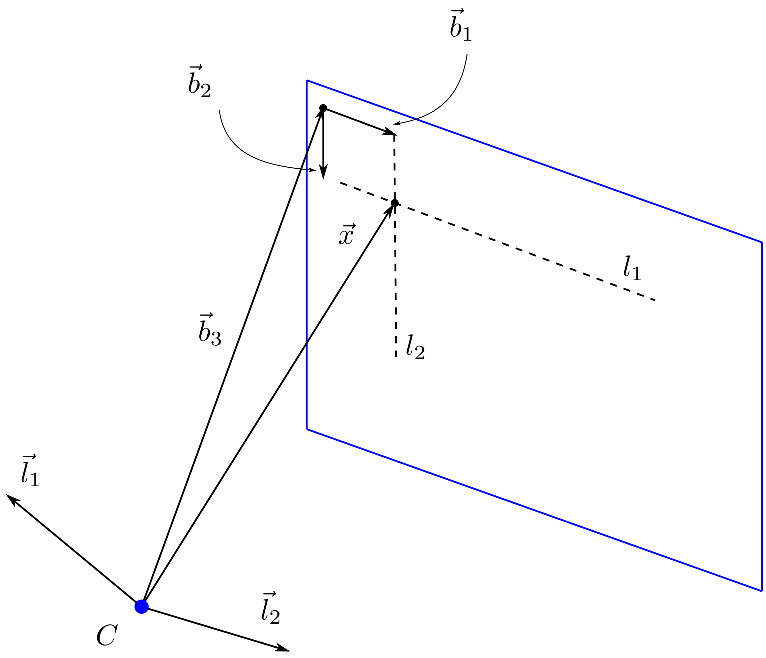
GVG Lab-08 Solution

Task 1. *Let us have two lines in the image l_1 and l_2 given by:*

$$l_1 : \ u = 1, \quad l_2 : \ v = 1.$$

Find their intersection in $\mathbb{A}^2$, if exists (using techniques of projective geometry).



Solution: Obviously, it is not necessary to use the techniques of projective geometry (the cross product rule for the intersection of 2 lines): we could simply setup the system of linear equations

$$\begin{cases} 1 \cdot u + 0 \cdot v = 1 \\ 0 \cdot u + 1 \cdot v = 1 \end{cases} \quad \Leftrightarrow \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\mathtt{M}} \begin{bmatrix} u \\ v \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\mathbf{b}}$$

and solve it by $\mathtt{M}^{-1}\mathbf{b}$. However, the above system will not have any solutions if the lines are parallel and not identical ($\mathbf{b}$ will not belong to rng $\mathtt{M}$).

We can also rewrite the above system as

$$\underbrace{\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}}_{\mathtt{A}} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and use Gaussian elimination to solve it. If the kernel of $\mathtt{A}$ has the generator with the last coordinate zero, then the system has no solutions.

Another way to find the kernel of $\mathtt{A}$ is to compute the cross product of the 2 rows of $\mathtt{A}$. The result may be interpreted as the intersection of l_1 and l_2 in $\mathbb{P}^2$, since now 3×1 vectors of numbers with the last coordinate zero represent points at infinity of $\mathbb{P}^2$. (We see that it is easier to work in $\mathbb{P}^2$ rather than in $\mathbb{A}^2$ since we don't need to distinguish the 2 cases of parallel and not parallel lines.) The homogeneous representatives of l_1 and l_2 in $\mathbb{P}^2$ are:

$$\mathbf{l}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{l}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Their cross product is

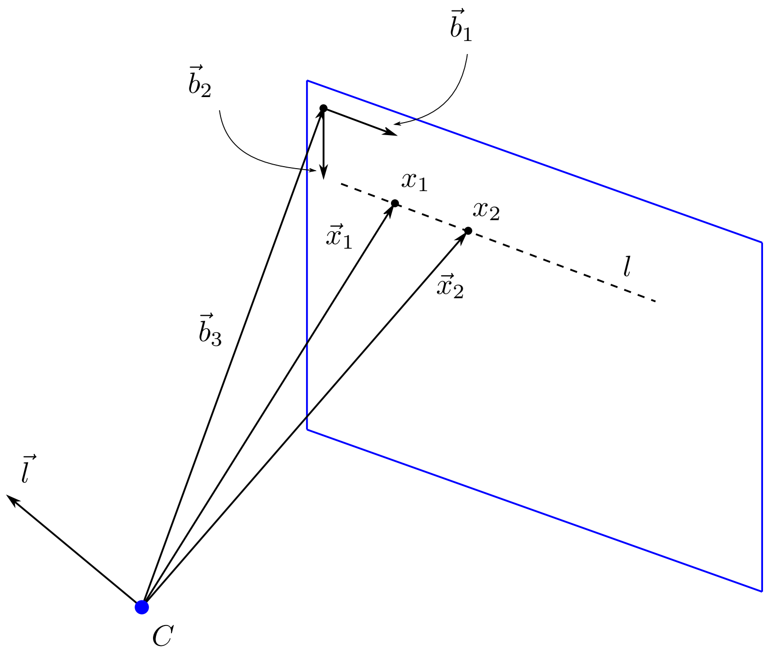
$$\mathbf{x} = \mathbf{l}_1 \times \mathbf{l}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

We see that the last coordinate is nonzero, which means that l_1 and l_2 intersect in $\mathbb{A}^2$. (If it was zero, then they would be parallel and wouldn't intersect in $\mathbb{A}^2$, but in $\mathbb{P}^2$.) To find the point of intersection in $\mathbb{A}^2$ we need to find the representative of $[\mathbf{x}]$ with the last coordinate 1 and take the first 2 coordinates, which are $\begin{bmatrix}1 & 1\end{bmatrix}^\top$. $\square$

Task 2. *Let us have two image points x_1 and x_2 defined by*

$$\vec{u}_{1\alpha} = \begin{bmatrix}1\\1\end{bmatrix}, \quad \vec{u}_{2\alpha} = \begin{bmatrix}2\\1\end{bmatrix}$$

Find the line in the image (in the form $au + bv + c = 0$) passing through them (using techniques of projective geometry).



Solution: Again, it is not necessary to use techniques of projective geometry (the cross product rule for the line passing through 2 points): we could simply setup the system of linear equations

$$\begin{cases} a \cdot 1 + b \cdot 1 + c = 0 \\ a \cdot 2 + b \cdot 1 + c = 0 \end{cases} \Leftrightarrow \underbrace{\begin{bmatrix}1 & 1 & 1\\2 & 1 & 1\end{bmatrix}}_{\mathtt{M}} \begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

and solve it by Gaussian elimination of $\mathtt{M}$.

The homogeneous representatives of x_1 and x_2 are

$$\mathbf{x}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix}2\\1\\1\end{bmatrix}$$

Their cross product is

$$\mathbf{l} = \mathbf{x}_1 \times \mathbf{x}_2 = \begin{bmatrix}1\\1\\1\end{bmatrix} \times \begin{bmatrix}2\\1\\1\end{bmatrix} = \begin{bmatrix}0\\1\\-1\end{bmatrix}$$

Passing to affine representation of the line given by $\mathbf{l}$ we get

$$l: \; 0 \cdot u + 1 \cdot v - 1 = 0 \Rightarrow l: \; v = 1.$$

The point at infinity of $\mathbb{P}^2$ represented by $\begin{bmatrix}1 & 0 & 0\end{bmatrix}^\top$ associated to l doesn't belong to l, but to its projective closure $\bar{l}$. $\square$

Task 3. *Consider points $\mathbf{x} = [1, 0, 1]^\top$, $\mathbf{y} = [1, 2, 0]^\top$ and $\mathbf{z} = [0, 1, 1]^\top$ in the real projective plane. Find the line l which is parallel (in the canonically associated affine plane) to the line passing through points $\mathbf{x}, \mathbf{y}$ and such that l passes through $\mathbf{z}$.*

Solution: The fact that l is parallel (in the canonically associated affine plane) to the line l' passing through points $\mathbf{x}, \mathbf{y}$ means that l and l' meet at a point at infinity. Since $y \in l'$ and the last coordinate of the representative $\mathbf{y}$ of y is zero, then $l \cap l' = y$, or $y \in l$. Since $z \in l$ by the task, then l is a line passing through y and z:

$$\mathbf{l} = \mathbf{y} \times \mathbf{z} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$$

□

Task 4. *Consider the homography with the following matrix*

$$\mathtt{H} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & a & 1 \end{bmatrix}$$

Find the parameter a, to get point $[1, 1]^\top$ mapped into a point at infinity.

Solution: The condition in the task may be rewritten algebraically as follows:

$$\lambda \begin{bmatrix} u \\ v \\ 0 \end{bmatrix} = \mathtt{H} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \lambda \neq 0,\ u \neq 0 \text{ or } v \neq 0$$

We can reparametrize the variables using substitution $u' = \lambda u$, $v' = \lambda v$. Then conditions $\lambda \neq 0,\ u \neq 0$ or $v \neq 0$ will be equivalently rewritten as $u' \neq 0$ or $v' \neq 0$. Thus, we have

$$\begin{bmatrix} u' \\ v' \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & a & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad u' \neq 0 \text{ or } v' \neq 0$$

$$\begin{cases} u' = 2 \\ v' = 1 \\ a + 1 = 0 \end{cases}, \quad u' \neq 0 \text{ or } v' \neq 0$$

Hence $a = -1$. We can see that

$$\mathtt{H} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

is a valid homography matrix (i.e. it is invertible). □

Task 5. *Find all constraints on parameters a, b such that the homography represented by*

$$\mathtt{H} = \begin{bmatrix} a & 0 & 1 \\ b & 0 & 1 \\ a & b & 1 \end{bmatrix}$$

maps line $\mathbf{l} = [0, 1, 1]^\top$ onto the line at infinity.

Solution: First of all, for $\mathtt{H}$ to be a valid homography matrix it must be invertible, i.e.

$$\det \mathtt{H} \neq 0 \Leftrightarrow b(b - a) \neq 0 \Leftrightarrow b \neq 0 \text{ and } a \neq b.$$

Suppose that $x, y \in l$, whose homogeneous coordinates in $\mathbb{P}^2$ are $\mathbf{x}$ and $\mathbf{y}$. Using the property of the cross product we can write

$$\underbrace{\mathtt{H}\mathbf{x} \times \mathtt{H}\mathbf{y}}_{\mathbf{l}'} = \frac{1}{\det \mathtt{H}^{-\top}} \mathtt{H}^{-\top} \underbrace{(\mathbf{x} \times \mathbf{y})}_{\mathbf{l}}$$

This means that having a line l in $\mathbb{P}^2$ with homogeneous coordinates $\mathbf{l}$ and a homography matrix $\mathtt{H}$, the homogeneous coordinates $\mathbf{l}'$ of the image l' of l by $\mathtt{H}$ may be obtained by $\mathtt{H}^{-\top}\mathbf{l}$ (since homogeneous coordinates are defined up to scale, we may forget about the scale $\frac{1}{\det \mathtt{H}^{-\top}}$).

The condition in the task may be rewritten algebraically as follows:

$$\lambda \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \mathtt{H}^{-\top} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \lambda \neq 0$$

We compute

$$\mathtt{H}^{-\top} = \frac{1}{b(b-a)} \begin{bmatrix} -b & a-b & b^2 \\ b & 0 & -ab \\ 0 & b-a & 0 \end{bmatrix}$$

Hence

$$\lambda \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{b(b-a)} \begin{bmatrix} -b & a-b & b^2 \\ b & 0 & -ab \\ 0 & b-a & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \lambda \neq 0$$

$$\begin{cases} 0 = a - b + b^2 \\ 0 = -ab \\ b(b-a)\lambda = b - a \end{cases}, \quad \lambda \neq 0$$

From the second equation $0 = -ab$ we conclude that $a = 0$, since $b \neq 0$. Substituting $a = 0$ into the first equation we get $0 = -b + b^2$ which means that $b = 1$ (since $b \neq 0$). We still need to verify if there is a nonzero solution to λ. For this we substitute $a = 0$ and $b = 1$ to the last equation and get $\lambda = 1$. Thus, $a = 0$ and $b = 1$ is indeed a solution which generates a valid homography matrix

$$\mathtt{H} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

□