Quantum Computing 2025 - Exercise Sheet 5

Introduction to Quantum Control

1. *(Time-independent systems)*
The time evolution of time-independent systems is modeled by the time-independent Schrödinger equation (TISE)

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = H |\psi(t)\rangle, \quad U(0) = I \tag{1}$$

where $|\psi(t)\rangle$ is the quantum state at time t and H is the Hamiltonian of the system.

a) *The quantum states evolve through unitary matrices such that $|\psi(t)\rangle = U(t)|\psi(0)\rangle$. Why does $U(t)$ have to be unitary?*

We want to ensure the probabilities to add to 1 at all times, so we have

$$\langle\psi(t)| |\psi(t)\rangle = \langle\psi(0)| U^\dagger(t)U(t) |\psi(0)\rangle = 1$$

Therefore, we conclude $U^\dagger(t)U(t) = I$, as $\langle\psi(0)| I |\psi(0)\rangle = 1$.

b) *Rewrite the TISE in terms of $U(t)$.*

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = H |\psi(t)\rangle \Rightarrow \quad i\hbar \frac{\partial}{\partial t} U(t) |\psi(0)\rangle = HU(t) |\psi(0)\rangle \Rightarrow \quad i\hbar \frac{\partial}{\partial t} U(t) = HU(t)$$

c) *Find the explicit formula for $U(t)$ that solves the TISE.*

If we solve the differential equation we readily see that the solution is the exponential of the Hamiltonian

$$U(T) = e^{-\frac{i}{\hbar} H \cdot T}$$

d) *Show that if H is Hermitian then $U(t)$ is unitary.*

If H is Hermitian we have $H^\dagger = H$, then

$$U^\dagger(t) = e^{\frac{i}{\hbar} H^\dagger t} = e^{\frac{i}{\hbar} H t}$$

Therefore, we have $U^\dagger(t)U(t) = e^{\frac{i}{\hbar} Ht} e^{-\frac{i}{\hbar} Ht} = I$

e) *If we let $H = \frac{\pi}{8}\sigma^z$, what is the state at time $t = 4\hbar$, $|\psi(4\hbar)\rangle = U(4\hbar)|\psi(0)\rangle$, where $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. Use the formula $e^{-i\theta\sigma^z} = \cos(\theta)I - i\sin(\theta)\sigma^z$.*

We have

$$U(4\hbar)|\psi(0)\rangle = e^{-\frac{i}{\hbar}\frac{\pi}{8}\sigma^z \cdot 4\hbar} |+\rangle = e^{-i\frac{\pi}{2}\sigma^z} |+\rangle = (\cos(\frac{\pi}{2})I - i\sin(\frac{\pi}{2})\sigma^z) |+\rangle = -i\sigma^z |+\rangle = -i |-\rangle$$

2. *(Time-dependent systems)*
The time evolution of time-dependent systems is modeled by the time-dependent Schrödinger equation (TDSE)

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = H(t) |\psi(t)\rangle. \tag{2}$$

If we consider controlled systems we can write the Hamiltonian as

$$H(t) = H_0 + u(t)H_c, \tag{3}$$

where H_0 is the free Hamiltonian, H_c is the control Hamiltonian, and $u(t)$ is a control function of any form.
The solution to the TDSE at some time T is also given by an exponential called the Magnus expansion, defined as

$$U(T) = \exp(\Omega^{(\infty)}), \qquad \Omega^{(n)} = \sum_{k=0}^{n} \Omega_k, \quad \text{we approximately write (for some } n\text{)} \quad U(T) \simeq \exp(\Omega^{(n)}). \tag{4}$$

and we define the first terms of the sum as

$$\Omega_1(T) = \frac{-i}{\hbar}\int_0^T dt_1 H(t_1) \tag{5}$$

$$\Omega_2(T) = \frac{-1}{2\hbar}\int_0^T dt_1 \int_0^{t_1} dt_2 [H(t_1), H(t_2)]. \tag{6}$$

a) *If we let $H(t) = \sigma^z + t\sigma^x$ find $\Omega^{(2)}$ for $T = 1$ (You have to calculate both terms of the series and add them together).*

Following the formula provided we can compute

$$\Omega_1(1) = \frac{-i}{\hbar}\int_0^1 dt_1(\sigma^z + t_1\sigma^x) = -\frac{i}{\hbar}(\sigma^z + \frac{1}{2}\sigma^x)$$

and

$$\Omega_2(1) = \frac{-1}{2\hbar}\int_0^1 dt_1 \int_0^{t_1} dt_2[\sigma^z + t_1\sigma^x, \sigma^z + t_2\sigma^x] \tag{7}$$

$$= -\frac{[\sigma^z, \sigma^x]}{2\hbar}\int_0^1 dt_1 \int_0^{t_1} dt_2(t_2 - t_1) \tag{8}$$

$$= -\frac{[\sigma^z, \sigma^x]}{2\hbar}\int_0^1 dt_1(\frac{t_1^2}{2} - t_1^2) \tag{9}$$

$$= \frac{-2i\sigma^y}{12\hbar} = \frac{-i\sigma^y}{6\hbar} \tag{10}$$

Therefore, we can write the truncated sum as

$$\Omega^{(2)}(1) = \Omega_1 + \Omega_2 = -\frac{i}{\hbar}(\sigma^z + \frac{1}{2}\sigma^x + \frac{1}{6}\sigma^y)$$

b) *What is $U(1)$ (approximate to the second order Magnus expansion)? Use the formula $e^{-i\theta(\mathbf{n}\cdot\sigma)} = \cos(\theta)I - i\sin(\theta)(\mathbf{n}\cdot\sigma)$ for **unit vector n** (make sure to normalize it and consider how this affects θ). Find the solution as $aI - i(b\sigma^x + c\sigma^y + d\sigma^z)$ for some constants a, b, c, d, and leave these in terms of cos and sin.*

We now know $U(1) \simeq e^{\Omega^{(2)}(1)} = \exp(-\frac{i}{\hbar}(\sigma^z + \frac{1}{2}\sigma^x + \frac{1}{6}\sigma^y))$. Here we can let $n = (1, \frac{1}{2}, \frac{1}{6})$ and we have to normalize this in order to use the given formula. We find $|n| = \sqrt{1 + \frac{1}{4} + \frac{1}{36}} = \sqrt{\frac{23}{18}}$, so the normalized vector is now $\hat{n} = (\frac{1}{|n|}, \frac{1}{2|n|}, \frac{1}{6|n|})$. We can then write

$$U(1) \simeq \exp(-\frac{i}{\hbar}|n|(\frac{1}{|n|}\sigma^z + \frac{1}{2|n|}\sigma^x + \frac{1}{6|n|}\sigma^y)).$$

Applying the formula we get

$$U(1) \simeq \cos(\frac{|n|}{\hbar})I - i\sin(\frac{|n|}{\hbar})\left(\frac{1}{|n|}\sigma^z + \frac{1}{2|n|}\sigma^x + \frac{1}{6|n|}\sigma^y\right) = aI - i(b\sigma^x + c\sigma^y + d\sigma^z),$$

for some constants a, b, c, d.

3. *(Piecewise constant systems)*
If we divide the time into m segments $[t_1, t_2, \ldots, t_m = T]$ where the Hamiltonian is constant (time-independent). We can write the unitary evolution as

$$U(T) = e^{\frac{-i}{\hbar}H(t_m)\Delta t}\ldots e^{\frac{-i}{\hbar}H(t_2)\Delta t}e^{\frac{-i}{\hbar}H(t_1)\Delta t},$$

where $\Delta t = T/m$. If we let $T = 2\pi\hbar$ and $m = 4$ with equally long segments, find the unitary matrix that encodes the evolution of the system, $U(T)$, for Hamiltonian $H(t_1) = \sigma^z, H(t_2) = 2\sigma^y, H(t_3) = I, H(t_4) = \frac{1}{2}\sigma^x$.
We compute each exponential and then we will take the product, and note that $\Delta t = \frac{\pi}{2\hbar}$.

$$e^{\frac{-i}{\hbar}H(t_1)\Delta t} = e^{-i\frac{\pi}{2}\sigma^z} = -i\sigma^z$$
$$e^{\frac{-i}{\hbar}H(t_2)\Delta t} = e^{-i\pi\sigma^y} = \cos(\pi)I = -I$$
$$e^{\frac{-i}{\hbar}H(t_3)\Delta t} = e^{-i\frac{\pi}{2}\sigma^z} = -iI$$
$$e^{\frac{-i}{\hbar}H(t_4)\Delta t} = e^{-i\frac{\pi}{4}\sigma^x} = \frac{\sqrt{2}}{2}I - i\frac{\sqrt{2}}{2}\sigma^x$$

We can now find the total unitary by taking the product

$$U(2\pi\hbar) = (\frac{\sqrt{2}}{2}I - i\frac{\sqrt{2}}{2}\sigma^x)(-iI)(-I)(-i\sigma^z) = \frac{\sqrt{2}}{2}(\sigma^z - \sigma^y)$$

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