# Reasons of Introducing the Language of Projective Geometry

GVG Lab 08

# Projection from $\mathbb{A}^3$ to $\mathbb{A}^2$

The algebraic model of perspective projection from (almost whole) $\mathbb{A}^3$ to $\mathbb{A}^2$ has the form

$$\eta \begin{bmatrix} \vec{u}_\alpha \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} \mathtt{A} \mid -\mathtt{A}\vec{C}_\delta \end{bmatrix}}_{\mathtt{P}_\beta} \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix}, \quad \eta \neq 0$$

It assumes that $X$ doesn't belong to the principal plane.

# What about points from the principal plane?

We can still evaluate the product $\mathtt{P}_\beta \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix}$ on the right-hand side for $X$ from the principal plane to see what happens:

$$\begin{bmatrix} u \\ v \\ 0 \end{bmatrix} = \mathtt{P}_\beta \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix}$$

We can see that the vector on the left doesn't have a representation as $\eta \begin{bmatrix} \vec{u}_\alpha \\ 1 \end{bmatrix}$ for $\eta \neq 0$.

# Extending affine plane $\mathbb{A}^2$ to projective plane $\mathbb{P}^2$

**Geometric construction**: Identify points in the image plane with rays passing through those points and the camera center (**finite points** of the projective plane, or points that are **visible in the image**), and add new rays passing through the camera center and lying in the principal plane (**ideal points** or **points at infinity** of the projective plane, or points that are **not visible in the image**).

**Algebraic construction**: For the equivalence relation $\sim$ on the set $\mathbb{R}^3 \backslash \{\mathbf{0}\}$ defined by

$$\mathbf{x}_1 \sim \mathbf{x}_2 \iff \exists \lambda \in \mathbb{R}\backslash\{0\} : \mathbf{x}_1 = \lambda \mathbf{x}_2$$

we define

$$\mathbb{P}^2 = \left(\mathbb{R}^3 \backslash \{\mathbf{0}\}\right) / \sim \; = \left\{ [\mathbf{x}] \mid \mathbf{x} \in \mathbb{R}^3 \backslash \{\mathbf{0}\} \right\},$$

where

$$[\mathbf{x}] = \{\lambda \mathbf{x} \mid \lambda \in \mathbb{R}\backslash\{0\}\}$$

is a **point** of the projective plane $\mathbb{P}^2$.

# Extending affine plane $\mathbb{A}^2$ to projective plane $\mathbb{P}^2$

**Advantages**:

1) Now any two lines in $\mathbb{P}^2$ intersect (even parallel ones) $\rightarrow$ studying points and lines becomes simpler;

2) Working with just 1 point (the camera center) which doesn't project to the camera is easier than with the plane of points which don't project (the principal plane). As a consequence, the back-projected plane of an image line is a plane in $\mathbb{A}^3$ with just 1 point removed (the camera center);

3) Despite the fact that points at infinity of $\mathbb{P}^2$ are not visible in the image, we can still get useful information from them algebraically, e.g. using vanishing points at infinity for camera calibration.

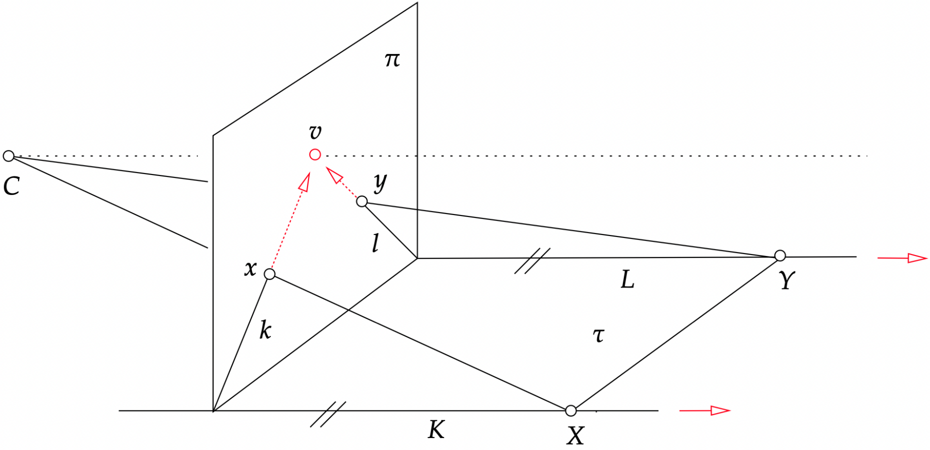
# Projection from $\mathbb{A}^3$ to $\mathbb{P}^2$

By extending $\mathbb{A}^2$ to $\mathbb{P}^2$ we extend the domain of definition of the projection map:

$$\mathbf{x} = \mathtt{P}_\beta \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix}, \quad [\mathbf{x}] \in \mathbb{P}^2$$

Now $X$ can be any point from $\mathbb{A}^3$ except for the camera projection center $C$.

# Vanishing point



# Extending affine space $\mathbb{A}^3$ to projective space $\mathbb{P}^3$

**Reason**: It happens that vanishing points can be used for camera calibration. We can introduce $3$ different definitions for the vanishing point:

(a) the intersection of the projections of 2 parallel lines;

(b) the limit of the projection of a line as the variable which parametrizes it goes to infinity;

(c) **the projection of the point at infinity of a line**.

(Of course, the last definition only makes sense after introducing the projective space). From some point of view the last definition is the most convenient to work with.

**Idea**: identify points of $\mathbb{A}^3$ with 1D subspaces of $\mathbb{A}^4$ generated by $\begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix}$ (**finite points** of the projective space) and add 1D subspaces generated by $\begin{bmatrix} \mathbf{d} \\ 0 \end{bmatrix}$ (**points at infinity** of the projective space).

# Projection from $\mathbb{P}^3$ to $\mathbb{P}^2$

By extending $\mathbb{A}^3$ to $\mathbb{P}^3$ we extend the domain of the projection map:

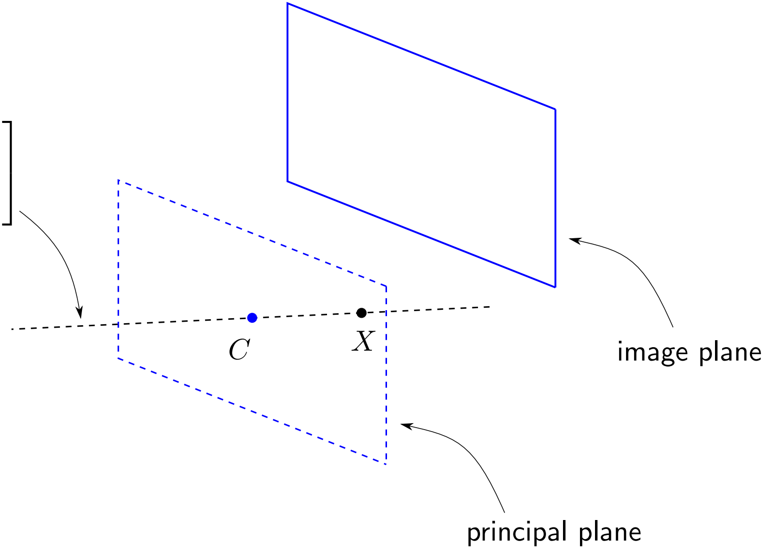
$$\mathbf{x} = \mathtt{P}_\beta \mathbf{X}, \quad [\mathbf{x}] \in \mathbb{P}^2, [\mathbf{X}] \in \mathbb{P}^3$$

The world point $X$ can now be any point from $\mathbb{P}^3$ (i.e. including points at infinity) except for the camera projection center $C$.

# Finite points of $\mathbb{P}^3 \to$ finite points of $\mathbb{P}^2$

# Finite points of $\mathbb{P}^3 \to$ points at infinity of $\mathbb{P}^2$

$$[\mathbf{x}] = \left[ \begin{bmatrix} u \\ v \\ 0 \end{bmatrix} \right]$$



# Points at infinity of $\mathbb{P}^3 \to$ finite points of $\mathbb{P}^2$

# Points at infinity of $\mathbb{P}^3 \to$ points at infinity of $\mathbb{P}^2$