\mathbb{P}^2$. (This is logical since the world lines $\overline{X_1X_2}$ and $\overline{X_3X_4}$ are parallel to the image plane of the camera).

To find the horizon we need to find the line passing through the points v_1 (visible in the image) and v_2 (not visible in the image). We do this again by the cross product:

$$\mathbf{h} = \mathbf{v}_1 \times \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} \times \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \\ 0 \end{bmatrix} \sim \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Passing to affine coordinates, we see that

$$h = \{(u, v) \in \mathbb{R}^2 \mid 1 \cdot u + 0 \cdot v + 0 = 0\}$$

is a line $u = 0$ in the image. Notice that the horizon of τ is (always) the intersection of the plane parallel to τ and passing through the camera center C and the image plane of the camera. □

Task 3. *Consider line l in $\mathbb{P}^2$ represented by $\mathbf{l} = [1, 0, 1]^\top$ and homography*

$$\mathtt{H} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which maps line l onto line l'. Find the point on the line l that is mapped onto itself.

Solution: We first determine all the points in $\mathbb{P}^2$ that are mapped onto themselves by $\mathtt{H}$:

$$\lambda \mathbf{x} = \mathtt{H}\mathbf{x}, \quad [\mathbf{x}] \in \mathbb{P}^2,\ \lambda \neq 0$$

This may be equivalently restated as finding eigenvectors of $\mathtt{H}$ (since $\mathtt{H}$ is invertible, then all its eigenvalues are nonzero). We first find the eigenvalues of $\mathtt{H}$:

$$\det(\lambda \mathtt{I} - \mathtt{H}) = 0 \Leftrightarrow (\lambda - 1)^3 = 0 \Leftrightarrow \lambda = 1.$$

To find the eigenspace corresponding to the eigenvalue $\lambda = 1$ we solve

$$(1 \cdot \mathtt{I} - \mathtt{H})\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

The set of solutions is a 2-dimensional linear space:

$$S = \left\langle \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{x}_1}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{x}_2} \right\rangle$$

In other words, every point in $\mathbb{P}^2$ in the form $a\mathbf{x}_1 + b\mathbf{x}_2$ for $a, b \in \mathbb{R}$ is mapped onto itself by $\mathtt{H}$. Notice that all these points are points at infinity, since the last coordinates of $\mathbf{x}_1$ and $\mathbf{x}_2$ (and thus of $a\mathbf{x}_1 + b\mathbf{x}_2$ for $a, b \in \mathbb{R}$) are zero. These points form the line at infinity k in $\mathbb{P}^2$ represented by

$$\mathbf{k} = \mathbf{x}_1 \times \mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

In order to find a point on the line l that is mapped onto itself by $\mathtt{H}$ we need to find the intersection of k and l:

$$\mathbf{p} = \mathbf{k} \times \mathbf{l} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \sim \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

□

Task 4. *Find all points in $\mathbb{P}^2$, which are projected into themselves by homography*

$$\mathtt{H} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution: See the first part of the solution to Task 3. □

Task 5. *Find centers of all cameras*

$$\mathtt{P}_\beta = \begin{bmatrix} a & 0 & 1 & 0 \\ 0 & 1 & 0 & c \\ 1 & b & 1 & 0 \end{bmatrix}$$

which project point $[1,1,1]^\top$ *in space into point* $[1,1]^\top$ *in the image.*

Solution: First of all, for $\mathtt{P}_\beta$ to be a valid image projection matrix it must take the form

$$\mathtt{P}_\beta = \begin{bmatrix} \mathtt{A} \mid -\mathtt{A}\vec{C}_\delta \end{bmatrix}$$

where $\mathtt{A}$ is invertible 3×3 matrix. Thus, there is a restriction on $\mathtt{P}_\beta$:

$$\det \mathtt{P}_{\beta_{1:3,1:3}} \neq 0 \Leftrightarrow a \neq 1.$$

By definition, a world point X projects into a point $[u,v]^\top$ in the image if there exists a unique line connecting X and the camera projection center C and this line intersects the image plane in $\mathbb{A}^3$ at x with $x_{(o,\alpha)} = [u,v]^\top$. This geometric definition may be rewritten algebraically in the equivalent form as follows: a world point X projects into a point $[u,v]^\top$ in the image if

$$\exists\, \eta \in \mathbb{R}\backslash\{0\}: \quad \eta \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \mathtt{P}_\beta \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix} \tag{1}$$

Remark. *Notice that the statement*

$$\exists \eta \in \mathbb{R}: \quad \eta \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \mathtt{P}_\beta \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix} \tag{2}$$

is not equivalent to (1). *It is true that* (1) $\Rightarrow$ (2) *since if* $\eta \in \mathbb{R}\backslash\{0\}$, *then* $\eta \in \mathbb{R}$. *However, the converse* (2) $\Rightarrow$ (1) *doesn't hold. To see why, take* $X = C$. *Then the right hand side of both* (1) *and* (2) *becomes the zero vector. While in* (2) *we can take* $\eta = 0$ *to make the matrix equation true, in* (1) *there is no such* η. *(In other words,* (2) *also enables* C *"to be projected" to the image point* $[u,v]^\top$, *while* (1) *does not.)*

Substituting known values to (1) we obtain

$$\eta \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a & 0 & 1 & 0 \\ 0 & 1 & 0 & c \\ 1 & b & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \eta \in \mathbb{R}\backslash\{0\}$$

$$\begin{cases} \eta = a + 1 \\ \eta = c + 1 \\ \eta = b + 2 \end{cases}, \quad \eta \in \mathbb{R}\backslash\{0\}$$

$$\begin{cases} a = \eta - 1 \\ b = \eta - 2 \\ c = \eta - 1 \end{cases}, \quad \eta \in \mathbb{R}\backslash\{0\}$$

Substituting a, b, c into $\mathtt{P}_\beta$ (and remembering that $a \neq 1$) we get the set S of all possible cameras

$$S = \left\{ \begin{bmatrix} \eta-1 & 0 & 1 & 0 \\ 0 & 1 & 0 & \eta-1 \\ 1 & \eta-2 & 1 & 0 \end{bmatrix} \;\middle|\; \eta \in \mathbb{R}\backslash\{0,2\} \right\}$$

which project point $[1,1,1]^\top$ in space into point $[1,1]^\top$ in the image. To find centers of these cameras we need to invert the left 3×3 block parametrized by η:

$$\begin{bmatrix} \eta-1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & \eta-2 & 1 \end{bmatrix}^{-1} = \frac{1}{\eta-2} \begin{bmatrix} 1 & 0 & -1 \\ \eta-2 & \eta-2 & -\eta^2+3\eta-2 \\ -1 & 0 & \eta-1 \end{bmatrix}^\top = \frac{1}{\eta-2} \begin{bmatrix} 1 & \eta-2 & -1 \\ 0 & \eta-2 & 0 \\ -1 & -\eta^2+3\eta-2 & \eta-1 \end{bmatrix}$$

$$\vec{C}_\delta = -\mathtt{P}^{-1}_{\beta_{1:3,1:3}}\mathtt{P}_{\beta_{1:3,4}} = -\frac{1}{\eta-2}\begin{bmatrix} 1 & \eta-2 & -1 \\ 0 & \eta-2 & 0 \\ -1 & -\eta^2+3\eta-2 & \eta-1 \end{bmatrix}\begin{bmatrix} 0 \\ \eta-1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1-\eta \\ 1-\eta \\ (1-\eta)^2 \end{bmatrix}$$

Thus, the set of camera centers of all cameras from S is described by

$$\left\{ \begin{bmatrix} 1-\eta \\ 1-\eta \\ (1-\eta)^2 \end{bmatrix} \middle| \, \eta \in \mathbb{R}\backslash\{0,2\} \right\}.$$

□

Task 6. *Let us have two lines L_1 and L_2 in $\mathbb{A}^3$ given by:*

$$L_1 : \vec{X}_{1\delta} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{X}_{2\delta} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}; \quad L_2 : \vec{X}_{3\delta} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{X}_{4\delta} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Find the intersection of L_1 and L_2 (if exists) in the projective space $\mathbb{P}^3$.

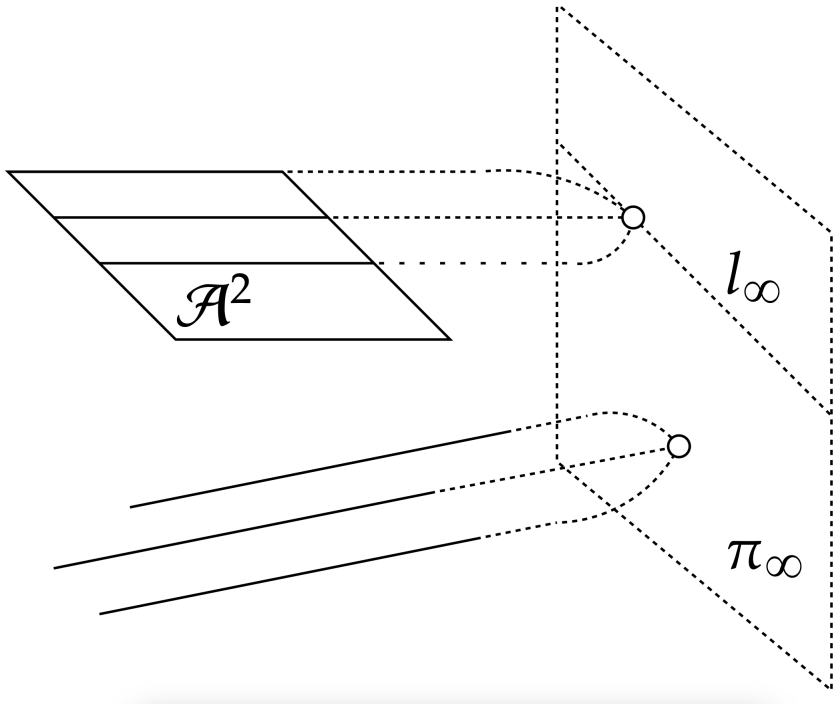

Figure 2: Lines in $\mathbb{P}^3$

Solution: While every two lines in $\mathbb{P}^2$ intersect, this is not the case in $\mathbb{P}^3$. Obviously, two lines L_1 and L_2 defined by 4 points from $\mathbb{A}^3$ intersect in $\mathbb{P}^3$ if and only if these lines lie in a plane. Algebraically, this condition can be expressed as

$$\det\left(\begin{bmatrix} \vec{X}_{1\delta}^\top & 1 \\ \vec{X}_{2\delta}^\top & 1 \\ \vec{X}_{3\delta}^\top & 1 \\ \vec{X}_{4\delta}^\top & 1 \end{bmatrix}\right) = 0.$$

Notice that the determinant of that matrix is zero if and only if it has a nontrivial kernel, whose generator (or generators if the 4 points are degenerate) defines the coefficients of the plane. For this task we can see that

$$\det\left(\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}\right) = 0$$

since the first and the last column are equal (meaning the 4 columns are linearly dependent). This means that the lines L_1 and L_2 intersect in $\mathbb{P}^3$ (however, they may not intersect in $\mathbb{A}^3$, which happens when they are parallel).

In order to find the intersection of L_1 and L_2 in $\mathbb{P}^3$ we need to construct a 3×4 matrix, whose rows represent 3 different planes: one passes through L_1, the second – through L_2, and the third contains them both. They can be constructed as follows:

$$\vec{X}_{1\delta} \times \vec{X}_{2\delta} = \begin{bmatrix}1\\0\\0\end{bmatrix} \times \begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}0\\-1\\0\end{bmatrix} \Rightarrow \sigma_1:\ 0\cdot x + (-1)\cdot y + 0\cdot z + 0 = 0 \Rightarrow \boldsymbol{\sigma}_1 = \begin{bmatrix}0\\-1\\0\\0\end{bmatrix}$$

$$\vec{X}_{3\delta} \times \vec{X}_{4\delta} = \begin{bmatrix}1\\1\\0\end{bmatrix} \times \begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}1\\-1\\0\end{bmatrix} \Rightarrow \sigma_2:\ 1\cdot x + (-1)\cdot y + 0\cdot z + 0 = 0 \Rightarrow \boldsymbol{\sigma}_2 = \begin{bmatrix}1\\-1\\0\\0\end{bmatrix}$$

$$\ker\left(\begin{bmatrix}1&0&0&1\\1&0&1&1\\1&1&0&1\\1&1&1&1\end{bmatrix}\right) = \left\langle\begin{bmatrix}-1\\0\\0\\1\end{bmatrix}\right\rangle \Rightarrow \boldsymbol{\sigma}_3 = \begin{bmatrix}-1\\0\\0\\1\end{bmatrix}$$

To find the intersection of L_1 and L_2 in $\mathbb{P}^3$ we need to find the kernel of the following matrix:

$$\begin{bmatrix}\boldsymbol{\sigma}_1^\top\\\boldsymbol{\sigma}_2^\top\\\boldsymbol{\sigma}_3^\top\end{bmatrix} = \begin{bmatrix}0&-1&0&0\\1&-1&0&0\\-1&0&0&1\end{bmatrix} \sim \begin{bmatrix}1&-1&0&0\\-1&0&0&1\\0&-1&0&0\end{bmatrix} \sim \begin{bmatrix}1&-1&0&0\\0&-1&0&1\\0&-1&0&0\end{bmatrix} \sim \begin{bmatrix}1&-1&0&0\\0&1&0&-1\\0&0&0&1\end{bmatrix}$$

$$\ker\left(\begin{bmatrix}\boldsymbol{\sigma}_1^\top\\\boldsymbol{\sigma}_2^\top\\\boldsymbol{\sigma}_3^\top\end{bmatrix}\right) = \left\langle\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\right\rangle \Rightarrow P = \overline{L_1} \cap \overline{L_2} = \left[\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\right] \in \mathbb{P}^3$$

Notice that the intersection of the projective closures of L_1 and L_2 is a point at infinity of $\mathbb{P}^3$ since L_1 and L_2 are parallel. Also notice that the first 3 coordinates $\begin{bmatrix}0 & 0 & 1\end{bmatrix}^\top$ of P define the direction vector $\mathbf{d}$ of the given lines. $\square$