Quantum Computing

Exercises: Quantum walks

1. At each time step, a quantum walk corresponds to a unitary map $U \in U(N)$ such that
$$\begin{aligned} U : \mathcal{H}_G &\to \mathcal{H}_G \\ |x\rangle &\mapsto a|x-1\rangle + b\,|x\rangle + c\,|x+1\rangle \end{aligned}$$
Show that U is unitary if and only if one of the following three conditions is true:
   (a) $|a| = 1, b = c = 0$,
   (b) $|b| = 1, a = c = 0$,
   (c) $|c| = 1, a = b = 0$.

2. Demonstrate that the shift operator S, as defined in
$$S = \left(|0\rangle\langle 0| \otimes \sum_{x=-\infty}^{\infty} |x+1\rangle\langle x| \right) + \left(|1\rangle\langle 1| \otimes \sum_{x=-\infty}^{\infty} |x-1\rangle\langle x| \right)$$
is equivalent to
$$S\,|i,x\rangle = \begin{cases} |0,x+1\rangle & \text{if } i=0, \\ |1,x-1\rangle & \text{if } i=1. \end{cases}$$

3. In the lecture notes, starting at the state $|\psi_0\rangle = |0\rangle\,|0\rangle$, we have seen how to obtain the succesive states up to $|\psi_3\rangle$ by using the unitary operator $U = S(H \otimes I)$. Derive $|\psi_4\rangle$ for the walker on the finite subset of $\mathbb{Z}$.

4. Show that the formula from the lecture notes, $H\,|k\rangle = 2\cos(k)\,|k\rangle$ holds, by performing the Fourier transform in the computational basis states.