

GVG Lab-02 Solution

Task 1. Find the image point $[u, v]^\top$ which is the projection of 3D point $\vec{X}_\delta = [1, 2, 3]^\top$ by the camera with the following image projection matrix:

$$\mathbf{P}_\beta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Solution: By the definition of the mathematical model of the projective camera we have

$$\eta \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \mathbf{P}_\beta \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix}, \quad \eta \neq 0$$

$$\eta \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$$

or, equivalently,

$$\eta u = 1, \quad \eta v = 2, \quad \eta = 3$$

from which it follows that

$$u = \frac{1}{\eta} = \frac{1}{3}, \quad v = \frac{2}{\eta} = \frac{1}{2}.$$

□

Task 2. Find the coordinates of the camera projection center \vec{C}_δ of a camera with the following scaled image projection matrix:

$$\mathbf{Q} = \xi \mathbf{P}_\beta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Solution: By definition,

$$\mathbf{P}_\beta = [\mathbf{A} \mid -\mathbf{A}\vec{C}_\delta]$$

where $\mathbf{A} = \mathbf{T}_{\delta \rightarrow \beta}$. Hence

$$\mathbf{Q} = \xi \mathbf{P}_\beta = \xi [\mathbf{A} \mid -\mathbf{A}\vec{C}_\delta] = [\xi \mathbf{A} \mid -\xi \mathbf{A}\vec{C}_\delta], \quad \xi \neq 0$$

The camera projection center can be computed in two ways.

1. The camera projection center \vec{C}_δ can be retrieved from the kernel of \mathbf{Q} , since

$$\mathbf{Q} \begin{bmatrix} \vec{C}_\delta \\ 1 \end{bmatrix} = [\xi \mathbf{A} \mid -\xi \mathbf{A}\vec{C}_\delta] \begin{bmatrix} \vec{C}_\delta \\ 1 \end{bmatrix} = \xi \mathbf{A}\vec{C}_\delta - \xi \mathbf{A}\vec{C}_\delta = \mathbf{0}$$

Computing the kernel of \mathbf{Q} means solving the system of linear equations

$$\mathbf{Q}\mathbf{x} = \mathbf{0}, \quad \mathbf{x} = [x_1 \ x_2 \ x_3 \ x_4]^\top \tag{1}$$

We solve it by applying Gaussian elimination method:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

From the last row $x_3 + 2x_4 = 0$ we conclude that $x_3 = -2x_4$ and we let x_4 to be any real number. From the second row we get $x_2 = 0$. From the first row we obtain $x_1 = 0$. Thus, the solutions to (1) are

$$S = \left\{ \left[\begin{array}{c} 0 \\ 0 \\ -2x_4 \\ x_4 \end{array} \right] \mid x_4 \in \mathbb{R} \right\}$$

We are interested in the representative of the kernel with last coordinate equal to 1. For this we take $x_4 = 1$ and get

$$\begin{bmatrix} \vec{C}_\delta \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix} \in S \Rightarrow \vec{C}_\delta = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$$

Remark. We know that the kernel of every scaled image projection matrix is one-dimensional, since rank of this matrix is always equal to 3. We also note that we will be always able to find the representative of the kernel of \mathbf{Q} with the last coordinate equal to 1 since otherwise there would exist a nontrivial kernel to the invertible matrix $\xi\mathbf{A}$.

2. Another way to compute the camera projection center \vec{C}_δ is the following:

$$\begin{aligned} \vec{C}_\delta &= (-\xi\mathbf{A})^{-1}(-\xi\mathbf{A})\vec{C}_\delta = -(\xi\mathbf{A})^{-1}(-\xi\mathbf{A}\vec{C}_\delta) = -\mathbf{Q}_{1:3,1:3}^{-1}\mathbf{Q}_{1:3,4} = \\ &= -\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = -\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} \end{aligned}$$

□

Task 3. Write down the coordinates of all three-dimensional points which project into image point $[2, 1]^\top$ by a camera with the following scaled image projection matrix

$$\mathbf{Q} = \xi\mathbf{P}_\beta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Solution: The equation which models the projection of the world point \vec{X}_δ which doesn't belong to the principal plane to the image point $[u, v]^\top$ has the form [1, Equation 6.12]:

$$\eta\vec{x}_\beta = \mathbf{P}_\beta \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix}, \quad \vec{x}_\beta = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix}, \quad \eta \in \mathbb{R}, \eta \neq 0 \quad (2)$$

In other words, the world point \vec{X}_δ projects to the image point $[u, v]^\top$ if and only if there exists $\eta \neq 0$ such that Equation (2) is satisfied. Obviously, $\xi \neq 0$, since otherwise \mathbf{Q} would be a zero matrix. Hence we can multiply (2) by ξ from both sides:

$$\begin{aligned} \xi\eta\vec{x}_\beta &= \xi\mathbf{P}_\beta \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix}, \quad \eta, \xi \neq 0 \\ \xi\eta\vec{x}_\beta &= \mathbf{Q} \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix} = \mathbf{Q}_{1:3,1:3}\vec{X}_\delta + \mathbf{Q}_{1:3,4} \end{aligned}$$

The whole ray p of world points \vec{X}_δ which project to the image point $[u, v]^\top$ can be written as

$$\begin{aligned} p &= \left\{ \mathbf{Q}_{1:3,1:3}^{-1}(\xi\eta\vec{x}_\beta - \mathbf{Q}_{1:3,4}) \mid \eta \in \mathbb{R}, \eta \neq 0 \right\} = \\ &= \left\{ \xi\eta\mathbf{Q}_{1:3,1:3}^{-1}\vec{x}_\beta - \mathbf{Q}_{1:3,1:3}^{-1}\mathbf{Q}_{1:3,4} \mid \eta \in \mathbb{R}, \eta \neq 0 \right\} \\ &\stackrel{(1)}{=} \left\{ \eta\mathbf{Q}_{1:3,1:3}^{-1}\vec{x}_\beta - \mathbf{Q}_{1:3,1:3}^{-1}\mathbf{Q}_{1:3,4} \mid \eta \in \mathbb{R}, \eta \neq 0 \right\} \end{aligned}$$

where (1) is due to the fact that $\{\xi\eta \mid \eta \in \mathbb{R}, \eta \neq 0\}$ and $\{\eta \mid \eta \in \mathbb{R}, \eta \neq 0\}$ are equal sets for $\xi \neq 0$. We compute

$$\mathbf{Q}_{1:3,1:3}^{-1}\vec{x}_\beta = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \xi^{-1}\mathbf{A}^{-1}\vec{x}_\beta = \xi^{-1}\vec{x}_\delta$$

$$-\mathbf{Q}_{1:3,1:3}^{-1}\mathbf{Q}_{1:3,4} = \vec{C}_\delta = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$$

Hence the ray p has the form

$$p = \left\{ \eta \vec{x}_\delta + \vec{C}_\delta \mid \eta \in \mathbb{R}, \eta \neq 0 \right\} = \left\{ \eta \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} \mid \eta \in \mathbb{R}, \eta \neq 0 \right\}$$

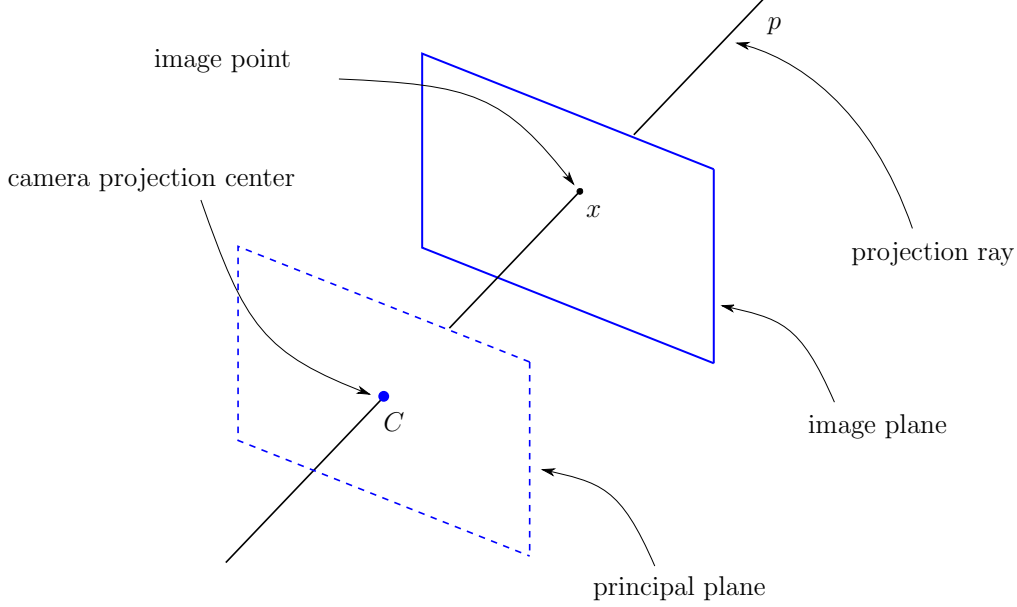


Figure 1: The back-projected line of the image point

We can see that p is indeed a line in space with the camera projection center removed (**since this is the only point in the principal plane which belongs to the line connecting the camera projection center and the image point**). \square

Task 4. Denote the image coordinates by $[u, v]^\top$. Write down coordinates of all points in the three-dimensional space that projects on the line $v = 0$ by a camera with the following scaled image projection matrix

$$\mathbf{Q} = \xi \mathbf{P}_\beta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Solution: The equation which models the projection of the world point \vec{X}_δ which doesn't belong to the principal plane to the image point $[u, 0]^\top$ has the form [1, Equation 6.12]:

$$\eta \vec{x}_\beta = \mathbf{P}_\beta \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix}, \quad \vec{x}_\beta = \begin{bmatrix} u \\ 0 \\ 1 \end{bmatrix}, \quad u \in \mathbb{R}, \eta \in \mathbb{R}, \eta \neq 0 \quad (3)$$

We apply the same steps as in the previous task. The set σ of points \vec{X}_δ which satisfy Equation (3) can be expressed as

$$\begin{aligned} \sigma &= \left\{ \eta \mathbf{Q}_{1:3,1:3}^{-1} \begin{bmatrix} u \\ 0 \\ 1 \end{bmatrix} - \mathbf{Q}_{1:3,1:3}^{-1} \mathbf{Q}_{1:3,4} \mid u \in \mathbb{R}, \eta \in \mathbb{R}, \eta \neq 0 \right\} = \\ &= \left\{ \eta \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} \mid u \in \mathbb{R}, \eta \in \mathbb{R}, \eta \neq 0 \right\} = \end{aligned}$$

$$\begin{aligned}
&= \left\{ \eta \begin{bmatrix} u \\ -u \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} \mid u \in \mathbb{R}, \eta \in \mathbb{R}, \eta \neq 0 \right\} = \\
&= \left\{ \eta u \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \eta \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} \mid u \in \mathbb{R}, \eta \in \mathbb{R}, \eta \neq 0 \right\} = \\
&= \left\{ a \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} \mid a, b \in \mathbb{R}, b \neq 0 \right\}
\end{aligned}$$

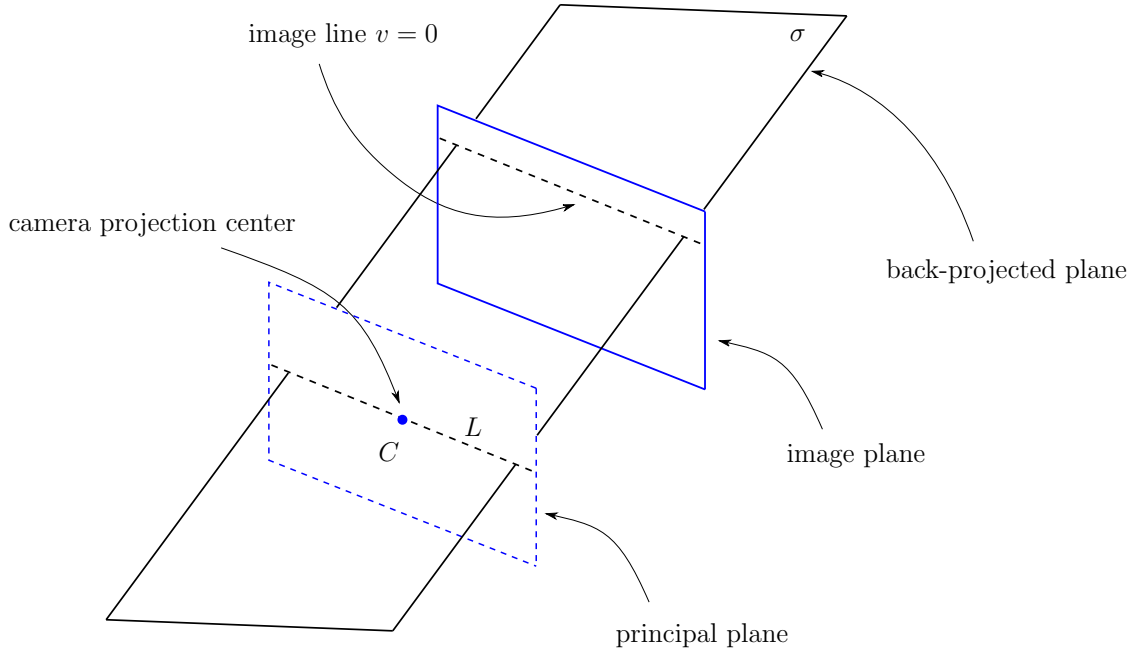


Figure 2: The back-projected plane of the image line

We can see that σ is a plane in space with the line passing through the camera projection center removed (since the intersection of the plane passing through the camera projection center and the image line with the principal plane is a line). To get the parametric equation of this line we need to substitute $b = 0$ into the parametric equation of σ :

$$L = \left\{ a \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} \mid a \in \mathbb{R} \right\}.$$

We can also see that the camera projection center belongs to L . □

References

- [1] Tomas Pajdla, *Elements of geometry for computer vision*, https://cw.fel.cvut.cz/wiki/_media/courses/gvg/pajdla-gvg-lecture-2021.pdf.