Quantum Computing

Exercises 3: Qubits

1. *Let us consider the set* $\{|0\rangle, |1\rangle\}$*, that forms a basis in* $\mathbb{C}^2$ *(the computational basis). Calculate the vectors in* $\mathbb{C}^4$*:*

$$|0\rangle \otimes |0\rangle \ , \ |0\rangle \otimes |1\rangle \ , \ |1\rangle \otimes |0\rangle \ , \ |1\rangle \otimes |1\rangle$$

and interpret the result.
b) Consider the Pauli matrices σ_x *and* σ_z *. Find* $\sigma_x \otimes \sigma_z$ *and* $\sigma_z \otimes \sigma_x$ *and discuss. Both* σ_x *and* σ_z *are hermitian. Are* $\sigma_x \otimes \sigma_z$ *and* $\sigma_z \otimes \sigma_x$ *hermitian? Both* σ_x *and* σ_z *are unitary. Is* $\sigma_x \otimes \sigma_z$ *and* $\sigma_z \otimes \sigma_x$ *unitary?*

2. *Consider again the computational basis for,* $\{|0\rangle, |1\rangle\}$*. The Walsh-Hadamard transform is a 1-qubit operation, denoted by* H*, and performs the linear transform*

$$|0\rangle \to \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \ , \ |1\rangle \to \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

a) Find the unitary operator U_H *which implements* H *with respect to the basis* $\{|0\rangle, |1\rangle\}$*.*
b) Find the inverse of this operator.
c) Find its matrix representation in the computational (standard) basis:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ , \ |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and in the Hadamard basis:

$$|0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \ , \ |1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

3. *[Nielsen & Chuang Ex. 4.1] Find the points on the Bloch sphere which correspond to the normalized eigenvectors of the different Pauli matrices.*

4. *Consider the qubit, which Hamiltonian is represented in the basis* $B = \{|0\rangle, |1\rangle\}$ *by the matrix:*

$$H = \begin{pmatrix} E_0 & v \\ v & E_0 \end{pmatrix}$$

a) Can E_0 *and* v *be complex numbers?*
b) Obtain the spectrum of H *and its eigenvalues*

5. *Consider a physical system whose state space con be subtended by the three vectors of the basis* $B = \{|u_1\rangle, |u_2\rangle, |u_3\rangle\}$*. In this space, the operators* L *and* S *are defined by:*

$$L|u_1\rangle = |u_1\rangle \ , \ L|u_2\rangle = 0 \ , \ L|u_3\rangle = -|u_3\rangle$$

$$S|u_1\rangle = |u_3\rangle \ , \ S|u_2\rangle = |u_2\rangle \ , \ S|u_3\rangle = |u_1\rangle$$

a) Write down the matrices that represent the four operators L, L^2, S, S^2 *in the basis* B*.*
b) Which pairs commute?

6. *A three-level system or qutrit is described by its Hamiltonian*

$$H = E_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{\epsilon}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

in the basis $B = \{|0\rangle, |1\rangle, |2\rangle\}$

a) Calculate the eigenvalues and eigenvectors of H.
b) Check that the Hamiltonian commutes with th operator Π, which matrix in the basis B is

$$\Pi = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

What does this mean?