

GVG Lab-03 Solution

Task 1. Assume a camera with the following image projection matrix

$$\mathbf{P}_\beta = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

Find the coordinates of the projection center.

Solution: By definition,

$$\mathbf{P}_\beta = [\mathbf{A} \quad -\mathbf{A}\vec{C}_\delta]$$

where $\mathbf{A} = \mathbf{T}_{\delta \rightarrow \beta}$. Thus, the camera projection center expressed in the world coordinate system \vec{C}_δ can be retrieved from the kernel of \mathbf{P}_β , since

$$\mathbf{P}_\beta \begin{bmatrix} \vec{C}_\delta \\ 1 \end{bmatrix} = [\mathbf{A} \quad -\mathbf{A}\vec{C}_\delta] \begin{bmatrix} \vec{C}_\delta \\ 1 \end{bmatrix} = \mathbf{A}\vec{C}_\delta - \mathbf{A}\vec{C}_\delta = \mathbf{0}$$

Computing the kernel of \mathbf{P}_β means solving the system of linear equations

$$\mathbf{P}_\beta \mathbf{x} = \mathbf{0}, \quad \mathbf{x} = [x_1 \quad x_2 \quad x_3 \quad x_4]^\top \quad (1)$$

We solve it by applying Gaussian elimination method:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

From the last row ($-x_3 + x_4 = 0$) we let x_4 to be any real number and conclude that $x_3 = x_4$. Substituting it to the second row we get $x_2 = -2x_4$. Further, substituting x_2 and x_3 to the first row we obtain $x_1 = -x_2 = x_4$. Thus, the solutions to (1) are

$$S = \left\{ \begin{bmatrix} x_4 \\ -2x_4 \\ x_4 \\ x_4 \end{bmatrix} \mid x_4 \in \mathbb{R} \right\}$$

We know that the kernel of every image projection matrix is one-dimensional, since rank of this matrix is always equal to 3. We are interested in the representative of the kernel with last coordinate equal to 1. For this we take $x_4 = 1$ and get

$$\begin{bmatrix} \vec{C}_\delta \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 1 \end{bmatrix} \in S \Rightarrow \vec{C}_\delta = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 1 \end{bmatrix}$$

Another way to compute \vec{C}_δ would be to invert \mathbf{A} . For this we stack \mathbf{A} and \mathbf{I} together into 3×6 matrix and apply Gaussian elimination so that the left 3×3 part becomes \mathbf{I} . The inverse \mathbf{A}^{-1} then appears as the right 3×3 block:

$$\begin{aligned} [\mathbf{A} \mid \mathbf{I}] &= \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array} \right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array} \right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array} \right] \sim \\ &\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array} \right] \end{aligned}$$

Thus,

$$\mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \Rightarrow \vec{C}_\delta = -\mathbf{A}^{-1} \mathbf{P}_{\beta_{1:3,4}} = - \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

□

Task 2. Assume the following camera projection matrix

$$P = \begin{bmatrix} 2 & 0 & 4 & 0 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Find the principal point of the image plane.

Solution: By definition,

$$P = [KR \mid -KR\vec{C}_\delta]$$

The principal point is defined to be the elements k_{13} and k_{23} of the camera calibration matrix K . We use [1, Equations (7.45) and (7.46)] to compute it. If we denote the matrix KR by B and its rows by column vectors \mathbf{b}_i , then we can rewrite those equations as

$$k_{13} = \mathbf{b}_1^\top \mathbf{b}_3 = [2 \quad 0 \quad 4] \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 4,$$

$$k_{23} = \mathbf{b}_2^\top \mathbf{b}_3 = [0 \quad 2 \quad 2] \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 2.$$

The principal point of the image plane equals

$$\vec{u}_{0\alpha} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

□

Task 3. Let us assume the following image projection matrix

$$P_\beta = \begin{bmatrix} 6 & -8 & 50 & 800 \\ 16 & 12 & 40 & -1200 \\ 0 & 0 & 0.1 & 0 \end{bmatrix}$$

Find K , R , \vec{C}_δ , f .

Solution: We first compute the focal length f of the given camera according to [1, Equation (7.25)]:

$$\|P_{\beta_{3,1:3}}\| = \frac{1}{f} \Rightarrow f = \frac{1}{\|P_{\beta_{3,1:3}}\|} = \frac{1}{0.1} = 10$$

According to [1, Equation (7.24)]:

$$P_{\beta_{1:3,1:3}} = \frac{1}{f}KR \Rightarrow KR = f \cdot P_{\beta_{1:3,1:3}} = \begin{bmatrix} 60 & -80 & 500 \\ 160 & 120 & 400 \\ 0 & 0 & 1 \end{bmatrix}$$

Using [1, Equations (7.45)-(7.49)] we decompose $B = KR$ into K and R :

$$k_{23} = \mathbf{b}_2^\top \mathbf{b}_3 = [160 \quad 120 \quad 400] \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 400,$$

$$k_{13} = \mathbf{b}_1^\top \mathbf{b}_3 = [60 \quad -80 \quad 500] \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 500,$$

$$k_{22}^2 + 400^2 = \mathbf{b}_2^\top \mathbf{b}_2 = [160 \quad 120 \quad 400] \begin{bmatrix} 160 \\ 120 \\ 400 \end{bmatrix} = 200000 \Rightarrow k_{22} = \sqrt{200000 - 160000} = \sqrt{40000} = 200,$$

$$k_{12} \cdot 200 + 500 \cdot 400 = \mathbf{b}_1^\top \mathbf{b}_2 = [60 \quad -80 \quad 500] \begin{bmatrix} 160 \\ 120 \\ 400 \end{bmatrix} = 200000 \Rightarrow k_{12} = \frac{200000 - 200000}{200} = 0,$$

$$k_{11}^2 + 0^2 + 500^2 = \mathbf{b}_1^\top \mathbf{b}_1 = \begin{bmatrix} 60 & -80 & 500 \end{bmatrix} \begin{bmatrix} 60 \\ -80 \\ 500 \end{bmatrix} = 260000 \Rightarrow k_{11} = \sqrt{260000 - 250000} = \sqrt{10000} = 100.$$

Thus,

$$\mathbf{K} = \begin{bmatrix} 100 & 0 & 500 \\ 0 & 200 & 400 \\ 0 & 0 & 1 \end{bmatrix}$$

The rotation matrix \mathbf{R} of the camera can be computed as

$$\mathbf{R} = \mathbf{K}^{-1} \mathbf{B} = \begin{bmatrix} 0.01 & 0 & -5 \\ 0 & 0.005 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 60 & -80 & 500 \\ 160 & 120 & 400 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.6 & -0.8 & 0 \\ 0.8 & 0.6 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The camera projection center \vec{C}_δ can be obtained as

$$\vec{C}_\delta = -\mathbf{P}_{\beta_{1:3,1:3}}^{-1} \mathbf{P}_{\beta_{1:3,4}} = - \begin{bmatrix} 6 & -8 & 50 \\ 16 & 12 & 40 \\ 0 & 0 & 0.1 \end{bmatrix}^{-1} \begin{bmatrix} 800 \\ -1200 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 100 \\ 0 \end{bmatrix}$$

□

Task 4. Write the transition matrix from basis δ to basis γ , coordinates of the camera projection center in the camera cartesian coordinate system $C_{(C,\gamma)}$ and \vec{C}_γ of the camera from the previous example.

Solution: According to [1, Equation (7.6)], the transition matrix $\mathbf{T}_{\delta \rightarrow \gamma}$ equals

$$\mathbf{T}_{\delta \rightarrow \gamma} = \frac{1}{f} \mathbf{R} = \frac{1}{10} \begin{bmatrix} 0.6 & -0.8 & 0 \\ 0.8 & 0.6 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.06 & -0.08 & 0 \\ 0.08 & 0.06 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}$$

Also,

$$C_{(C,\gamma)} = \overrightarrow{C} \vec{C}_\gamma = \vec{0}_\gamma = \mathbf{0},$$

$$\vec{C}_\gamma = \mathbf{T}_{\delta \rightarrow \gamma} \vec{C}_\delta = \begin{bmatrix} 0.06 & -0.08 & 0 \\ 0.08 & 0.06 & 0 \\ 0 & 0 & 0.1 \end{bmatrix} \begin{bmatrix} 0 \\ 100 \\ 0 \end{bmatrix} = \begin{bmatrix} -8 \\ 6 \\ 0 \end{bmatrix}$$

□

References

- [1] Tomas Pajdla, *Elements of geometry for computer vision*, https://cw.fel.cvut.cz/wiki/_media/courses/gvg/pajdla-gvg-lecture-2021.pdf.