Quantum Computing 2025 - Exercise Sheet 6

Quantum Phase Estimation

Quantum Phase Estimation is an algorithm developed to estimate the phase θ of given Unitary U with eigenvalues $e^{2i\pi\theta}$.

1. *Implement the Quantum Phase Estimation algorithm for the T-gate*

$$T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$$

considering the target state $|1\rangle$, to show that it correctly estimates the phase as $\theta = \frac{1}{8}$ only using n=3 ancilla qubits. Additionally show that we get the same result with $n = 4$ qubits, and that the precision is worse with $n = 2$ qubits.

To find the phase we want our target state to be $|1\rangle$, and we let $n = 3$. This way the phase will appear in the phase-kickback when we apply our sequence of control unitaries. The initial state is $|0\rangle^{\otimes 3}|1\rangle$.
After applying the Hadmard to the first register we get the state

$$\frac{1}{\sqrt{8}}(|0\rangle + |1\rangle)^{\otimes 3}|1\rangle = \frac{1}{\sqrt{8}}\sum_{j=0}^{7} |j\rangle\,|1\rangle\,.$$

Now we apply the controlled-T gates to induce the phase kickback, so we get

$$\frac{1}{\sqrt{8}}\sum_{k=0}^{7} e^{i\frac{\pi}{4}k}\,|k\rangle\,|1\rangle\,.$$

What follows is to apply the inverse QFT, which results in

$$\frac{1}{\sqrt{8}}\sum_{k=0}^{7} e^{i\frac{\pi}{4}k}\sum_{k=0}^{7} e^{-2\pi ikx/8}\,|x\rangle\,|1\rangle = \frac{1}{\sqrt{8}}\sum_{x=0}^{7}\sum_{k=0}^{7} e^{-2\pi ik(x-1)/8}\,|x\rangle\,|1\rangle\,.$$

Therefore we see that $\delta = 0$, and so $\Pr(a = 1) = 1$, which tells us that the phase is $\theta = \frac{a}{2^n} = \frac{1}{8}$.
If $n = 4$, we only have a significant change in the QFT step, which gives

$$\frac{1}{4}\sum_{k=0}^{15} e^{i\frac{\pi}{4}k}\sum_{x=0}^{15} e^{-2\pi ikx/16}\,|x\rangle\,|1\rangle = \frac{1}{4}\sum_{x=0}^{15}\sum_{k=0}^{15} e^{-2\pi ik(x-2)/16}\,|x\rangle\,|1\rangle\,.$$

In this case we have $\delta = 0$, again, and we have $\Pr(a = 2) = 1$, which tells us that the phase is $\theta = \frac{a}{2^n} = \frac{2}{16} = \frac{1}{8}$.
If $n = 2$ the QFT step changes, and in this case we have

$$\frac{1}{2}\sum_{k=0}^{3} e^{i\frac{\pi}{4}k}\sum_{x=0}^{3} e^{-2\pi ikx/4}\,|x\rangle\,|1\rangle = \frac{1}{2}\sum_{x=0}^{3}\sum_{k=0}^{3} e^{-2\pi ik(x-0)/4}e^{2\pi i\frac{1}{8}k}\,|x\rangle\,|1\rangle\,.$$

Here we see how the closest integer is $a = 0$, so we have $\delta = \frac{1}{8}$. If we measure $a = 0$ then we are deducing $\theta = 0$, which is within a range of $\frac{1}{2^2}$ of the answer $\frac{1}{8}$. This shows that is very important to have enough qubits to get high accuracy of the phase estimate.