Quantum Computing

Exercises 3: Qubits

1. *Let us consider the set $\{|0\rangle, |1\rangle\}$, that forms a basis in $\mathbb{C}^2$ (the computational basis). Calculate the vectors in $\mathbb{C}^4$:*

$$|0\rangle \otimes |0\rangle \ , \ |0\rangle \otimes |1\rangle \ , \ |1\rangle \otimes |0\rangle \ , \ |1\rangle \otimes |1\rangle$$

and interpret the result.
b) Consider the Pauli matrices σ_x and σ_z . Find $\sigma_x \otimes \sigma_z$ and $\sigma_z \otimes \sigma_x$ and discuss. Both σ_x and σ_z are hermitian. Are $\sigma_x \otimes \sigma_z$ and $\sigma_z \otimes \sigma_x$ hermitian? Both σ_x and σ_z are unitary. Is $\sigma_x \otimes \sigma_z$ and $\sigma_z \otimes \sigma_x$ unitary?

a)

$$|0\rangle \otimes |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ 0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = |00\rangle$$

$$|0\rangle \otimes |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = |01\rangle$$

$$|1\rangle \otimes |0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ 1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = |10\rangle$$

$$|1\rangle \otimes |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = |11\rangle$$

These represent the two qubit computational basis states.
b)

$$\begin{aligned} \sigma_x \otimes \sigma_z &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & 1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ 1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & 0 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \\ \sigma_z \otimes \sigma_x &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & 0 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ 0 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & -1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \end{aligned} \tag{1}$$

Simply checking by Hermicity, by taking the conjugate transpose, and unitarity, by multiplying with its inverse we can clearly see the above matrices are Hermitian and unitary.

In general, for $C = A \otimes B$, and if we know that both A, B are unitary ($AA^{-1} = BB^{-1} = I$), and Hermitian ($A = A^\dagger, B = B^\dagger$) then we can show C is always unitary and hermitian respectively

$$CC^{-1} = (A \otimes B)(A^{-1} \otimes B) = (AA^{-1}) \otimes (BB^{-1}) = I \otimes I = I$$

$$C^\dagger = (A \otimes B)^\dagger = A^\dagger \otimes B^\dagger = A \otimes B = C$$

2. *Consider again the computational basis for, $\{|0\rangle, |1\rangle\}$. The Walsh-Hadamard transform is a 1-qubit operation, denoted by H, and performs the linear transform*

$$|0\rangle \to \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \ , \ |1\rangle \to \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

a) Find the unitary operator U_H which implements H with respect to the basis $\{|0\rangle, |1\rangle\}$.
b) Find the inverse of this operator.
c) Find its matrix representation in the computational (standard) basis:

$$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix} \ , \ |1\rangle = \begin{pmatrix}0\\1\end{pmatrix},$$

and in the Hadamard basis:

$$|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \ , \ |1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}.$$

a) For an operator, X, that performs the mapping on an orthonormal basis $\{|x_1,\rangle|x_2\rangle\}$:

$$|x_1\rangle \to |y_1\rangle, \quad |x_2\rangle \to |y_2\rangle$$

we may simply write it as

$$X = |y_1\rangle\langle x_1| + |y_2\rangle\langle x_2|$$

hence

$$U_H = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)\langle 0| + \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)\langle 1| = \frac{1}{\sqrt{2}}\left(|0\rangle\langle 0 + |1\rangle\langle 0| + |0\rangle\langle 1| - |1\rangle\langle 1|\right)$$

.

b) The inverse reverse the mapping i.e:

$$\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \to |0\rangle, \quad \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) \to |1\rangle.$$

Performing the same procedure as above, we have the result

$$U_H^{-1} = \frac{1}{\sqrt{2}}|0\rangle\left(\langle 0| + \langle 1|\right) + \frac{1}{\sqrt{2}}|1\rangle\left(\langle 0| - \langle 1|\right) = \frac{1}{\sqrt{2}}\left(|0\rangle\langle 0 + |1\rangle\langle 0| + |0\rangle\langle 1| - |1\rangle\langle 1|\right) = U_H$$

The Hadamard gate is its own inverse!

c) i) in the computational basis:

$$U_H = \begin{pmatrix}\langle 0|U_H|0\rangle & \langle 0|U_H|1\rangle\\ \langle 1|U_H|0\rangle & \langle 1|U_H|1\rangle\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

ii) in the Hadarmard basis (also denoted as $\{|+\rangle, |-\rangle\}$):

$$U_H = \begin{pmatrix}\langle +|U_H|+\rangle & \langle +|U_H|-\rangle\\ \langle -|U_H|+\rangle & \langle -|U_H|-\rangle\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

3. *[Nielsen & Chuang Ex. 4.1] Find the points on the Bloch sphere which correspond to the normalized eigenvectors of the different Pauli matrices. For the eigenvectors of σ_z we have $\{|0\rangle, |1\rangle\}$ by comparing with the general formula for a wavefuction written in spherical coordinates (θ, ϕ) $|\psi\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle$*

we see that for $|0\rangle$:

$$\cos(\theta/2) = 1 \to \theta/2 = 0 \to \theta = 0$$

and ϕ is arbitrary.
Contintuing in the same way we see

$$|1\rangle \to \theta = \pi, \phi \textit{ is arbitrary}$$

$$|+\rangle \to \theta = \pi/2, \phi = 0$$

$$|-\rangle \to \theta = \pi/2, \phi = \pi$$

$$|+i\rangle \to \theta = \pi/2, \phi = \pi/2$$

$$|-i\rangle \to \theta = \pi/2, \phi = -\pi/2$$