

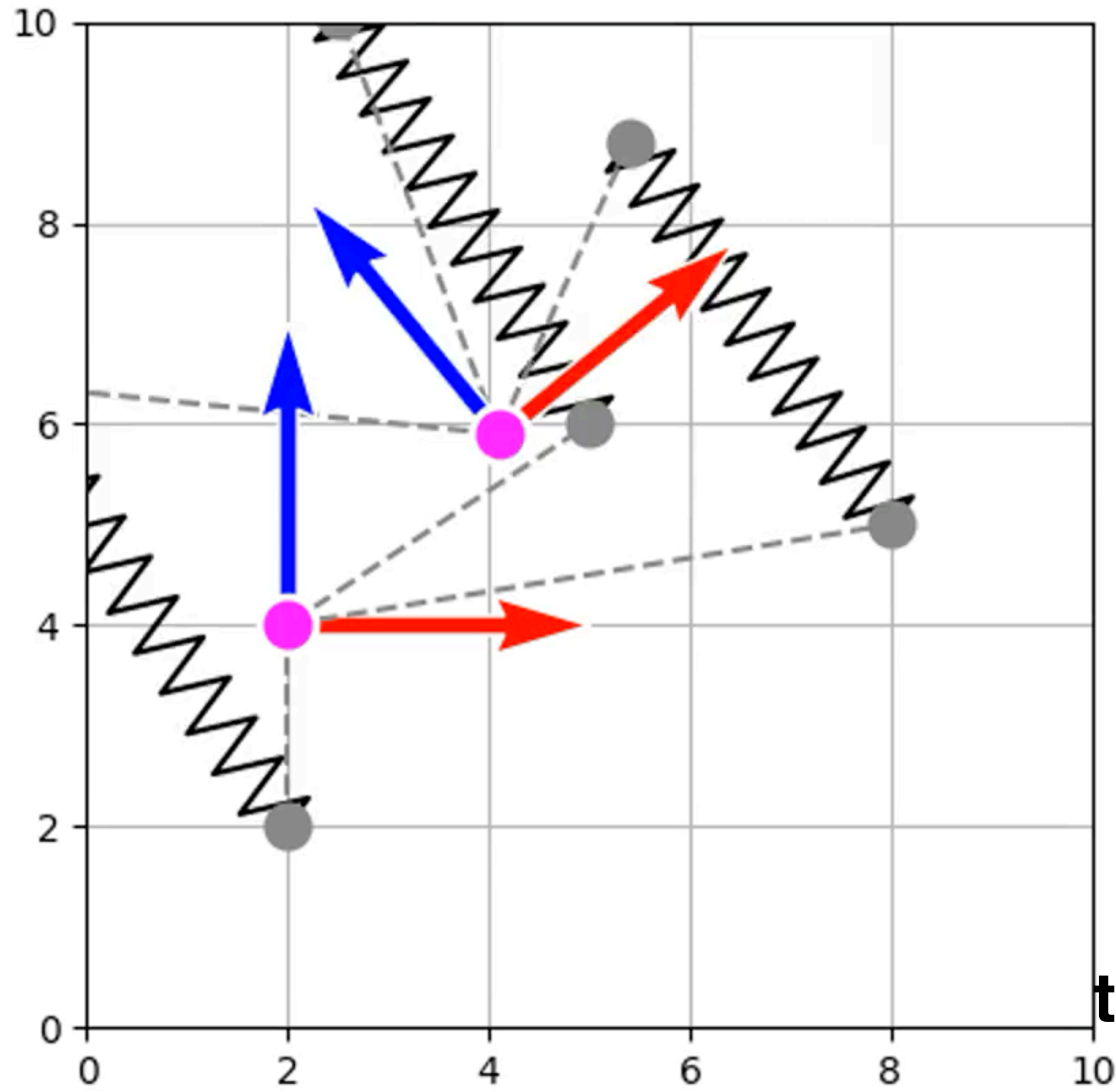
# Relative motion from known correspondences

Absolute orientation on  $SE(2)$  and  $SE(3)$  manifolds and its closed-form solution

Karel Zimmermann

# Relative motion from two point cloud scans

$$= \arg \min_{x,y,\theta} \sum_i \left\| \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \mathbf{z}_1^{\mathbf{m}_i} + \begin{bmatrix} x \\ y \end{bmatrix} - \mathbf{z}_2^{\mathbf{m}_i} \right\|^2 = \arg \min_{\mathbf{t},\theta} \sum_i \left\| \mathbf{R}_\theta \mathbf{z}_1^{\mathbf{m}_i} + \mathbf{t} - \mathbf{z}_2^{\mathbf{m}_i} \right\|^2$$



- $\mathbf{x}_t$  ... robot poses
- $\mathbf{z}_t^{\mathbf{m}_i}$  ... 2D points from pointcloud
- ↗ ↘ ..... local coordinate frame

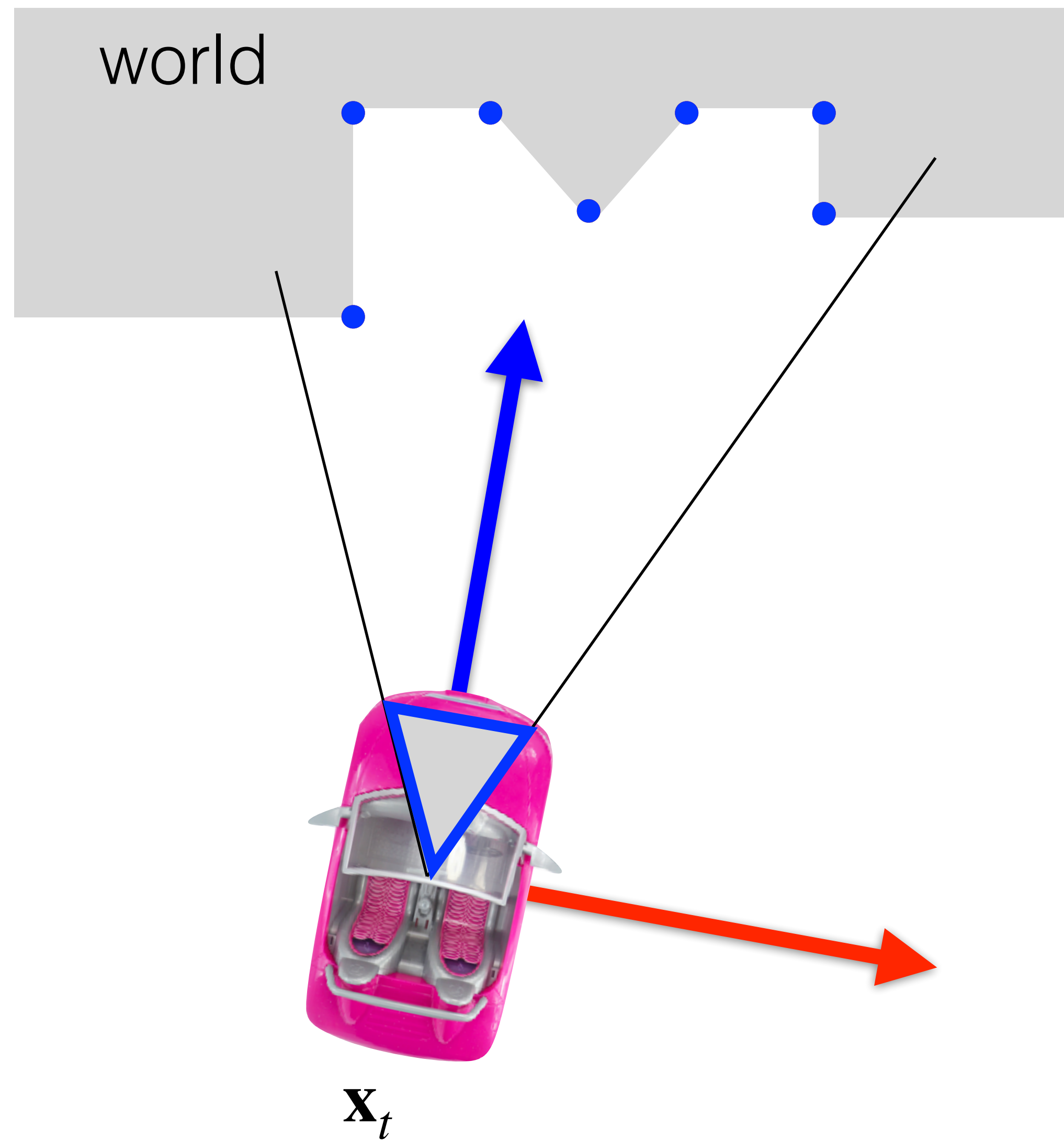
**Can I do it?**

**Correspondences typically missing.**

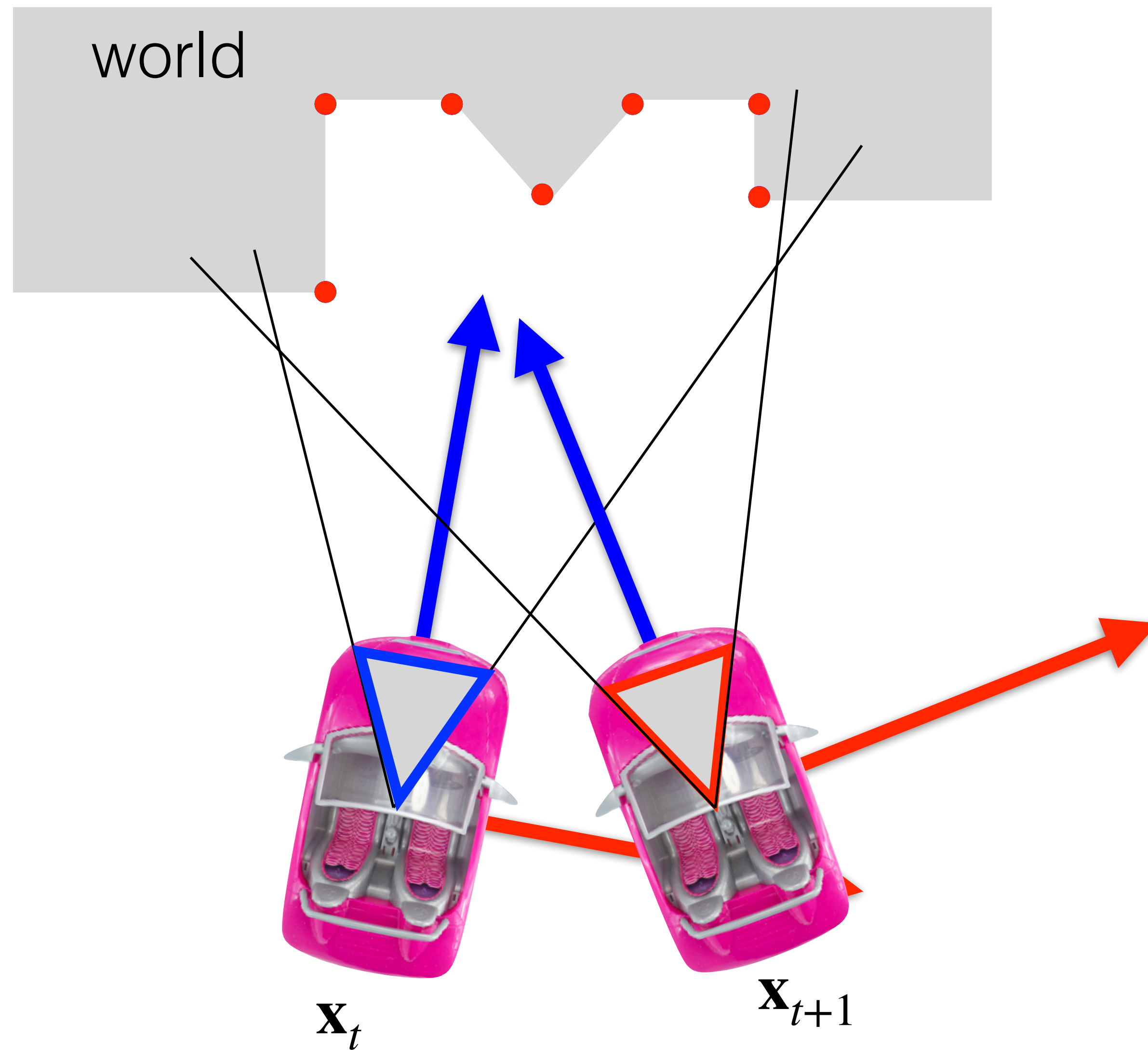
**How many points are in a usual pointcloud?**

**Do we really want to keep all these factors/equations in the factorgraph?**

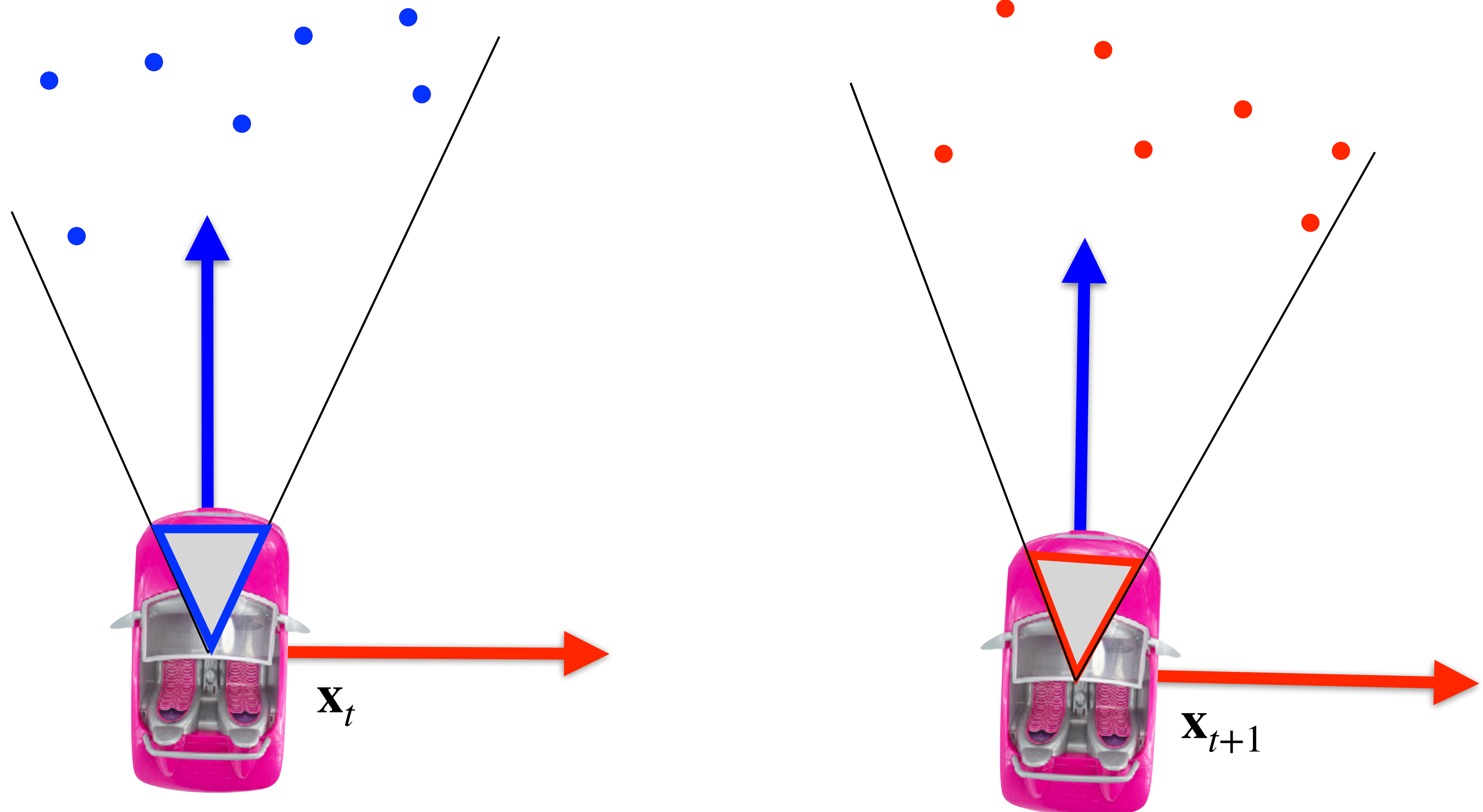
# Pose estimation from known correspondences



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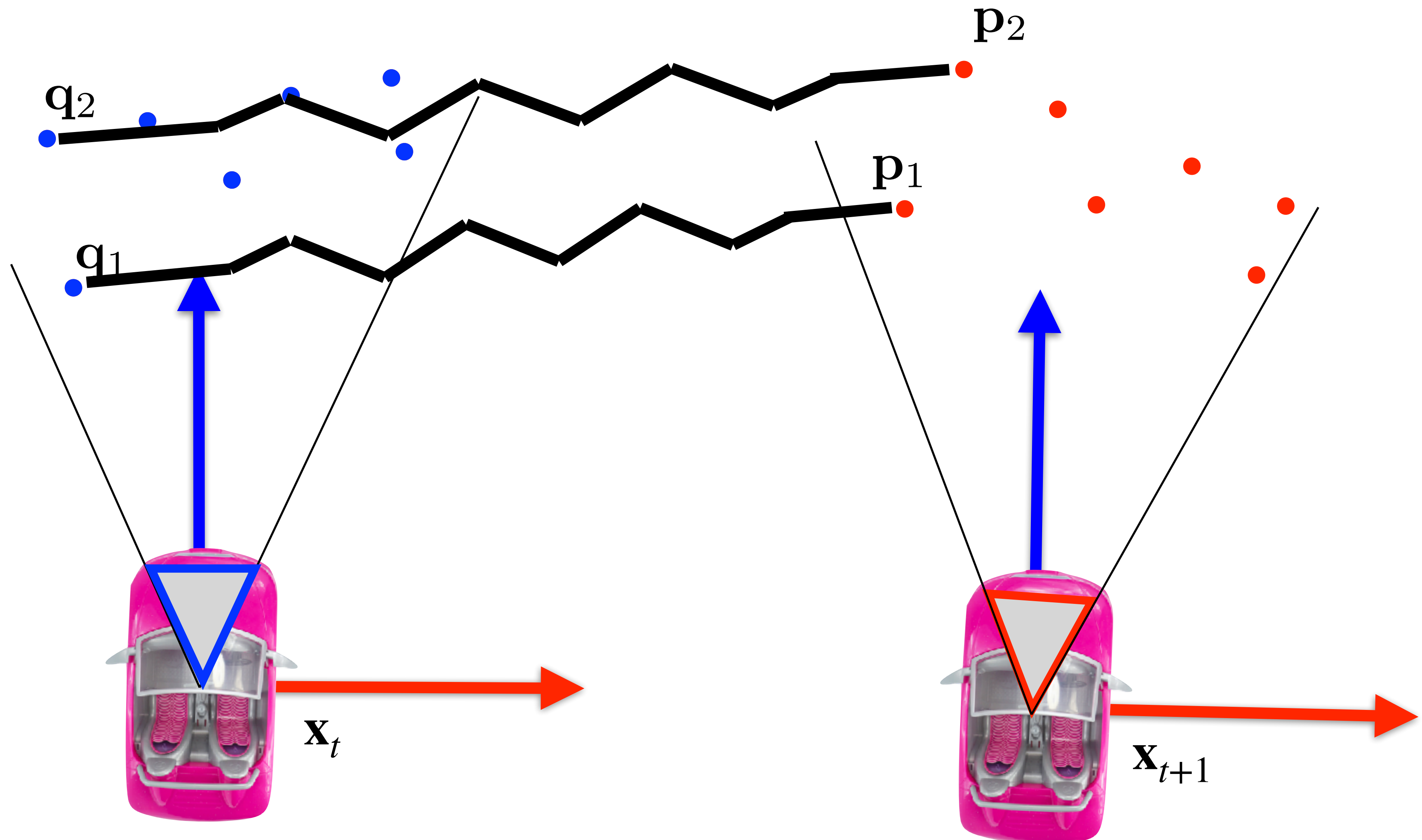


# Pose estimation from known correspondences



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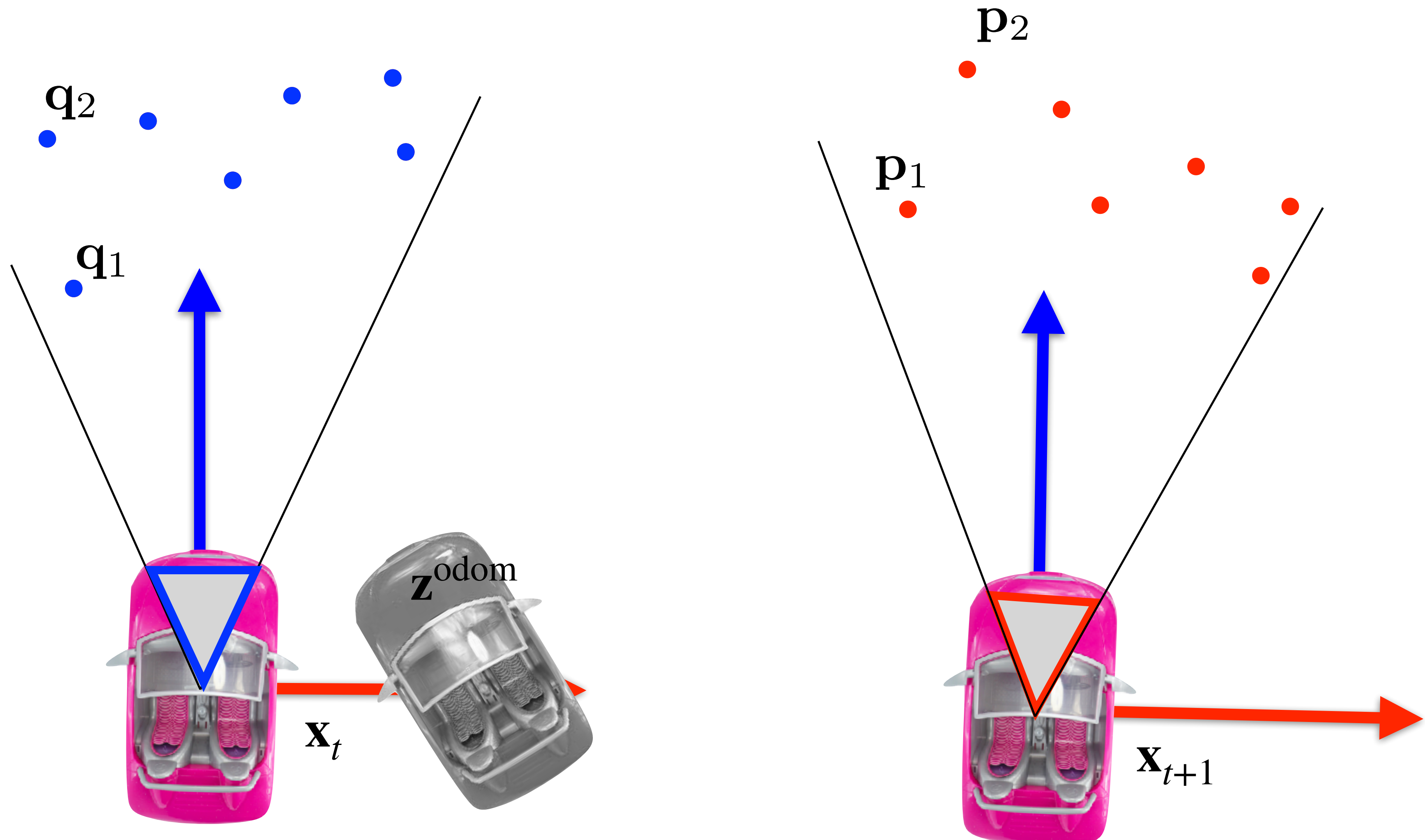
Estimate odometry measurement:  $\mathbf{z}^{\text{odom}} = \arg \min_{\mathbf{t}, \theta} \sum_i \left\| \mathbf{R}_\theta \mathbf{p}_i + \mathbf{t} - \mathbf{q}_i \right\|^2$





# Pose estimation from known correspondences

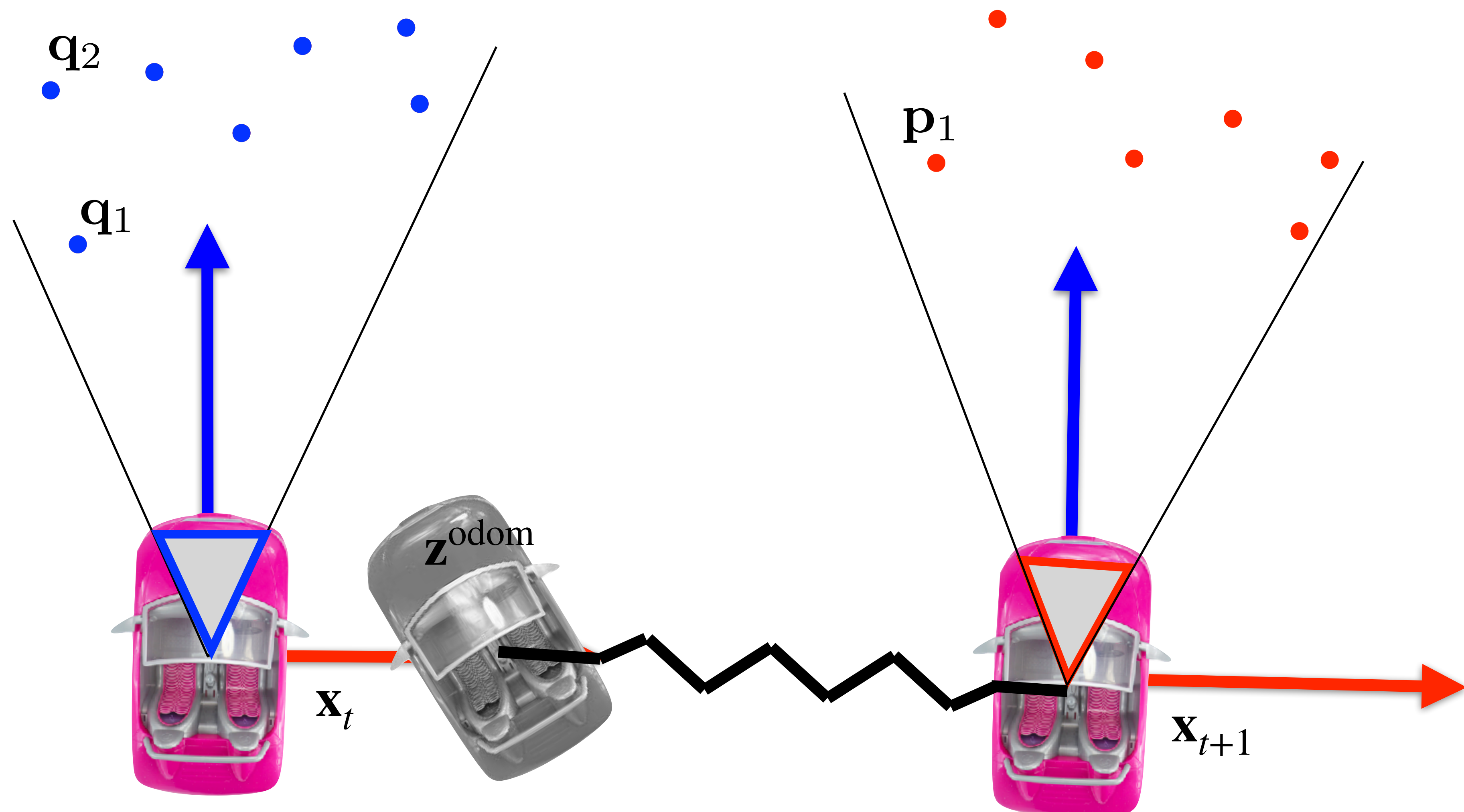
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# Pose estimation from known correspondences

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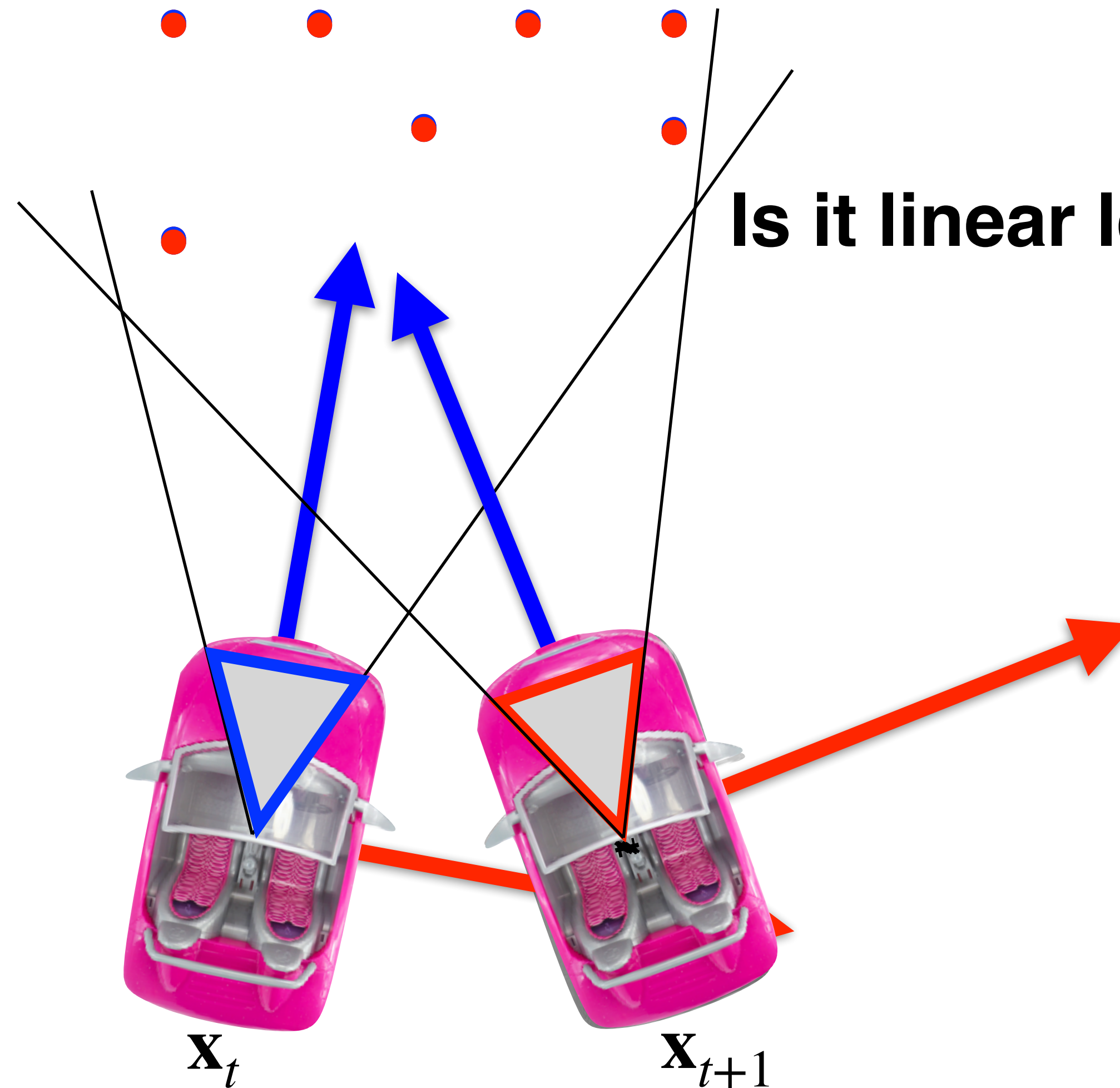
Apply only single odometry factor:  $\arg \min_{\mathbf{x}_t, \mathbf{x}_{t+1}} \left\| \mathbf{w}2r(\mathbf{x}_{t+1}, \mathbf{x}_t) - \mathbf{z}^{\text{odom}} \right\|^2$



# Pose estimation from known correspondences

Estimate odometry measurement:  $\mathbf{z}^{\text{odom}} = \arg \min_{\mathbf{t}, \theta} \sum_i \left\| \mathbf{R}_{\theta} \mathbf{p}_i + \mathbf{t} - \mathbf{q}_i \right\|^2$

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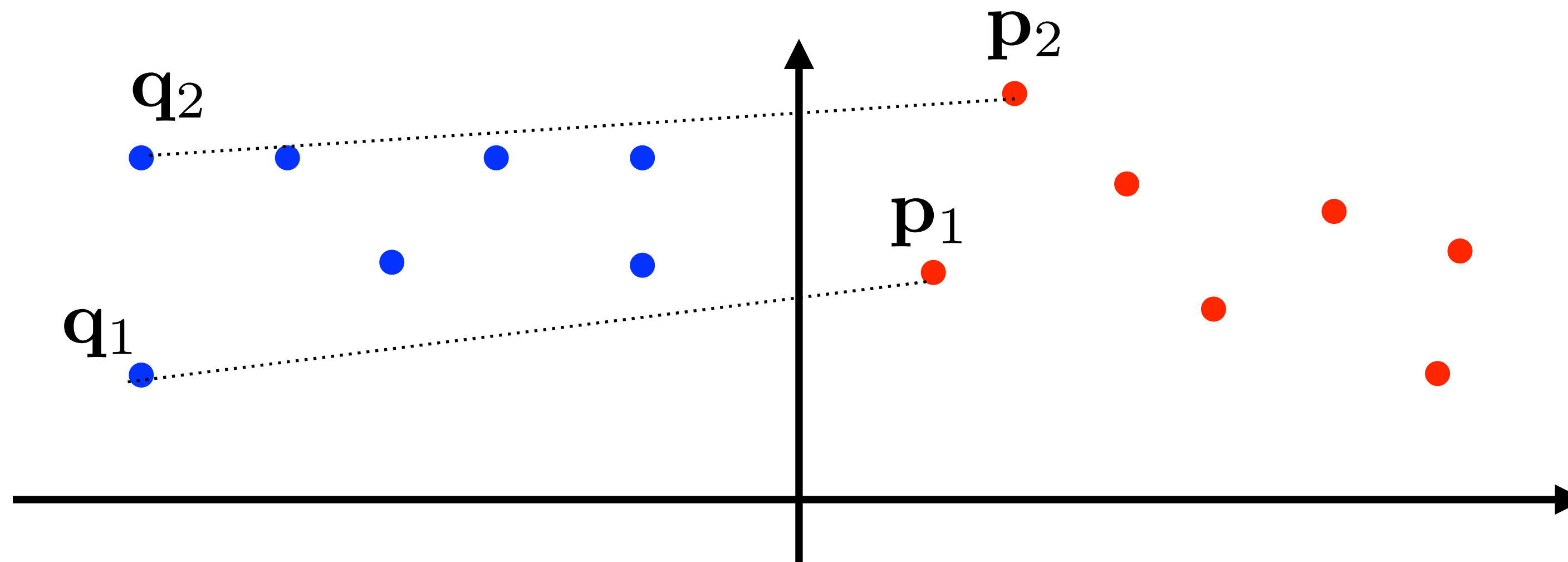


**Is it linear least squares problem**

# Absolute orientation problem in SE(2)

$$\mathbf{z}^{\text{odom}} = \arg \min_{\mathbf{t}, \theta} \sum_i \left\| \mathbf{R}_\theta \mathbf{p}_i + \mathbf{t} - \mathbf{q}_i \right\|^2$$

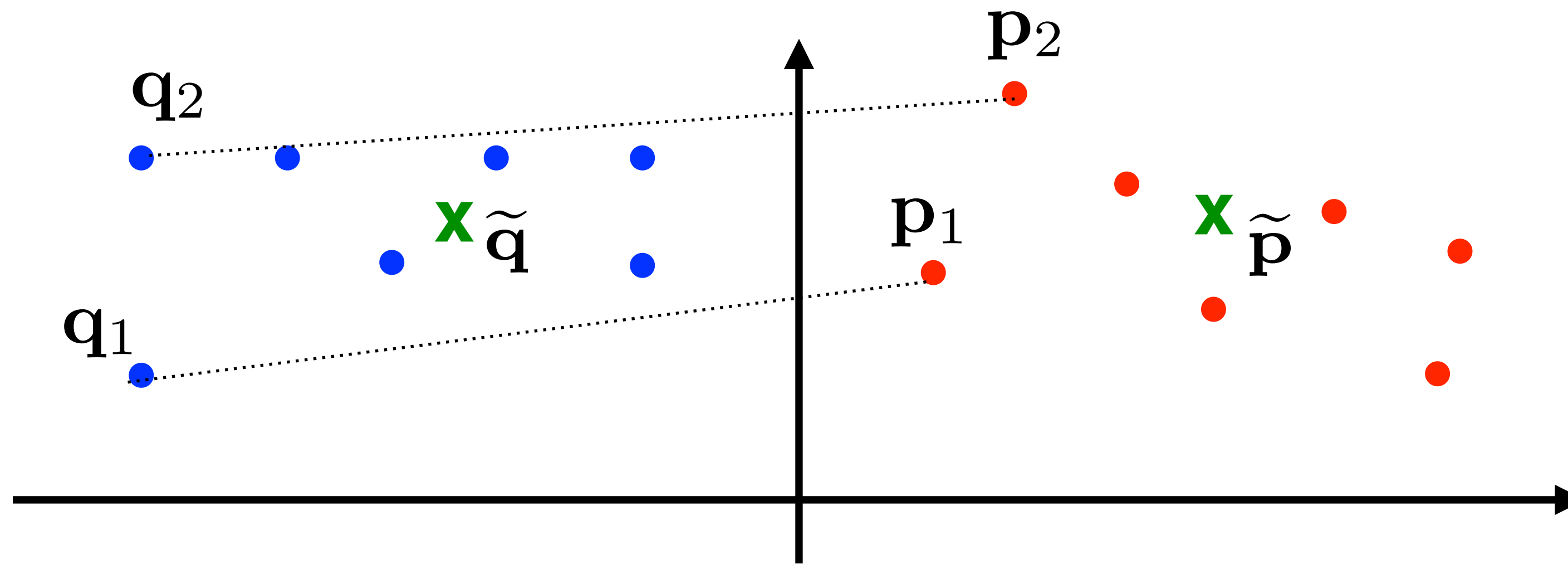
Substitution:



# Absolute orientation problem in SE(2)

$$\mathbf{z}^{\text{odom}} = \arg \min_{\mathbf{t}, \theta} \sum_i \left\| \mathbf{R}_\theta \mathbf{p}_i + \mathbf{t} - \mathbf{q}_i \right\|^2$$

Substitution:  $\mathbf{p}'_i = \mathbf{p}_i - \underbrace{\frac{1}{N} \sum_i \mathbf{p}_i}_{\tilde{\mathbf{p}}}$ ,  $\mathbf{q}'_i = \mathbf{q}_i - \underbrace{\frac{1}{N} \sum_i \mathbf{q}_i}_{\tilde{\mathbf{q}}}$



Absolute orientation problem in SE(2)

**Do you see any special properties?**

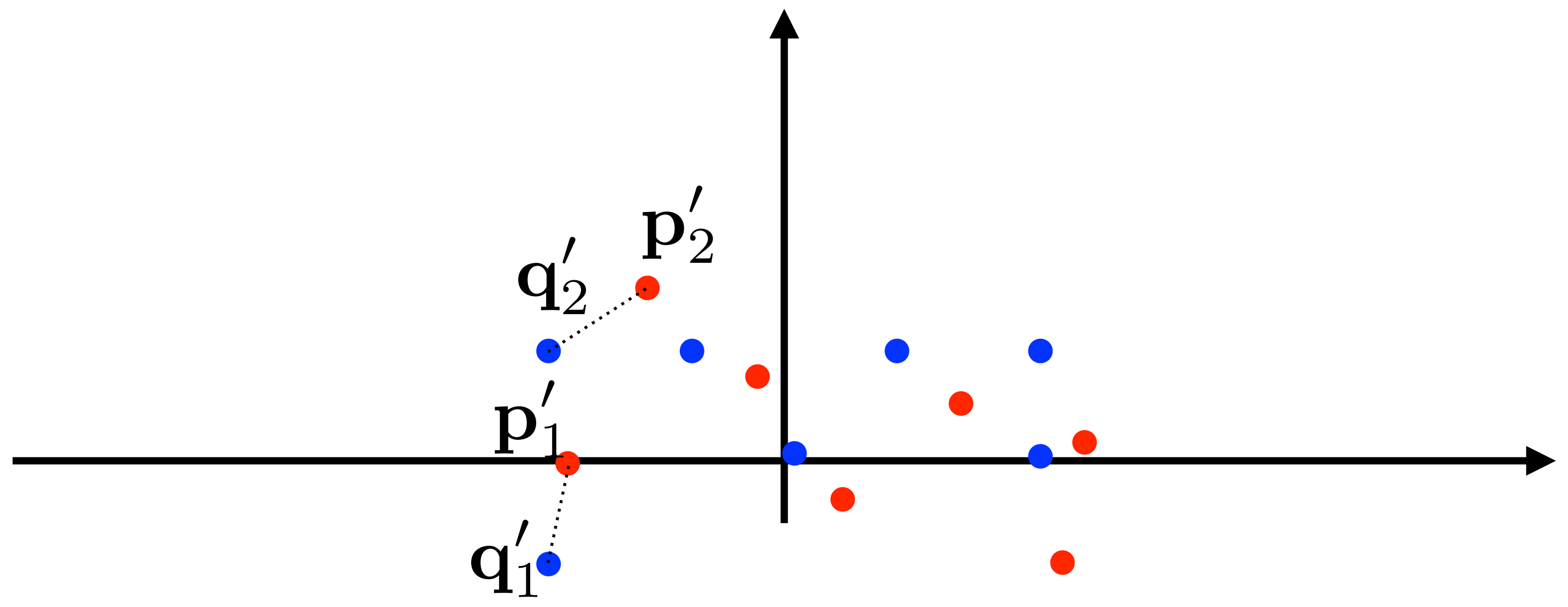
$$\mathbf{z}^{\text{odom}} = \arg \min_{\mathbf{t}, \theta} \sum_i \left\| \mathbf{R}_\theta \mathbf{p}_i + \mathbf{t} - \mathbf{q}_i \right\|^2 = \arg \min_{\mathbf{t}, \theta} \sum_i \left\| \mathbf{R}_\theta \mathbf{p}'_i - \mathbf{q}'_i \right\|^2 + \left\| \mathbf{R}_\theta \tilde{\mathbf{p}} + \mathbf{t} - \tilde{\mathbf{q}} \right\|^2$$

Substitution:  $\mathbf{p}'_i = \mathbf{p}_i - \underbrace{\frac{1}{N} \sum_i \mathbf{p}_i}_{\tilde{\mathbf{p}}}$ ,  $\mathbf{q}'_i = \mathbf{q}_i - \underbrace{\frac{1}{N} \sum_i \mathbf{q}_i}_{\tilde{\mathbf{q}}}$

Can be always zero by appropriate choice of  $\mathbf{t}$

Depends only on  $\theta$

Solution:  $\theta^* = \arg \min_{\theta} \sum_i \left\| \mathbf{R}_\theta \mathbf{p}'_i - \mathbf{q}'_i \right\|^2$



# Absolute orientation problem in SE(2)

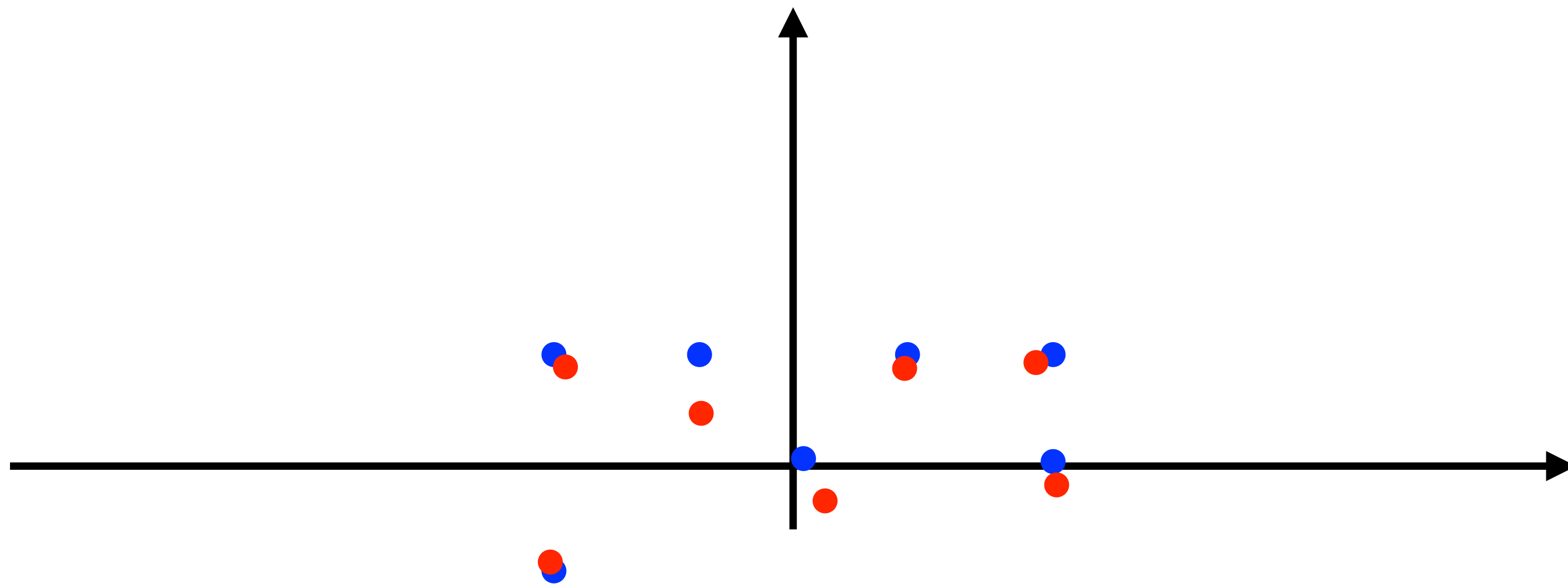
$$\mathbf{z}^{\text{odom}} = \arg \min_{\mathbf{t}, \theta} \sum_i \left\| \mathbf{R}_\theta \mathbf{p}_i + \mathbf{t} - \mathbf{q}_i \right\|^2 \stackrel{\text{Optimize for } \theta}{=} \arg \min_{\mathbf{t}, \theta} \sum_i \left\| \mathbf{R}_\theta \mathbf{p}'_i - \mathbf{q}'_i \right\|^2 + \left\| \mathbf{R}_\theta \tilde{\mathbf{p}} + \mathbf{t} - \tilde{\mathbf{q}} \right\|^2 \quad \text{Set } \mathbf{t} \text{ to make it zero}$$

Substitution:  $\mathbf{p}'_i = \mathbf{p}_i - \underbrace{\frac{1}{N} \sum_i \mathbf{p}_i}_{\tilde{\mathbf{p}}}$ ,  $\mathbf{q}'_i = \mathbf{q}_i - \underbrace{\frac{1}{N} \sum_i \mathbf{q}_i}_{\tilde{\mathbf{q}}}$

Can be always zero by appropriate choice of  $\mathbf{t}$

Solution:  $\theta^* = \arg \min_{\theta} \sum_i \left\| \mathbf{R}_\theta \mathbf{p}'_i - \mathbf{q}'_i \right\|^2$   $\mathbf{t}^* = \arg \min_{\mathbf{t}} \left\| \mathbf{R}_{\theta^*} \tilde{\mathbf{p}} + \mathbf{t} - \tilde{\mathbf{q}} \right\|^2 = \tilde{\mathbf{q}} - \mathbf{R}_{\theta^*} \tilde{\mathbf{p}}$

Depends only on  $\theta$



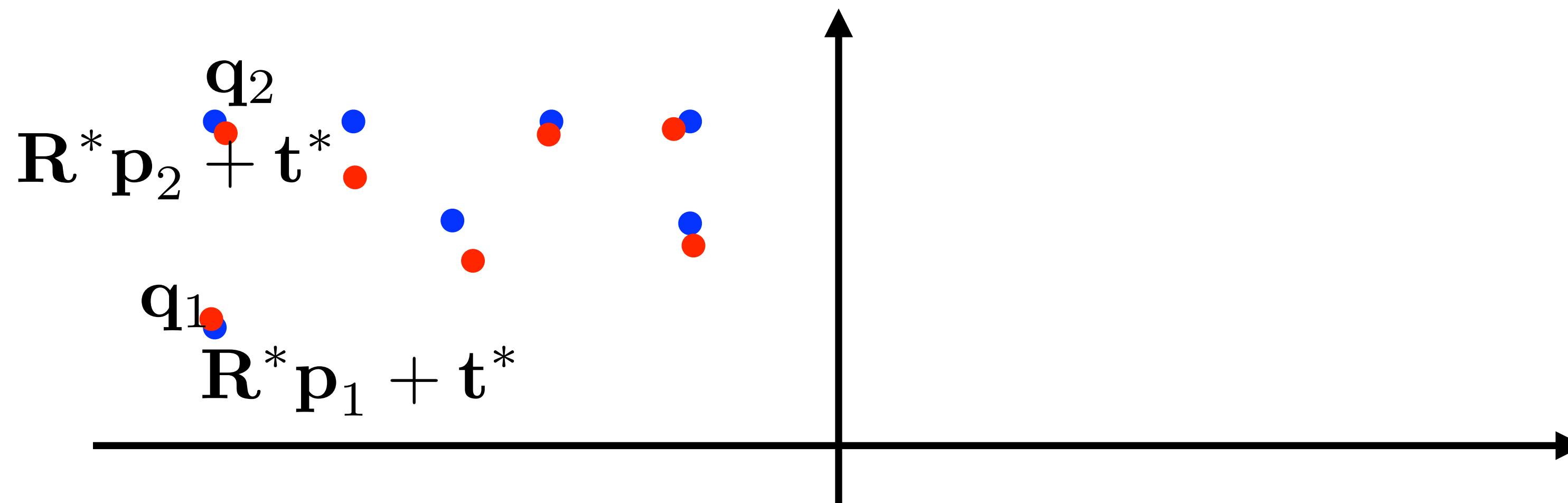
# Absolute orientation problem in SE(2)

$$\mathbf{z}^{\text{odom}} = \arg \min_{\mathbf{t}, \theta} \sum_i \left\| \mathbf{R}_\theta \mathbf{p}_i + \mathbf{t} - \mathbf{q}_i \right\|^2 = \arg \min_{\mathbf{t}, \theta} \sum_i \left\| \mathbf{R}_\theta \mathbf{p}'_i - \mathbf{q}'_i \right\|^2 + \left\| \mathbf{R}_\theta \tilde{\mathbf{p}} + \mathbf{t} - \tilde{\mathbf{q}} \right\|^2$$

Substitution:  $\mathbf{p}'_i = \mathbf{p}_i - \underbrace{\frac{1}{N} \sum_i \mathbf{p}_i}_{\tilde{\mathbf{p}}}$ ,  $\mathbf{q}'_i = \mathbf{q}_i - \underbrace{\frac{1}{N} \sum_i \mathbf{q}_i}_{\tilde{\mathbf{q}}}$

Can be always zero by appropriate choice of  $\mathbf{t}$

Solution:  $\theta^* = \arg \min_{\theta} \sum_i \left\| \mathbf{R}_\theta \mathbf{p}'_i - \mathbf{q}'_i \right\|^2$        $\mathbf{t}^* = \arg \min_{\mathbf{t}} \left\| \mathbf{R}_{\theta^*} \tilde{\mathbf{p}} + \mathbf{t} - \tilde{\mathbf{q}} \right\|^2 = \tilde{\mathbf{q}} - \mathbf{R}_{\theta^*} \tilde{\mathbf{p}}$

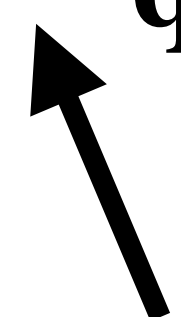


# Absolute orientation problem in SE(2)

$$\mathbf{z}^v = \arg \min_{\mathbf{t}, \theta} \sum_i \left\| \mathbf{R}_\theta \mathbf{p} + \mathbf{t} - \mathbf{q} \right\|^2 = \arg \min_{\mathbf{t}, \theta} \sum_i \left\| \mathbf{R}_\theta \mathbf{p}'_i - \mathbf{q}'_i \right\|^2 + \left\| \mathbf{R}_\theta \tilde{\mathbf{p}} + \mathbf{t} - \tilde{\mathbf{q}} \right\|^2$$

$$\text{Substitution: } \mathbf{p}'_i = \mathbf{p}_i - \underbrace{\frac{1}{N} \sum_i \mathbf{p}_i}_{\tilde{\mathbf{p}}}, \quad \mathbf{q}'_i = \mathbf{q}_i - \underbrace{\frac{1}{N} \sum_i \mathbf{q}_i}_{\tilde{\mathbf{q}}}$$

Can be always zero by appropriate choice of  $\mathbf{t}$



$$\text{Solution: } \theta^* = \arg \min_{\theta} \sum_i \left\| \mathbf{R}_\theta \mathbf{p}'_i - \mathbf{q}'_i \right\|^2 = \text{???}$$

$$\mathbf{t}^* = \arg \min_{\mathbf{t}} \left\| \mathbf{R}_{\theta^*} \tilde{\mathbf{p}} + \mathbf{t} - \tilde{\mathbf{q}} \right\|^2 = \tilde{\mathbf{q}} - \mathbf{R}_{\theta^*} \tilde{\mathbf{p}}$$

# Absolute orientation problem in SE(2)

$$\theta^* = \arg \min_{\theta} \sum_i \|\mathbf{R}_{\theta} \mathbf{p}'_i - \mathbf{q}'_i\|^2$$

$$= \arg \min_{\theta} \sum_{\mathbf{p}', \mathbf{q}'} \left\| \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} p'_x \\ p'_y \end{bmatrix} - \begin{bmatrix} q'_x \\ q'_y \end{bmatrix} \right\|^2$$

$$= \arg \min_{\theta} \sum_{\mathbf{p}', \mathbf{q}'} \left\| \begin{array}{l} \cos(\theta)p'_x - \sin(\theta)p'_y - q'_x \\ \sin(\theta)p'_x + \cos(\theta)p'_y - q'_y \end{array} \right\|^2$$

# Absolute orientation problem in SE(2)

$$\theta^* = \arg \min_{\theta} \sum_i \|\mathbf{R}_{\theta} \mathbf{p}'_i - \mathbf{q}'_i\|^2$$

$$= \arg \min_{\theta} \sum_{\mathbf{p}', \mathbf{q}'} \left\| \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} p'_x \\ p'_y \end{bmatrix} - \begin{bmatrix} q'_x \\ q'_y \end{bmatrix} \right\|^2$$

$$= \arg \min_{\theta} \sum_{\mathbf{p}', \mathbf{q}'} \left\| \begin{array}{l} \cos(\theta)p'_x - \sin(\theta)p'_y - q'_x \\ \sin(\theta)p'_x + \cos(\theta)p'_y - q'_y \end{array} \right\|^2$$

$$= \arg \min_{\theta} \sum_{\mathbf{p}', \mathbf{q}'} \left( p'_x \cos(\theta) - p'_y \sin(\theta) - q'_x \right)^2 + \left( p'_x \sin(\theta) + p'_y \cos(\theta) - q'_y \right)^2$$

# Absolute orientation problem in SE(2)

$$\theta^* = \arg \min_{\theta} \sum_{\mathbf{p}', \mathbf{q}'} \left( p'_x \cos(\theta) - p'_y \sin(\theta) - q'_x \right)^2 + \left( p'_x \sin(\theta) + p'_y \cos(\theta) - q'_y \right)^2$$

Derivative:

$$\sum_{\mathbf{p}', \mathbf{q}'} 2(p'_x \cos(\theta) - p'_y \sin(\theta) - q'_x) \cdot (-p'_x \sin(\theta) - p'_y \cos(\theta))$$

$$+ 2(p'_x \sin(\theta) + p'_y \cos(\theta) - q'_y) \cdot (p'_x \cos(\theta) - p'_y \sin(\theta)) = 0$$

Simplify:

$$\sum_{\mathbf{p}', \mathbf{q}'} p_x'^2 \cdot (-\cos(\theta)\sin(\theta) + \sin(\theta)\cos(\theta))$$

$$+ p_y'^2 \cdot (\sin(\theta)\cos(\theta) - \cos(\theta)\sin(\theta))$$

$$+ p'_x p'_y \cdot (-\cos^2(\theta) + \sin^2(\theta) + \cos^2(\theta) - \sin^2(\theta))$$

$$+ p'_x \cdot (q'_x \sin(\theta) - q'_y \cos(\theta))$$

$$+ p'_y \cdot (q'_x \cos(\theta) + q'_y \sin(\theta)) = 0$$

**Expand brackets + factor out p-monomials**

**Is there anything that cancels out?**

# Absolute orientation problem in SE(2)

$$\theta^* = \arg \min_{\theta} \sum_{\mathbf{p}', \mathbf{q}'} \left( p'_x \cos(\theta) - p'_y \sin(\theta) - q'_x \right)^2 + \left( p'_x \sin(\theta) + p'_y \cos(\theta) - q'_y \right)^2$$

Derivative:

$$\sum_{\mathbf{p}', \mathbf{q}'} 2(p'_x \cos(\theta) - p'_y \sin(\theta) - q'_x) \cdot (-p'_x \sin(\theta) - p'_y \cos(\theta))$$

$$+ 2(p'_x \sin(\theta) + p'_y \cos(\theta) - q'_y) \cdot (p'_x \cos(\theta) - p'_y \sin(\theta)) = 0$$

Simplify:

$$\sum_{\mathbf{p}', \mathbf{q}'} p_x'^2 \cdot \left( -\cos(\theta)\sin(\theta) + \sin(\theta)\cos(\theta) \right)$$

$$+ p_y'^2 \cdot \left( \sin(\theta)\cos(\theta) - \cos(\theta)\sin(\theta) \right)$$

$$+ p_x' p_y' \cdot \left( -\cos^2(\theta) + \sin^2(\theta) + \cos^2(\theta) - \sin^2(\theta) \right)$$

$$+ p_x' \cdot \left( q'_x \sin(\theta) - q'_y \cos(\theta) \right)$$

$$+ p_y' \cdot \left( q'_x \cos(\theta) + q'_y \sin(\theta) \right) = 0$$

**Expand brackets + factor out p-monomials**

**Is there anything that cancels out?**

# Absolute orientation problem in SE(2)

$$\theta^* = \arg \min_{\theta} \sum_{\mathbf{p}', \mathbf{q}'} \left( p'_x \cos(\theta) - p'_y \sin(\theta) - q'_x \right)^2 + \left( p'_x \sin(\theta) + p'_y \cos(\theta) - q'_y \right)^2$$

Derivative: 
$$\sum_{\mathbf{p}', \mathbf{q}'} p'_x \cdot (q'_x \sin(\theta) - q'_y \cos(\theta)) + p'_y \cdot (q'_x \cos(\theta) + q'_y \sin(\theta)) = 0$$

Solve: 
$$\sum_{\mathbf{p}', \mathbf{q}'} p'_x \cdot (q'_x \tan(\theta) - q'_y) + p'_y \cdot (q'_x + q'_y \tan(\theta)) = 0$$

$$\sum_{\mathbf{p}', \mathbf{q}'} \tan(\theta) \cdot (p'_x q'_x + p'_y q'_y) + (p'_y q'_x - p'_x q'_y) = 0$$

$$\theta^* = \arctan \left( \frac{\sum_{\mathbf{p}', \mathbf{q}'} p'_x q'_y - p'_y q'_x}{\sum_{\mathbf{p}', \mathbf{q}'} p'_x q'_x + p'_y q'_y} \right) = \arctan \left( \frac{H_{xy} - H_{yx}}{H_{xx} + H_{yy}} \right)$$

$$\mathbf{H} = \sum_i \mathbf{p}'_i \mathbf{q}'_i{}^\top \dots \text{covariance matrix}$$

## Absolute orientation problem in **SE(2)**

$$\mathbf{z}^v = \arg \min_{\mathbf{t}, \theta} \sum_i \left\| \mathbf{R}_\theta \mathbf{p} + \mathbf{t} - \mathbf{q} \right\|^2 = \arg \min_{\mathbf{t}, \theta} \sum_i \left\| \mathbf{R}_\theta \mathbf{p}'_i - \mathbf{q}'_i \right\|^2 + \left\| \mathbf{R}_\theta \tilde{\mathbf{p}} + \mathbf{t} - \tilde{\mathbf{q}} \right\|^2$$

Substitution:  $\mathbf{p}'_i = \mathbf{p}_i - \underbrace{\frac{1}{N} \sum_i \mathbf{p}_i}_{\tilde{\mathbf{p}}}$ ,  $\mathbf{q}'_i = \mathbf{q}_i - \underbrace{\frac{1}{N} \sum_i \mathbf{q}_i}_{\tilde{\mathbf{q}}}$

Can be always zero by appropriate choice of  $\mathbf{t}$

Depends only on  $\theta$

Solution:  $\mathbf{H} = \sum_i \mathbf{p}'_i \mathbf{q}'_i{}^\top$  ... covariance matrix

$$\theta^* = \arg \min_{\theta} \sum_i \left\| \mathbf{R}_\theta \mathbf{p}'_i - \mathbf{q}'_i \right\|^2 = \arctan \left( \frac{H_{xy} - H_{yx}}{H_{xx} + H_{yy}} \right)$$

$$\mathbf{t}^* = \arg \min_{\mathbf{t}} \left\| \mathbf{R}_{\theta^*} \tilde{\mathbf{p}} + \mathbf{t} - \tilde{\mathbf{q}} \right\|^2 = \tilde{\mathbf{q}} - \mathbf{R}_{\theta^*} \tilde{\mathbf{p}}$$

**We solved absolute orientation problem for SE(2),  
can we do it for SE(3)**

## Absolute orientation problem in **SE(3)**

$$\mathbf{z}^v = \arg \min_{\mathbf{t}, \mathbf{R}} \sum_i \left\| \mathbf{R}\mathbf{p} + \mathbf{t} - \mathbf{q} \right\|^2 = \arg \min_{\mathbf{t}, \mathbf{R}} \sum_i \left\| \mathbf{R}\mathbf{p}'_i - \mathbf{q}'_i \right\|^2 + \left\| \mathbf{R}\tilde{\mathbf{p}} + \mathbf{t} - \tilde{\mathbf{q}} \right\|^2$$

Substitution:  $\mathbf{p}'_i = \mathbf{p}_i - \underbrace{\frac{1}{N} \sum_i \mathbf{p}_i}_{\tilde{\mathbf{p}}}$ ,  $\mathbf{q}'_i = \mathbf{q}_i - \underbrace{\frac{1}{N} \sum_i \mathbf{q}_i}_{\tilde{\mathbf{q}}}$

Can be always zero by appropriate choice of  $\mathbf{t}$

Depends only on  $\mathbf{R}$

Solution:  $\mathbf{H} = \sum_i \mathbf{p}'_i \mathbf{q}'_i{}^T$  ... covariance matrix with SVD decomposition  $\mathbf{H} = \mathbf{U}\mathbf{S}\mathbf{V}^T$

$$\mathbf{R}^* = \arg \min_{\mathbf{R}} \sum_i \left\| \mathbf{R}\mathbf{p}'_i - \mathbf{q}'_i \right\|^2 = \mathbf{V}\mathbf{U}^T$$

$$\mathbf{t}^* = \arg \min_{\mathbf{t}} \left\| \mathbf{R}^* \tilde{\mathbf{p}} + \mathbf{t} - \tilde{\mathbf{q}} \right\|^2 = \tilde{\mathbf{q}} - \mathbf{R}^* \tilde{\mathbf{p}}$$

Python:

```
H = P @ Q.T
U, S, V = np.linalg.svd(H, full_matrices=True)
```

# Summary

- **Static environment + known correspondences** is required assumption
- Given 3D-3D (or 2D-2D) correspondences, **globally optimal alignment in L2 has closed-form solution** (i.e. least-squares solution constrained on  $SE(3)$  manifold)
- **Applications:**
  - Lidar-Lidar or Lidar-Robot Calibration
  - Localization from (un)known correspondences
  - Computer graphics for alignment of 3D models
- **Next:** Localization from unknown correspondences ICP

# Proof [Arun-TPAMI-87]

$$\mathbf{R}^*, \mathbf{t}^* = \arg \min_{\mathbf{R} \in SO(3), \mathbf{t} \in \mathcal{R}^3} \sum_i \|\mathbf{R}\mathbf{p}_i + \mathbf{t} - \mathbf{q}_i\|_2^2 =$$

## Proof [Arun-TPAMI-87]

$$\begin{aligned}\mathbf{R}^*, \mathbf{t}^* &= \arg \min_{\mathbf{R} \in SO(3), \mathbf{t} \in \mathcal{R}^3} \sum_i \|\mathbf{R}\mathbf{p}_i + \mathbf{t} - \mathbf{q}_i\|_2^2 = \\ &= \arg \min_{\mathbf{R} \in SO(3), \mathbf{t} \in \mathcal{R}^3} \sum_i \|\mathbf{R}(\mathbf{p}'_i + \tilde{\mathbf{p}}) + \mathbf{t} - \mathbf{q}'_i - \tilde{\mathbf{q}}\|_2^2 =\end{aligned}$$

## Proof [Arun-TPAMI-87]

$$\begin{aligned}
 \mathbf{R}^*, \mathbf{t}^* &= \arg \min_{\mathbf{R} \in SO(3), \mathbf{t} \in \mathcal{R}^3} \sum_i \|\mathbf{R}\mathbf{p}_i + \mathbf{t} - \mathbf{q}_i\|_2^2 = \\
 &= \arg \min_{\mathbf{R} \in SO(3), \mathbf{t} \in \mathcal{R}^3} \sum_i \|\mathbf{R}(\mathbf{p}'_i + \tilde{\mathbf{p}}) + \mathbf{t} - \mathbf{q}'_i - \tilde{\mathbf{q}}\|_2^2 = \\
 &= \arg \min_{\mathbf{R} \in SO(3), \mathbf{t} \in \mathcal{R}^3} \sum_i \|\mathbf{R}\mathbf{p}'_i - \mathbf{q}'_i + \underbrace{\mathbf{R}\tilde{\mathbf{p}} + \mathbf{t} - \tilde{\mathbf{q}}}_{\mathbf{t}'}\|_2^2 =
 \end{aligned}$$

## Proof [Arun-TPAMI-87]

$$\begin{aligned}
 \mathbf{R}^*, \mathbf{t}^* &= \arg \min_{\mathbf{R} \in SO(3), \mathbf{t} \in \mathcal{R}^3} \sum_i \|\mathbf{R}\mathbf{p}_i + \mathbf{t} - \mathbf{q}_i\|_2^2 = \\
 &= \arg \min_{\mathbf{R} \in SO(3), \mathbf{t} \in \mathcal{R}^3} \sum_i \|\mathbf{R}(\mathbf{p}'_i + \tilde{\mathbf{p}}) + \mathbf{t} - \mathbf{q}'_i - \tilde{\mathbf{q}}\|_2^2 = \\
 &= \arg \min_{\mathbf{R} \in SO(3), \mathbf{t} \in \mathcal{R}^3} \sum_i \|\mathbf{R}\mathbf{p}'_i - \mathbf{q}'_i + \underbrace{\mathbf{R}\tilde{\mathbf{p}} + \mathbf{t} - \tilde{\mathbf{q}}}_{\mathbf{t}'}\|_2^2 = \\
 &= \arg \min_{\mathbf{R} \in SO(3), \mathbf{t} \in \mathcal{R}^3} \sum_i (\mathbf{R}\mathbf{p}'_i - \mathbf{q}'_i + \mathbf{t}')^\top (\mathbf{R}\mathbf{p}'_i - \mathbf{q}'_i + \mathbf{t}') =
 \end{aligned}$$

## Proof [Arun-TPAMI-87]

$$\begin{aligned}
 \mathbf{R}^*, \mathbf{t}^* &= \arg \min_{\mathbf{R} \in SO(3), \mathbf{t} \in \mathcal{R}^3} \sum_i \|\mathbf{R}\mathbf{p}_i + \mathbf{t} - \mathbf{q}_i\|_2^2 = \\
 &= \arg \min_{\mathbf{R} \in SO(3), \mathbf{t} \in \mathcal{R}^3} \sum_i \|\mathbf{R}(\mathbf{p}'_i + \tilde{\mathbf{p}}) + \mathbf{t} - \mathbf{q}'_i - \tilde{\mathbf{q}}\|_2^2 = \\
 &= \arg \min_{\mathbf{R} \in SO(3), \mathbf{t} \in \mathcal{R}^3} \sum_i \|\mathbf{R}\mathbf{p}'_i - \mathbf{q}'_i + \underbrace{\mathbf{R}\tilde{\mathbf{p}} + \mathbf{t} - \tilde{\mathbf{q}}}_{\mathbf{t}'}\|_2^2 = \\
 &= \arg \min_{\mathbf{R} \in SO(3), \mathbf{t} \in \mathcal{R}^3} \sum_i (\mathbf{R}\mathbf{p}'_i - \mathbf{q}'_i + \mathbf{t}')^\top (\mathbf{R}\mathbf{p}'_i - \mathbf{q}'_i + \mathbf{t}') = \\
 &= \arg \min_{\mathbf{R} \in SO(3), \mathbf{t} \in \mathcal{R}^3} \sum_i \|\mathbf{R}\mathbf{p}'_i - \mathbf{q}'_i\|_2^2 + \underbrace{\sum_i 2(\mathbf{R}\mathbf{p}'_i - \mathbf{q}'_i)\mathbf{t}'}_{=0} + \|\mathbf{t}'\|_2^2 =
 \end{aligned}$$

## Proof [Arun-TPAMI-87]

$$\begin{aligned}
\mathbf{R}^*, \mathbf{t}^* &= \arg \min_{\mathbf{R} \in SO(3), \mathbf{t} \in \mathcal{R}^3} \sum_i \|\mathbf{R}\mathbf{p}_i + \mathbf{t} - \mathbf{q}_i\|_2^2 = \\
&= \arg \min_{\mathbf{R} \in SO(3), \mathbf{t} \in \mathcal{R}^3} \sum_i \|\mathbf{R}(\mathbf{p}'_i + \tilde{\mathbf{p}}) + \mathbf{t} - \mathbf{q}'_i - \tilde{\mathbf{q}}\|_2^2 = \\
&= \arg \min_{\mathbf{R} \in SO(3), \mathbf{t} \in \mathcal{R}^3} \sum_i \|\mathbf{R}\mathbf{p}'_i - \mathbf{q}'_i + \underbrace{\mathbf{R}\tilde{\mathbf{p}} + \mathbf{t} - \tilde{\mathbf{q}}}_{\mathbf{t}'}\|_2^2 = \\
&= \arg \min_{\mathbf{R} \in SO(3), \mathbf{t} \in \mathcal{R}^3} \sum_i (\mathbf{R}\mathbf{p}'_i - \mathbf{q}'_i + \mathbf{t}')^\top (\mathbf{R}\mathbf{p}'_i - \mathbf{q}'_i + \mathbf{t}') = \\
&= \arg \min_{\mathbf{R} \in SO(3), \mathbf{t} \in \mathcal{R}^3} \sum_i \|\mathbf{R}\mathbf{p}'_i - \mathbf{q}'_i\|_2^2 + \underbrace{\sum_i 2(\mathbf{R}\mathbf{p}'_i - \mathbf{q}'_i)\mathbf{t}' + \|\mathbf{t}'\|_2^2}_{=0} = \\
&= \arg \min_{\mathbf{R} \in SO(3), \mathbf{t} \in \mathcal{R}^3} \sum_i \|\mathbf{R}\mathbf{p}'_i - \mathbf{q}'_i\|_2^2 + \|\mathbf{t}'\|_2^2
\end{aligned}$$

we can reach second term zero by  $\mathbf{t} = \tilde{\mathbf{q}} - \mathbf{R}\tilde{\mathbf{p}} = \mathbf{t}^*$

Proof [Arun-TPAMI-87]

$$= \arg \min_{\mathbf{R} \in SO(3), \mathbf{t} \in \mathcal{R}^3} \sum_i \|\mathbf{R}\mathbf{p}'_i - \mathbf{q}'_i\|_2^2 + \|\mathbf{t}'\|_2^2$$

we can reach second term zero by  $\mathbf{t} = \tilde{\mathbf{q}} - \mathbf{R}\tilde{\mathbf{p}} = \mathbf{t}^*$

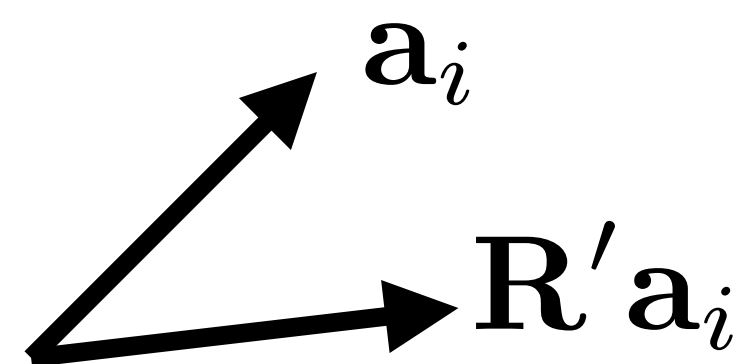
$$\arg \min_{\mathbf{R} \in SO(3)} \sum_i \|\mathbf{R}\mathbf{p}'_i - \mathbf{q}'_i\|_2^2 = \arg \max_{\mathbf{R} \in SO(3)} \sum_i \mathbf{q}'_i{}^\top \mathbf{R}\mathbf{p}'_i =$$

$$= \arg \max_{\mathbf{R} \in SO(3)} \sum_i \underbrace{\mathbf{q}'_i{}^\top}_{\mathbf{a}_i} \underbrace{\mathbf{R}\mathbf{p}'_i}_{\mathbf{b}_i} = \arg \max_{\mathbf{R} \in SO(3)} \text{trace } \mathbf{R} \underbrace{\mathbf{P}\mathbf{Q}^\top}_{\mathbf{H}} = \mathbf{V}\mathbf{U}^\top$$

$$\arg \max_{\mathbf{R}', \mathbf{R}^* \in SO(3)} \text{trace } \mathbf{R}' \mathbf{R}^* \mathbf{U}\mathbf{S}\mathbf{V}^\top \quad \dots \text{ expand into two rotations}$$

$$\arg \max_{\mathbf{R}' \in SO(3)} \text{trace } \mathbf{R}' \underbrace{\mathbf{V}\mathbf{U}^\top}_{\mathbf{R}^*} \underbrace{\mathbf{U}\mathbf{S}\mathbf{V}^\top}_{\mathbf{H}} = \arg \max_{\mathbf{R}' \in SO(3)} \text{trace } \mathbf{R}' \underbrace{(\mathbf{V}\sqrt{\mathbf{S}})}_{\mathbf{A}} \underbrace{(\sqrt{\mathbf{S}}\mathbf{V})^\top}_{\mathbf{A}^\top} =$$

$$= \arg \max_{\mathbf{R}' \in SO(3)} \sum_i \mathbf{a}_i{}^\top \mathbf{R}' \mathbf{a}_i = \mathbf{E}$$



$$\text{trace } \mathbf{B}\mathbf{A}^\top = \sum_i \mathbf{a}_i{}^\top \mathbf{b}_i$$