Quantum Computing - Exercise Sheet 7

Quantum Random Walks

1. *At each time step, a quantum walk corresponds to a unitary map $U \in U(N)$ such that*

$$U : \mathcal{H}_G \to \mathcal{H}_G$$
$$|x\rangle \mapsto a|x-1\rangle + b|x\rangle + c|x+1\rangle$$

Show that U is unitary if and only if one of the following three conditions is true: (a) $|a| = 1, b = c = 0$, (b) $|b| = 1, a = c = 0$, (c) $|c| = 1, a = b = 0$.

Using the unitarity of the operator we know that:

$$\langle x| \underbrace{U^\dagger U}_{1} |y\rangle = \delta_{xy}$$

So, for instance, for the following states, we have:

$$\left\langle x-1 \left| U^\dagger U \right| x+1 \right\rangle = (a\langle x-2| + b\langle x-1| + c\langle x|)(a|x\rangle + b|x+1\rangle + c|x+2\rangle) = 0$$

The only term surviving being $c\langle x|a|x\rangle = ac = 0$

$$\left\langle x \left| U^\dagger U \right| x+1 \right\rangle = (a\langle x-1| + b\langle x| + c\langle x+1|)(a|x\rangle + b|x+1\rangle + c|x+2\rangle) = 0$$

The non-vanishing terms now are

$$\begin{cases} b\langle x|a|x\rangle \Rightarrow ab \\ c\langle x+1|b|x+1\rangle \Rightarrow bc \end{cases} \Rightarrow ab + bc = 0$$

$$\left\langle x \left| U^\dagger U \right| x \right\rangle = (a\langle x-1| + b\langle x| + c\langle x+1|)(a|x-1\rangle + b|x\rangle + c|x+1\rangle) = 1$$

Lastly, the system to be solved is:

$$\begin{cases} ac = 0 \\ ab + bc = 0 \\ a^2 + b^2 + c^2 = 1 \end{cases}$$

2. *Demonstrate that the shift operator S, as defined as*

$$S = (|0\rangle\langle 0| \otimes \sum_{x=-\infty}^{+\infty} |x+1\rangle\langle x|) + (|1\rangle\langle 1| \otimes \sum_{x=-\infty}^{+\infty} |x-1\rangle\langle x|)$$

equivalent to

$$S|i,x\rangle = \begin{cases} |0,x+1\rangle & \textit{if } i = 0, \\ |1,x-1\rangle & \textit{if } i = 1. \end{cases}$$

Applying directly the first definition of the operator to the state $|i,x\rangle$, we get the second one:

$$S|i,x\rangle = (|0\rangle \overbrace{\langle 0||i\rangle}^{\delta_0} \otimes \underbrace{\sum_{k=-\infty}^{\infty} |k+1\rangle \overbrace{\langle k||x\rangle}^{\delta_k}}_{|x+1\rangle}) + (|1\rangle \overbrace{\langle 1||i\rangle}^{\delta_1} \otimes \underbrace{\sum_{k=-\infty}^{\infty} |k-1\rangle \overbrace{\langle k||x\rangle}^{\delta_{kx}}}_{|x-1\rangle})$$

$$= \begin{cases} |0\rangle \otimes |x+1\rangle \text{ if } i = 0, \\ |1\rangle \otimes |x-1\rangle \text{ if } i = 1. \end{cases} = \begin{cases} |0,x+1\rangle & \text{if } i = 0, \\ |1,x-1\rangle & \text{if } i = 1. \end{cases}$$

3. *Consider a one-dimensional quantum walk on $\mathbb{Z}$ where the coin operator C is parameterized by an angle θ as :*

$$c(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

The walker starts at $x = 0$ with initial state $|\psi_0\rangle = |i\rangle \otimes |x = 0\rangle$ and $|i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. Use the shift operator as defined before

a) Calculate the state of the system $|\psi_1\rangle$ after 1 step.

b) Compute the probabilities $p(x = 1)$ and $p(x = -1)$ that the walker is at $x = 1$ or $x = -1$.

c) Determine for which values of θ (if they exist) the quantum walks are unbiased ($p(x = 1) = p(x = -1)$)

d) Prove whether the Hadamard walker is biased a not.

a) Apply C on $|i\rangle$ to get:

$$\left(\frac{1}{\sqrt{2}}(\cos\theta+\sin\theta)|0\rangle+\frac{1}{\sqrt{2}}(\sin\theta-\cos\theta)|1\rangle\right)\otimes|x=0\rangle$$

Then apply S to get

$$\frac{1}{\sqrt{2}}(\cos\theta+\sin\theta)|0,1\rangle+\frac{1}{\sqrt{2}}(\sin\theta-\cos\theta)|1,-1\rangle$$

b) The probabilities are the modulus square of each of the amplitudes which give us

$$P(x=1)=|\frac{\cos\theta+\sin\theta}{\sqrt{2}}|^2,$$

$$P(x=-1)=|\frac{\sin\theta-\cos\theta}{\sqrt{2}}|^2$$

c) $P(x=1)=P(x=-1)$ for an unbiased walker. So

$$P(x=1)=\frac{1}{2}|\cos\theta+\sin\theta|^2=\frac{1}{2}(\cos^2\theta+\sin^2\theta+2\cos\theta\sin\theta)=\frac{1}{2}(1+2\cos\theta\sin\theta).$$

Similarly

$$P(x=-1)=\frac{1}{2}|\cos\theta-\sin\theta|^2=\frac{1}{2}(\cos^2\theta+\sin^2\theta-2\cos\theta\sin\theta)=\frac{1}{2}(1-2\cos\theta\sin\theta).$$

This leads to

$$\cos\theta\sin\theta=0.$$

Which is true when either $\sin\theta=0$ or $\cos\theta=0$. This occurs at $|\theta|=n\pi$ or $|\theta|=\frac{\pi}{2}+n\pi$ respectively, for integer values of n.
d) The Hadamard walker has for

$$H=\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}.$$

Comparing to the parameterized coin operator can see that this is the same but with $\theta=\frac{\pi}{4}$, so can substitute this value into the probability formulas without having to go through the full calculations as before.

$$P(x=1)=\frac{1}{2}|\cos\frac{\pi}{4}+\sin\frac{\pi}{4}|^2=\frac{1}{2}|\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}|^2=1.$$

Clearly this means $P(x=-1)=0$ and so the Hadamard walker is biased.

4. *Starting at the state $|\psi_0\rangle=|0\rangle|0\rangle$, obtain the successive states up $|\psi_4\rangle$ for the Hadamard walker on the finite subset of $\mathbb{Z}$.*

The states are

$$|\psi_1\rangle=\frac{1}{\sqrt{2}}(|0\rangle_i|1\rangle_x+|1\rangle_i|{-1}\rangle_x)$$

$$|\psi_2\rangle=\frac{1}{2}(|0\rangle_i|2\rangle_x+|1\rangle_i|0\rangle_x+|0\rangle_i|0\rangle_x-|1\rangle_i|{-2}\rangle_x)$$

$$|\psi_3\rangle=\frac{1}{2\sqrt{2}}(|0\rangle_i|3\rangle_x+|1\rangle_i|1\rangle_x+2|0\rangle_i|1\rangle_x-|0\rangle_i|{-1}\rangle_x+|1\rangle_i|3\rangle_x)$$

$$|\psi_4\rangle=\frac{1}{4}(|0\rangle_i|4\rangle_x+|1\rangle_i|2\rangle_x+3|0\rangle_i|2\rangle_x+|1\rangle_i|0\rangle_x-|0\rangle_i|0\rangle_x-|1\rangle_i|{-2}\rangle_x+|0\rangle_i|{-2}\rangle_x-|1\rangle_i|{-4}\rangle_x)$$

5. *Consider*

$$H_G^{(2)}=\sum_{\omega=1}^{2}\sum_{(i,j)\in E(G)}(|i\rangle\langle j|_\omega+|j\rangle\langle i|_w)$$

where $E(G)=\{(1,2)$ and $(2,1)\}$. This is the Hamiltonian for 2 particles on this G.
a) Assume we have distinguishable walkers. Compute the evolution of the initial state $|\psi_0\rangle=|1,2\rangle$ under the Hamiltonian $H_G^{(2)}$.
b) Assuming the walkers are distinguishable, now compute the evolution for the fermionic state $|\psi_0\rangle=\frac{1}{\sqrt{2}}(|1,2\rangle-|2,1\rangle)$

a)

$$
\begin{aligned}
H_G^{(2)}|\psi_0\rangle &= (|1\rangle_1\langle 2|_1 + |2\rangle_1\langle 1|_1 + |1\rangle_2\langle 2|_2 + |2\rangle_2\langle 1|_2)|1,2\rangle && (1)\\
&= (|1\rangle_1\langle 2|_1 + |2\rangle_1\langle 1|_1)|1\rangle \otimes |2\rangle + (|1\rangle_1\langle 2|_2 + |2\rangle_2\langle 1|_2)|1\rangle \otimes |2\rangle && (2)\\
&= (|1\rangle_1\langle 2|_1|1\rangle \otimes |2\rangle + |2\rangle_1\langle 1|_1|1\rangle \otimes |2\rangle) + (|1\rangle \otimes |1\rangle_2\langle 2|_2|2\rangle + |1\rangle \otimes |2\rangle_2\langle 1|_2|2\rangle) && (3)\\
&= (0 + |2\rangle_1 \otimes |2\rangle) + (|1\rangle \otimes |1\rangle_2 + 0) && (4)\\
&= |2,2\rangle + |1,1\rangle && (5)
\end{aligned}
$$

b) The computation for the fermionic state $|\psi_0 = \frac{1}{\sqrt{2}}(|1,2\rangle - |2,1\rangle)$ can be split into two parts for each of the terms. Both terms give the same result as shown above which leads to

$$H_G^{(2)}|\psi_0\rangle = \frac{1}{\sqrt{2}}(|2,2\rangle + |1,1\rangle - |2,2\rangle - |1,1\rangle) = 0.$$