and substitute them into Equation 1.50 to get

$$
\begin{aligned}
\exp\,[\theta\,\vec{v}]_\times &= \mathtt{I} + \left(\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\theta^{2(n-1)}}{(2n)!}\right)[\theta\,\vec{v}]_\times^2 + \left(\sum_{n=0}^{\infty}\frac{(-1)^{n}\theta^{2n}}{(2n+1)!}\right)[\theta\,\vec{v}]_\times \\
&= \mathtt{I} + \left(\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\theta^{2n}}{(2n)!}\right)[\vec{v}]_\times^2 + \left(\sum_{n=0}^{\infty}\frac{(-1)^{n}\theta^{2n+1}}{(2n+1)!}\right)[\vec{v}]_\times \\
&= \mathtt{I} - \left(\sum_{n=0}^{\infty}\frac{(-1)^{n}\theta^{2n}}{(2n)!} - 1\right)[\vec{v}]_\times^2 + \sin\theta\,[\vec{v}]_\times \\
&= \mathtt{I} - (\cos\theta - 1)\,[\vec{v}]_\times^2 + \sin\theta\,[\vec{v}]_\times \\
&= \mathtt{I} + \sin\theta\,[\vec{v}]_\times + (1-\cos\theta)\,[\vec{v}]_\times^2 \\
&= \mathtt{I} + \sin\|\vec{e}\|\left[\frac{\vec{e}}{\|\vec{e}\|}\right]_\times + (1-\cos\|\vec{e}\|)\left[\frac{\vec{e}}{\|\vec{e}\|}\right]_\times^2 \\
&= \mathtt{R}([\theta,\vec{v}])
\end{aligned}
\tag{1.54}
$$

by the comparison with Equation 1.25.

1.3 Quaternion representation of rotation

1.3.1 Quaternion parameterization

We shall now introduce another parameterization of $\mathtt{R}$ by four numbers but this time we will not use goniometric functions but polynomials only. We shall see later that this parameterization has other useful properties.

This paramterization is known as *unit quaternion* parameterization of rotations since rotations are represented by unit vectors from $\mathbb{R}^4$. In general, it may sense to talk about non-unit quaternions and we will see how to use them later when applying rotations represented by unit quaternions on points represented by non-unit quaternions. To simplify

$\mathbb{R}$ $a, b, c > 0$ $\quad \cdot_{\mathbb{R}}$

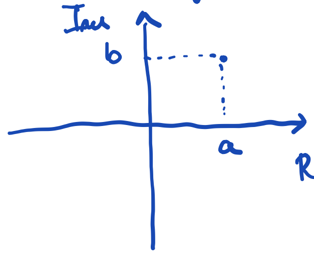
$a < b \Rightarrow ac < bc$

$\mathbb{R}^2$, $\cdot_{\mathbb{C}}$ ... NO SO NICE ORDERING

$\cancel{\mathbb{R}^3}$, $\cdot_{T}$ ... ? nice mult ?

$\mathbb{R}^4$, $\cdot_{Q}$ ... 4 (quaternions)

Generalize complex numbers to higher dimensions



$a + bi \equiv \mathbb{R}^2 \quad (a, b)$

$(a+bi)(c+di) =$
$= ac + adi + bci + bdi^2$
$= ac - bd + (ad+bc)i$

$(a,b)\cdot_{\mathbb{C}}(c,d) = (ac-bd,\ ad+bc)$

$\left.\begin{array}{l} +_{\mathbb{R}} = +_{\mathbb{C}} \\ \cdot_{\mathbb{R}} \end{array}\right\} \mathbb{R}^2$

$\cdot_{\mathbb{C}} : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$

our notation, we will often write “quaternions” insted of more correct “unit quaternions”.

Let us do a seemingly unnecessary trick. We will pass from θ to $\frac{\theta}{2}$ and introduce

$$\underset{\in\mathbb{R}^4}{\vec{q}} = \begin{bmatrix} \cos\frac{\theta}{2} \\ \vec{v}\sin\frac{\theta}{2} \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} \cos\frac{\theta}{2} \\ v_1\sin\frac{\theta}{2} \\ v_2\sin\frac{\theta}{2} \\ v_3\sin\frac{\theta}{2} \end{bmatrix} \tag{1.55}$$

$\vec{v}$, θ ↓ $\vec{v}$, $\frac{\theta}{2}$

There still holds

$$\|\vec{q}\|^2 = q_1^2+q_2^2+q_3^2+q_4^2 = \cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}\,v_1^2+\sin^2\frac{\theta}{2}\,v_2^2+\sin^2\frac{\theta}{2}\,v_3^2 = \cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2} = 1 \tag{1.56}$$

true. We can verify that the following identities

angle & axis

elegans

$$\cos\theta = 2\cos^2\frac{\theta}{2} - 1 = 2\,q_1^2 - 1 \tag{1.57}$$

$$\sin\theta = 2\cos\frac{\theta}{2}\,\sin\frac{\theta}{2} \tag{1.58}$$

$$\sin\theta\,\vec{v} = 2\cos\frac{\theta}{2}\,\sin\frac{\theta}{2}\,\vec{v} = 2\,q_1\,[q_2 \quad q_3 \quad q_4]^\top \tag{1.59}$$

$$\cos\theta = 1 - 2\sin^2\frac{\theta}{2} = 1 - 2\,(q_2^2+q_3^2+q_4^2) = q_1^2 - q_2^2 - q_3^2 - q_4^2 \tag{1.60}$$

$$1 - \cos\theta = 2\sin^2\frac{\theta}{2} = 2\,(q_2^2+q_3^2+q_4^2) \tag{1.61}$$

Q e law

i ... variable ≡ $ac-bd + (ad+bc)i = r(i)$

$(a+bi)(c+di) \overset{?}{=} ac + adi + bci + bd\,i^2$ $\quad \frac{(i^2+1)}{g(i)}$, $i^2 \to -1$

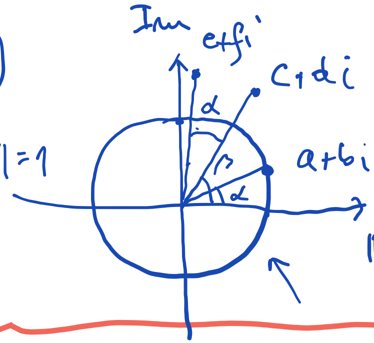
$f(i) : g(i) \equiv f(i) = q(i)\cdot g(i) + r(i)$ $\quad$ ↑

$\deg r(i) < \deg g(i)$, $\deg g(i) \geq 0$

$SO(3)$

$\| \|$

$\mathbb{R}^4$, $\| \| = 1$



$\|a+bi\| = 1$, $a^2+b^2=1$

$(a+bi)(c+di) =$

$= (e+fi)$ $\quad \mathbb{C} \longrightarrow \mathbb{Q}$, i → $\vec{i}, \vec{j}, \vec{k}$, $i, j, \ldots$

$\measuredangle(e+fi) = \alpha+\beta$

$\|e+fi\| = 1\cdot\|c+di\|$

$a+bi$ $(a^2+b^2=1)$

act as a rotation on $\mathbb{R}^2 \equiv \mathbb{C}$

$(a,b)(c,d) = (ac-bd, ad+bc)$

$$\boxed{\begin{bmatrix} a & -b \\ b & c \end{bmatrix}} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} ac-bd \\ bc+cd \end{bmatrix}$$

hold true. We can now substitute the above into Equation 1.23 to get

$$\begin{aligned}
\mathtt{R} &= \mathtt{I} + \sin\theta\,[\vec{v}]_\times + (1-\cos\theta)\,[\vec{v}]_\times^2 \quad \longleftarrow \text{angle-axis} && (1.62)\\
&= \mathtt{I} + 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}\,[\vec{v}]_\times + 2\sin^2\frac{\theta}{2}\,[\vec{v}]_\times^2 && (1.63)\\
&= \mathtt{I} + 2\cos\frac{\theta}{2}\left[\sin\frac{\theta}{2}\vec{v}\right]_\times + 2\left[\sin\frac{\theta}{2}\vec{v}\right]_\times^2 && (1.64)\\
&= \mathtt{I} + 2\cos\frac{\theta}{2}\left[\sin\frac{\theta}{2}\vec{v}\right]_\times + 2\left(\left[\sin\frac{\theta}{2}\vec{v}\right]_{\|} - \mathtt{I}\right) && (1.65)\\
&= \mathtt{I} + 2q_1\left[\begin{bmatrix} q_2\\ q_3\\ q_4\end{bmatrix}\right]_\times + 2\left(\left[\begin{bmatrix} q_2\\ q_3\\ q_4\end{bmatrix}\right]_{\|} - \mathtt{I}\right) && (1.66)\\
&= \begin{bmatrix} 1 & -2\,q_1q_4 & 2\,q_1q_3\\ 2\,q_1q_4 & 1 & -2\,q_1q_2\\ -2\,q_1q_3 & 2\,q_1q_2 & 1\end{bmatrix} + \begin{bmatrix} 2\,q_2q_2-2 & 2\,q_2q_3 & 2\,q_2q_4\\ 2\,q_3q_2 & 2\,q_3q_3-2 & 2\,q_3q_4\\ 2\,q_4q_2 & 2\,q_4q_3 & 2\,q_4q_4-2\end{bmatrix}\\
\mathtt{R} &= \begin{bmatrix} q_1^2+q_2^2-q_3^2-q_4^2 & 2\,(q_2q_3-q_1q_4) & 2\,(q_2q_4+q_1q_3)\\ 2\,(q_2q_3+q_1q_4) & q_1^2-q_2^2+q_3^2-q_4^2 & 2\,(q_3q_4-q_1q_2)\\ 2\,(q_2q_4-q_1q_3) & 2\,(q_3q_4+q_1q_2) & q_1^2-q_2^2-q_3^2+q_4^2\end{bmatrix} && (1.67)
\end{aligned}$$

def

gonio → poly

which uses only second order polynomials in elements of $\vec{q}$.

$\frac{1}{\|\vec{q}\|}$

1.3.2 Computing quaternions from R

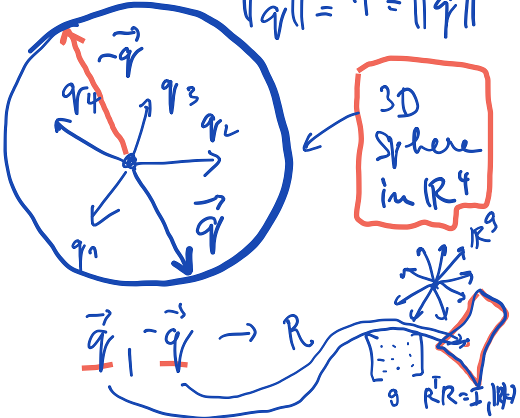
To get the quaternions representing a rotation matrix R, we start with Equation 1.64. Let us first confine θ to the real interval $(-\pi, \pi]$ as we did for the angle-axis parameterization.

Matrix R either is or it is not symmetric.

If R is symmetric, then either $\sin\theta/2\,\vec{v} = \vec{0}$ or $\cos\theta/2 = 0$. If $\sin\theta/2\,\vec{v} = \vec{0}$, then $\sin\theta/2 = 0$ since $\|\vec{v}\| = 1$ and thus $\cos\theta/2 = \pm 1$. However,

$Q \equiv \mathbb{R}^4$

$\|\vec{q}\| = 1 = \|\vec{q}\|^2$



3D sphere in $\mathbb{R}^4$

$\vec{q}, -\vec{q} \rightarrow R$

$\mathbb{R}^9$

$R^TR = I, |R|$

Unit quaternions $\equiv SO(3)$

$q_1^2 + q_2^2 + q_3^2 + q_4^2 = \|\vec{q}\|^2 = 1$

Polynomials deg 2

all monomials deg 2

→ homogeneous polynomials deg 2

$R(\vec{q}) = R(-\vec{q})$

$\cos\theta/2 = -1$ for no $\theta \in (-\pi, \pi]$ and hence $\cos\theta/2 = 1$. This corresponds to $\theta = 0$ and hence to $\mathtt{R} = \mathtt{I}$ which is thus represented by quaternion

$$\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^\top \tag{1.68}$$

If $\cos\theta/2 = 0$, then $\sin\theta/2 = \pm 1$ but $\sin\theta/2 = -1$ for no $\theta \in (-\pi, \pi]$ and hence $\sin\theta/2 = 1$. This corresponds to the rotation the by $\theta = \pi$ around the axis given by unit $\vec{v} = [v_1, v_2, v_3]^\top$. This rotation is thus represented by quaternion

$$\begin{bmatrix} 0 & v_1 & v_2 & v_3 \end{bmatrix}^\top \tag{1.69}$$

Notice that $\vec{v}$ and $-\vec{v}$ generate the same rotation matrix $\mathtt{R}$ and hence every rotation by $\theta = \pi$ is represented by two quaternions.

If $\mathtt{R}$ is not symmetric, then $\mathtt{R} - \mathtt{R}^\top \neq \mathtt{0}$ and hence we are geting a useful relationship

$$\mathtt{R} - \mathtt{R}^\top \quad = \quad 4\cos\frac{\theta}{2}\left[\sin\frac{\theta}{2}\vec{v}\right]_\times \tag{1.70}$$

and next continue with writing

$$\cos^2\frac{\theta}{2} = 1-\sin^2\frac{\theta}{2} = 1-\frac{1}{2}(1-\cos\theta) = 1-\frac{1}{2}\left(1-\frac{1}{2}(\operatorname{trace}\mathtt{R}-1)\right) = \frac{1}{4}(1+\operatorname{trace}\mathtt{R}) \tag{1.71}$$

using $\operatorname{trace}\mathtt{R}$, and thus

$$q_1 = \cos\frac{\theta}{2} = \frac{s}{2}\sqrt{\operatorname{trace}\mathtt{R}+1} \tag{1.72}$$

with $s = \pm 1$. We can form equation

$$\begin{bmatrix} 0 & r_{12}-r_{21} & r_{13}-r_{31} \\ r_{21}-r_{12} & 0 & r_{23}-r_{32} \\ r_{31}-r_{13} & r_{32}-r_{23} & 0 \end{bmatrix} = \left[\begin{bmatrix} r_{32}-r_{23} \\ r_{13}-r_{31} \\ r_{21}-r_{12} \end{bmatrix}\right]_\times = s\sqrt{\operatorname{trace}\mathtt{R}+1}\left[\begin{bmatrix} q_2 \\ q_3 \\ q_4 \end{bmatrix}\right]_\times \tag{1.73}$$

which gives the following two quaternions

$$\frac{+1}{2\sqrt{\operatorname{trace} R + 1}}\begin{bmatrix}\operatorname{trace} R + 1\\ r_{32} - r_{23}\\ r_{13} - r_{31}\\ r_{21} - r_{12}\end{bmatrix}, \quad \frac{-1}{2\sqrt{\operatorname{trace} R + 1}}\begin{bmatrix}\operatorname{trace} R + 1\\ r_{32} - r_{23}\\ r_{13} - r_{31}\\ r_{21} - r_{12}\end{bmatrix} \tag{1.74}$$

which represent the same rotation as R.

We see that all rotations are represented by the above by two quaternions $\vec{q}$ and $-\vec{q}$ except for the identity, which is represented by exactly one quaternion.

The quaternion representation of rotation presented above represents every rotation by a finite number of quaternions whereas angle-axis repesentation allowed for an infinite number of angle-axis pairs to correspond to the indentity. Yet, even this still has an "aesthetic flaw" at the identity, which has only one quaternion whereas all other rotations have two quaternions. The "flaw" can be removed by realizing that $\vec{q} = [-1, 0, 0, 0]^\top$ also maps to the identity. However, if we look for θ that corresponds to $\cos\theta/2 = -1$ we see that such $\theta/2 = \pm k\,\pi$ and hence $\theta = \pm 2\,k\,\pi$ for $k = 1, 2, \ldots$, which are points isolated from $(-\pi, \pi]$. Now, if we allow θ to be in interval $(-2\,\pi, +2\,\pi]$, then the set

$$\left\{\left[\begin{array}{c}\cos\theta/2\\ \vec{v}\,\sin\theta/2\end{array}\right] \;\middle|\; \theta\in[-2\,\pi, +2\,\pi],\, \vec{v}\in\mathbb{R}^3,\, \|\vec{v}\| = 1\right\} \tag{1.75}$$

of quaternions contains exactly two quaternions for every rotation matrix R and is obtained by a continuous mapping of a closed interval of angles, which is boundend, times a sphere in $\mathbb{R}^3$, which is also closed and bounded.

1.3.3 Quaternion composition

Consider two rotations represented by $\vec{q}_1$ and $\vec{q}_2$. The respective rotation matrices R_1, R_2 can be composed into rotation matrix $R_{21} = R_2\,R_1$, which

$$\begin{array}{ccccc} R_1 & \cdot & R_2 & = & R_3\\ \updownarrow & & \updownarrow & & \updownarrow\\ q_1 & \cdot_Q & q_2 & = & q_3\end{array}$$

1) special cases

2) a guess

3) verify by computation (long) → Maple

can be represented by $\vec{q}_{21}$. Let us investigate how to obtain $\vec{q}_{21}$ from $\vec{q}_1$ and $\vec{q}_2$. We shall use Equation 1.76 to relate $\mathtt{R}_1$ to $\vec{q}_1$ and $\mathtt{R}_2$ to $\vec{q}_1$, then evaluate $\mathtt{R}_{21} = \mathtt{R}_2\,\mathtt{R}_1$ and recover $\vec{q}_{21}$ from $\mathtt{R}_{21}$. We use Equation 1.23 to write

$$\mathtt{R} = 2\,\sin^2\frac{\theta}{2}\,\vec{v}\,\vec{v}^\top + (2\,\cos^2\frac{\theta}{2} - 1)\,\mathtt{I} + 2\,\cos\frac{\theta}{2}\,\sin\frac{\theta}{2}\,[\vec{v}]_\times \tag{1.76}$$

and

$$\mathtt{R}_1 = 2\,(s_1\vec{v}_1)\,(s_1\vec{v}_1)^\top + (2\,c_1^2 - 1)\,\mathtt{I} + 2\,c_1\,[s_1\vec{v}_1]_\times \tag{1.77}$$

$$\mathtt{R}_2 = 2\,(s_2\vec{v}_2)\,(s_2\vec{v}_2)^\top + (2\,c_2^2 - 1)\,\mathtt{I} + 2\,c_2\,[s_2\vec{v}_2]_\times \tag{1.78}$$

$$\mathtt{R}_{21} = 2\,(s_{21}\vec{v}_{21})\,(s_{21}\vec{v}_{21})^\top + (2\,c_{21}^2 - 1)\,\mathtt{I} + 2\,c_{21}\,[s_{21}\vec{v}_{21}]_\times$$

with shortcuts

$$c_1 = \cos\frac{\theta_1}{2},\ s_1 = \sin\frac{\theta_1}{2},\ c_2 = \cos\frac{\theta_2}{2},\ s_2 = \sin\frac{\theta_2}{2},\ c_{21} = \cos\frac{\theta_{21}}{2},\ s_{21} = \sin\frac{\theta_{21}}{2}$$

Let us next assume that both $\mathtt{R}_1$, $\mathtt{R}_2$ are not identities. Then $\theta_1 \neq 0$ and $\theta_2 \neq 0$ and rotation axes $\vec{v}_1 \neq \vec{0}$, $\vec{v}_2 \neq \vec{0}$ are well defined. We can now distinguish two cases. Either $\vec{v}_1 = \pm\vec{v}_2$, and then $\vec{v}_{21} = \vec{v}_1 = \pm\vec{v}_2$, or $\vec{v}_1 \neq \pm\vec{v}_2$, and then

$$[\vec{v}_1,\, \vec{v}_2,\, \vec{v}_2 \times \vec{v}_1] \tag{1.79}$$

forms a basis of $\mathbb{R}^3$. We also notice that $\vec{v}_1$, $\vec{v}_2$ always appear in $\mathtt{R}_1$, $\mathtt{R}_2$ in the product with s_1, s_2.

We can thus write

$$\sin\frac{\theta_{21}}{2}\,\vec{v}_{21} = a_1\sin\frac{\theta_1}{2}\,\vec{v}_1 + a_2\sin\frac{\theta_2}{2}\,\vec{v}_2 + a_3\,(\sin\frac{\theta_2}{2}\,\vec{v}_2 \times \sin\frac{\theta_1}{2}\,\vec{v}_1) \tag{1.80}$$

with coefficients $a_1, a_2, a_3 \in \mathbb{R}$. To find coefficients a_1, a_2, a_3, we will consider the following special situations:

1. $\vec{v}_1 = \pm\vec{v}_2$ implies $\vec{v}_{21} = \vec{v}_1 = \pm\vec{v}_2$ and $\theta_{21} = \theta_1 \pm \theta_2$ for all real θ_1 and θ_2.

2. $\vec{v}_2^\top \vec{v}_1 = 0$ and $\theta_1 = \theta_2 = \pi$ implies

$$\begin{aligned} \mathtt{R}_1 &= 2\,\vec{v}_1\vec{v}_1^\top - \mathtt{I} && (1.81)\\ \mathtt{R}_2 &= 2\,\vec{v}_2\vec{v}_2^\top - \mathtt{I} && (1.82)\\ \mathtt{R}_{21} &= (2\,\vec{v}_2\vec{v}_2^\top - \mathtt{I})(2\,\vec{v}_1\vec{v}_1^\top - \mathtt{I}) = \mathtt{I} - 2\,(\vec{v}_2\vec{v}_2^\top + \vec{v}_1\vec{v}_1^\top) && (1.83) \end{aligned}$$

We see that in the former case we are getting

$$\sin\frac{\theta_{21}}{2}\,\vec{v}_1 = (a_1\sin\frac{\theta_1}{2} + a_2\sin\frac{\theta_2}{2})\,\vec{v}_1 \quad \text{for all } \theta_1, \theta_2 \in \mathbb{R} \tag{1.84}$$

which for $\vec{v}_1 \neq \vec{0}$ leads to

$$\begin{aligned} \sin\frac{\theta_{21}}{2} &= a_1\sin\frac{\theta_1}{2} + a_2\sin\frac{\theta_2}{2} && (1.85)\\ \sin\frac{\theta_1+\theta_2}{2} &= a_1\sin\frac{\theta_1}{2} + a_2\sin\frac{\theta_2}{2} && (1.86)\\ \sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2} + \cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2} &= a_1\sin\frac{\theta_1}{2} + a_2\sin\frac{\theta_2}{2} && (1.87) \end{aligned}$$

for all $\theta_1, \theta_2 \in \mathbb{R}$. But that means that

$$a_1 = \cos\frac{\theta_2}{2} \quad \text{and} \quad a_2 = \cos\frac{\theta_1}{2} \tag{1.88}$$

In the latter case we find that $\vec{v}_{21}$ is a non-zero multiple of $\vec{v}_2 \times \vec{v}_1$ since

$$\begin{aligned} \mathtt{R}_{21}\,(\vec{v}_2 \times \vec{v}_1) &= (\mathtt{I} - 2\,(\vec{v}_2\vec{v}_2^\top + \vec{v}_1\vec{v}_1^\top))\,(\vec{v}_2 \times \vec{v}_1) && (1.89)\\ &= \vec{v}_2 \times \vec{v}_1 - 2\,\vec{v}_2\vec{v}_2^\top\,(\vec{v}_2 \times \vec{v}_1) - 2\,\vec{v}_1\vec{v}_1^\top\,(\vec{v}_2 \times \vec{v}_1) && (1.90)\\ &= \vec{v}_2 \times \vec{v}_1 && (1.91) \end{aligned}$$

But that means that

$$\sin\frac{\theta_{21}}{2}\,\vec{v}_{21} = a_3\,(\sin\frac{\theta_2}{2}\vec{v}_2 \times \vec{v}_1 \sin\frac{\theta_1}{2}) \tag{1.92}$$

We next get θ_{21} using Equation **??** as

$$\cos\theta_{21} = \frac{1}{2}(\operatorname{trace}\mathtt{R} - 1) = \frac{1}{2}(3 - 2\left(\|\vec{v}_2\|^2 + \|\vec{v}_1\|^2\right) - 1) = \frac{1}{2}(3-4-1) = -1 \tag{1.93}$$

and hence $\theta_{21} = \pm\pi$ and thus

$$\vec{v}_{21} = a_3\left(\vec{v}_1 \times \vec{v}_2\right) \tag{1.94}$$

but since $\vec{v}_1$ is perpendicular to $\vec{v}_2$, $\vec{v}_1 \times \vec{v}_2$ is a unit vector and thus $a_3 = 1$.
We can thus hypothesize that in general

$$\sin\frac{\theta_{21}}{2}\,\vec{v}_{21} = \cos\frac{\theta_2}{2}\left(\sin\frac{\theta_1}{2}\,\vec{v}_1\right) + \cos\frac{\theta_1}{2}\left(\sin\frac{\theta_2}{2}\,\vec{v}_2\right) + \left(\sin\frac{\theta_2}{2}\,\vec{v}_2\right)\times\left(\sin\frac{\theta_1}{2}\,\vec{v}_1\right) \tag{1.95}$$

Let's next find $\cos\frac{\theta_{21}}{2}$ consistent with the above hypothesis. We see that

$$\cos^2\frac{\theta_{21}}{2} = 1 - \sin^2\frac{\theta_{21}}{2} \tag{1.96}$$

and hence we evaluate

$$\begin{aligned}
\sin^2\frac{\theta_{21}}{2} &= \sin^2\frac{\theta_{21}}{2}\,\vec{v}_{21}^\top\vec{v}_{21} = \left(\sin\frac{\theta_{21}}{2}\,\vec{v}_{21}\right)^\top\left(\sin\frac{\theta_{21}}{2}\,\vec{v}_{21}\right) && (1.97)\\
&= \cos^2\frac{\theta_2}{2}\,\sin^2\frac{\theta_1}{2} + \cos^2\frac{\theta_1}{2}\,\sin^2\frac{\theta_2}{2} && (1.98)\\
&+ 2\cos\frac{\theta_2}{2}\cos\frac{\theta_1}{2}\left(\sin\frac{\theta_2}{2}\,\vec{v}_2\right)^\top\left(\sin\frac{\theta_1}{2}\,\vec{v}_1\right) && (1.99)\\
&+ \left[\left(\sin\frac{\theta_2}{2}\,\vec{v}_2\right)\times\left(\sin\frac{\theta_1}{2}\,\vec{v}_1\right)\right]^\top\left[\left(\sin\frac{\theta_2}{2}\,\vec{v}_2\right)\times\left(\sin\frac{\theta_1}{2}\,\vec{v}_1\right)\right] && (1.100)
\end{aligned}$$

We used the fact that $\vec{v}_1$, $\vec{v}_2$ are perpendicular to their vector product.

To move further, we will use that for every two unit vectors $\vec{u}$, $\vec{v}$ in $\mathbb{R}^3$ there holds

$$\begin{aligned}
(\vec{u}\times\vec{v})^\top(\vec{u}\times\vec{v}) &= \|(\vec{u}\times\vec{v})\|^2 = \|\vec{u}\|^2\|\vec{v}\|^2\sin^2\angle(\vec{u},\vec{v}) && (1.101)\\
&= \|\vec{u}\|^2\|\vec{v}\|^2(1-\cos^2\angle(\vec{u},\vec{v})) = \|\vec{u}\|^2\|\vec{v}\|^2 - (\vec{u}^\top\vec{v})^2 && (1.102)
\end{aligned}$$

true.

Applying this to the last summand in Equation 1.100, we get

$$
\begin{aligned}
\sin^2\frac{\theta_{21}}{2} &= \cos^2\frac{\theta_2}{2}\,\sin^2\frac{\theta_1}{2} + \cos^2\frac{\theta_1}{2}\,\sin^2\frac{\theta_2}{2} && (1.103)\\
&+ 2\cos\frac{\theta_2}{2}\,\cos\frac{\theta_1}{2}\,\left(\sin\frac{\theta_2}{2}\,\vec{v}_2\right)^{\top}\left(\sin\frac{\theta_1}{2}\,\vec{v}_1\right) && (1.104)\\
&+ \sin^2\frac{\theta_2}{2}\,\sin^2\frac{\theta_1}{2} - \left[\left(\sin\frac{\theta_2}{2}\,\vec{v}_2\right)^{\top}\left(\sin\frac{\theta_1}{2}\,\vec{v}_1\right)\right]^2 && (1.105)\\
&= \sin^2\frac{\theta_1}{2} + \cos^2\frac{\theta_1}{2}\,\sin^2\frac{\theta_2}{2} && (1.106)\\
&+ 2\cos\frac{\theta_2}{2}\,\cos\frac{\theta_1}{2}\,\left(\sin\frac{\theta_2}{2}\,\vec{v}_2\right)^{\top}\left(\sin\frac{\theta_1}{2}\,\vec{v}_1\right) - \left[\left(\sin\frac{\theta_2}{2}\,\vec{v}_2\right)^{\top}\left(\sin\frac{\theta_1}{2}\,\vec{v}_1\right)\right]^2\\
&= 1 - \cos^2\frac{\theta_1}{2}\,\cos^2\frac{\theta_2}{2} && (1.107)\\
&+ 2\cos\frac{\theta_2}{2}\,\cos\frac{\theta_1}{2}\,\left(\sin\frac{\theta_2}{2}\,\vec{v}_2\right)^{\top}\left(\sin\frac{\theta_1}{2}\,\vec{v}_1\right) - \left[\left(\sin\frac{\theta_2}{2}\,\vec{v}_2\right)^{\top}\left(\sin\frac{\theta_1}{2}\,\vec{v}_1\right)\right]^2
\end{aligned}
$$

where we used the fact that

$$
\begin{aligned}
\sin^2\frac{\theta_1}{2} + \cos^2\frac{\theta_1}{2}\,\sin^2\frac{\theta_2}{2} &= 1 - \cos^2\frac{\theta_1}{2} + \cos^2\frac{\theta_1}{2}\,\sin^2\frac{\theta_2}{2} && (1.108)\\
&= 1 + \cos^2\frac{\theta_1}{2}\,\left(\sin^2\frac{\theta_2}{2} - 1\right) = 1 - \cos^2\frac{\theta_1}{2}\,\cos^2\frac{\theta_2}{2}
\end{aligned}
$$

We are thus obtaining

complicated

$$\begin{aligned}
\cos^2\frac{\theta_{21}}{2} &= 1-\sin^2\frac{\theta_{21}}{2} && (1.109)\\
&= \cos^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2} && (1.110)\\
&\quad - 2\cos\frac{\theta_2}{2}\cos\frac{\theta_1}{2}\left(\sin\frac{\theta_2}{2}\vec{v}_2\right)^\top\left(\sin\frac{\theta_1}{2}\vec{v}_1\right)+\left[\left(\sin\frac{\theta_2}{2}\vec{v}_2\right)^\top\left(\sin\frac{\theta_1}{2}\vec{v}_1\right)\right]^2\\
&= \left(\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}-\left(\sin\frac{\theta_2}{2}\vec{v}_2\right)^\top\left(\sin\frac{\theta_1}{2}\vec{v}_1\right)\right)^2 && (1.111)
\end{aligned}$$

Our complete hypothesis will be

$$\begin{aligned}
\sin\frac{\theta_{21}}{2}\vec{v}_{21} &= \cos\frac{\theta_2}{2}\left(\sin\frac{\theta_1}{2}\vec{v}_1\right)+\cos\frac{\theta_1}{2}\left(\sin\frac{\theta_2}{2}\vec{v}_2\right)+\left(\sin\frac{\theta_2}{2}\vec{v}_2\right)\times\left(\sin\frac{\theta_1}{2}\vec{v}_1\right)\\
\cos\frac{\theta_{21}}{2} &= \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}-\left(\sin\frac{\theta_2}{2}\vec{v}_2\right)^\top\left(\sin\frac{\theta_1}{2}\vec{v}_1\right) && (1.112)
\end{aligned}$$

$$\vec{q}_3 = \begin{bmatrix} \cos\frac{\theta_{21}}{2} \\ \sin\frac{\theta_{21}}{2}\vec{v}_{21} \end{bmatrix}$$

$$\vec{q}_{21} = \vec{q}_2 \cdot_{Q} \vec{q}_1$$

To verify this, we will run the following Maple [3] program

```
> restart:
> with(LinearAlgebra):
> E:=IdentityMatrix(3):
> X_:=proc(u) <<0|-u[3]|u[2]>,<u[3]|0|-u[1]>,<-u[2]|u[1]|0>>
end proc:
> v1:=<x1,y1,z1>:
> v2:=<x2,y2,z2>:
> R1:=2*(s1*v1).Transpose(s1*v1)+(2*c1^2-1)*E+2*c1*X_(s1*v1):
> R2:=2*(s2*v2).Transpose(s2*v2)+(2*c2^2-1)*E+2*c2*X_(s2*v2):
> R21:=expand~(R2.R1):
```

Verification by direct computation Maple

```
> c21:=c2*c1-Transpose(s2*v2).(s1*v1);
```

$$c21 := c2\,c1 - s1\,x1\,s2\,x2 - s1\,y1\,s2\,y2 - s1\,z1\,s2\,z2$$

```
> s21v21:=c2*s1*v1+s2*c1*v2+X_(s2*v2).(s1*v1);
```

$$s21v21 := \begin{bmatrix} c2\,s1\,x1 + s2\,c1\,x2 - s2\,z2\,s1\,y1 + s2\,y2\,s1\,z1 \\ c2\,s1\,y1 + s2\,c1\,y2 + s2\,z2\,s1\,x1 - s2\,x2\,s1\,z1 \\ c2\,s1\,z1 + s2\,c1\,z2 - s2\,y2\,s1\,x1 + s2\,x2\,s1\,y1 \end{bmatrix}$$

```
> RR21:=2*s21v21.Transpose(s21v21)+(2*c21^2-1)*E+2*c21*X_(s21v21):
```

angle - axis

quaternion

```
> simplify(expand~(RR21-R21),[x1^2+y1^2+z1^2=1,x2^2+y2^2+z2^2=1,
                                c1^2+s1^2=1,c2^2+s2^2=1]);
```

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which verifies that our hypothesis was correct.

Considering two unit quaternions

$$\vec{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix}, \quad \text{and} \quad \vec{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} \tag{1.113}$$

we can now give their composition as

$$\vec{q}_{21} = \vec{q}\,\vec{p} = \underbrace{\begin{bmatrix} q_1 p_1 - q_2 p_2 - q_3 p_3 - q_4 p_4 \\ q_1 p_2 + q_2 p_1 + q_3 p_4 - q_4 p_3 \\ q_1 p_3 + q_3 p_1 + q_4 p_2 - q_2 p_4 \\ q_1 p_4 + q_4 p_1 + q_2 p_3 - q_3 p_2 \end{bmatrix}}_{4\times 1} \tag{1.114}$$

unit

$$= \begin{bmatrix} q_1 p_1 - q_2 p_2 - q_3 p_3 - q_4 p_4 \\ q_2 p_1 + q_1 p_2 - q_4 p_3 + q_3 p_4 \\ q_3 p_1 + q_4 p_2 + q_1 p_3 - q_2 p_4 \\ q_4 p_1 - q_3 p_2 + q_2 p_3 + q_1 p_4 \end{bmatrix} \tag{1.115}$$

$$= \underbrace{\begin{bmatrix} q_1 & -q_2 & -q_3 & -q_4 \\ q_2 & q_1 & -q_4 & q_3 \\ q_3 & q_4 & q_1 & -q_2 \\ q_4 & -q_3 & q_2 & q_1 \end{bmatrix}}_{\tilde{\vec{q}}} \underbrace{\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix}}_{\vec{p}} \tag{1.116}$$

$(a+bi)(c+di) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix}$

$= aI + \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix}$

$\vec{q} = \begin{bmatrix} q_1 \\ \vec{w} \end{bmatrix} = \begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\,\vec{v} \end{bmatrix}$

$\underset{\sim}{\cdot} \quad \underset{\sim}{\times}$

$\begin{bmatrix} q_1 & -\vec{w}^\top \\ \vec{w} & \underbrace{\boxed{q_1 I + [\vec{w}]_\times}}_{3\times 3} \end{bmatrix} = q_1 I + \begin{bmatrix} 0 & -\vec{w}^\top \\ \vec{w} & [\vec{w}]_\times \end{bmatrix}$

1.3.4 Application of quaternions to vectors

Consider a rotation by angle θ around an axis with direection $\vec{v}$ represented by a unit quaternion $\vec{q} = \left[\cos\frac{\theta}{2} \quad \sin\frac{\theta}{2}\,\vec{v}\right]$ and a vector $\vec{x} \in \mathbb{R}^3$. To rotate the vector, we may construct the rotation matrix $\mathtt{R}(\vec{q})$ and apply it to the vector $\vec{x}$ as $\mathtt{R}(\vec{q})\,\vec{x}$.

Interestingly enough, it is possible to accomplish this in somewhat different and more efficient way by first "embedding" vector $\vec{x}$ into a (non-unit!) quaternion

$$\vec{p}(\vec{x}) = \begin{bmatrix} 0 \\ \vec{x} \end{bmatrix} = \begin{bmatrix} 0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{1.117}$$

and then composing it with quaternion $\vec{q}$ from both sides

$$\boxed{\vec{q}\,\vec{p}(\vec{x})\,\vec{q}^{-1}} = \begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\,\vec{v} \end{bmatrix}\begin{bmatrix} 0 \\ \vec{x} \end{bmatrix}\begin{bmatrix} \cos\frac{\theta}{2} \\ -\sin\frac{\theta}{2}\,\vec{v} \end{bmatrix} \tag{1.118}$$

$\frac{\theta}{2} \to -\frac{\theta}{2}$

$\vec{x} \in \mathbb{R}^3, \quad R \in SO(3)$

$R\vec{x}$

$\vec{q}, \quad \|\vec{q}\| = 1$

✓ $\vec{q}(\vec{x}) \longrightarrow R(\vec{q})\,\vec{x}$

✓ $\vec{p}(\vec{x}) \in \mathbb{R}^4 \longrightarrow \begin{bmatrix} 0 \\ \vec{x} \end{bmatrix}$

$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}^{-1} = \begin{bmatrix} q_1 \\ -q_2 \\ -q_3 \\ -q_4 \end{bmatrix} \vdots$

One can verify that the following

$$\begin{bmatrix} 0 \\ \mathtt{R}(\vec{q})\,\vec{x} \end{bmatrix} = \vec{q}\,\vec{p}(\vec{x})\,\vec{q}^{-1} \tag{1.119}$$

holds true.

1.4 "Cayley transform" parameterization

We see that unit quaternions provide a nice parameterization. It is given as a matrix with polynomial entries of four parameters. However, unit quaternions still are somewhat redundant since every rotation is represented twice.

Let us now mention yet another classical rotation parameterization, which is known as "Cayley transform". This parameterization uses only three parameters to represent three-dimensional rotations. In a sense, it is as ecconomic as it can be. On the other hand, it can't represent rotations by 180°.

$\mathbb{Q}$ 2:1

Actually, it can be proven [4] that there is no mapping (parameterization), which could be (i) continuous, (ii) one-to-one, (iii) onto, and (iv) three-dimensional (i.e. mapping a "three-dimensional box" onto all three-dimensional rotations).

Axis-angle parameterization is continuous and onto but not one-to-one and not three-dimensional. Euler vector parameterization is continuous, onto, three-dimensional but not one-to one. Unit quaternions are continuous, onto but not three-dimensional and not one-to one (although they are close to that by being two-to-one). Finally, Cayley transform parameterization is continuous, one-to-one, three-dimensional but it not onto.

In addition, unit quaternions and Cayley transform parameterizations are "finite" in the sense that they are polynomial rational functions of their parameters while other above mentioned representations require

Exact computations

HW ... $\subset \mathbb{Q}$ rational numbers

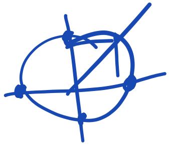
$\sqrt{2} \equiv +$root of $x^2 = 2 < \begin{matrix} \sqrt{2} \\ -\sqrt{2} \end{matrix}$

R in a computer $R^T R = I$

$R(\theta, \vec{v}) = f(\sin\theta, \cos\theta, \vec{v})$

$\in \mathbb{Q}$

Almost never $\in \mathbb{Q}$



→ Rational rotations

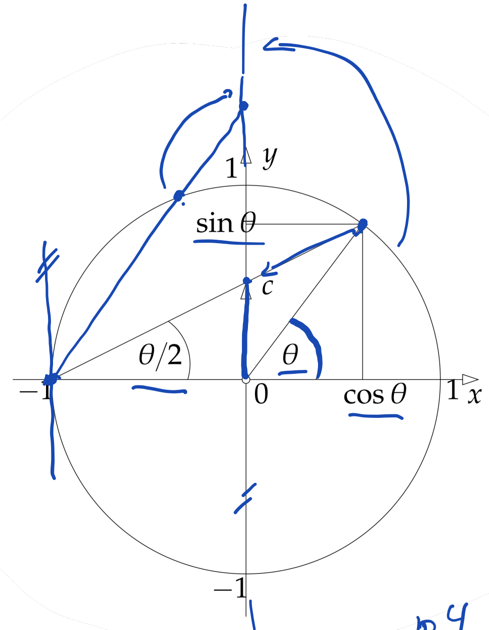
→ Cayley param

$\mathbb{Q} \subset \mathbb{R}^4 \rightsquigarrow SO(3)$

some "infinite" process for computing goniometric functions. This may be no probelem if approximate evaluation of functions is acceptable but, as we will see, it is a findamental obstackle to solving interestign engineering problems using computational algebra.

1.4.1 Cayley transform parameterization of two-dimensional rotations

Let us first look at two-dimesional roations. Figure 1.2 shows an illustartion of the relationship between parameter c and $\cos\theta$, $\sin\theta$ on the unit circle. We see that, using the similarity of triangles, $\frac{\sin\theta}{\cos\theta+1} = \frac{c}{1}$. Considering that $(\cos\theta)^2 + (\sin\theta)^2 = 1$ we are getting



$$1 - (\cos\theta)^2 = (\sin\theta)^2 = c^2(\cos\theta + 1)^2 = c^2((\cos\theta)^2 + 2\cos\theta + 1) \quad (1.120)$$

$$0 = (c^2+1)(\cos\theta)^2 + 2\,c^2\cos\theta + c^2 - 1 \quad (1.121)$$

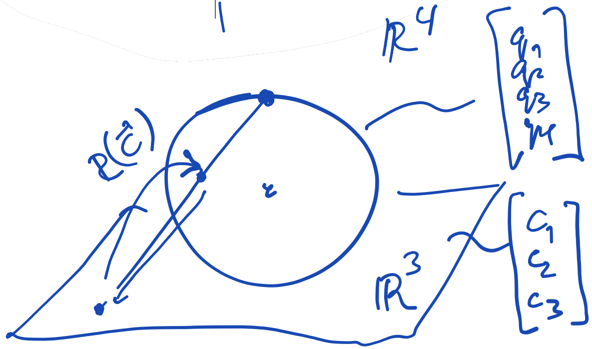
and thus

$$\cos\theta = \frac{-2\,c^2 \pm \sqrt{4\,c^4 - 4\,(c^2+1)(c^2-1)}}{2(c^2+1)} = \frac{-c^2 \pm \sqrt{c^4 - (c^4-1)}}{c^2+1} = \frac{\pm 1 - c^2}{1+c^2} \quad (1.122)$$

gives either $\cos\theta = -1$ or

$$\cos\theta = \frac{1-c^2}{1+c^2} \quad (1.123)$$

The former case corresponds to point $[-1\ 0]^\top$. In the latter case, we have



$$(\sin\theta)^2 = 1 - (\cos\theta)^2 = 1 - (\frac{1-c^2}{1+c^2})^2 = \frac{(1+c^2)^2 - (1-c^2)^2}{(1+c^2)^2} \quad (1.124)$$

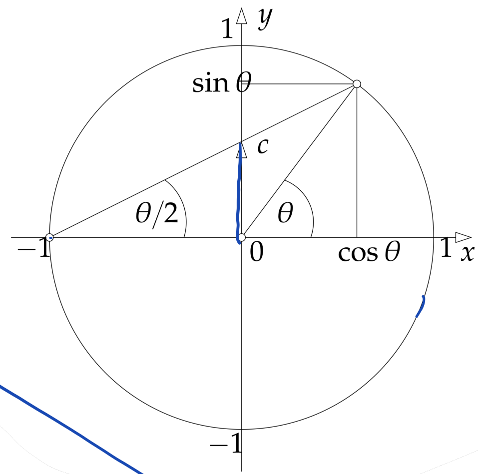
$$= \frac{(1+2\,c^2+c^4) - (1-2\,c^2+c^4)}{(1+c^2)^2} = \frac{4\,c^2}{(1+c^2)^2} = \left(\frac{2\,c}{1+c^2}\right)^2 \quad (1.125)$$

and thus $\sin\theta = \pm\frac{2c}{1+c^2}$. Now, we see from Figure 1.2 that we want $\sin\theta$ to be positive for positive c. Therefore, we conclude that

$$\boxed{\sin\theta = \frac{2\,c}{1+c^2}} \tag{1.126}$$

It is impotant to notice that with the parameterization given by Equation 1.123, we can never get $\cos\theta = -1$ for a real c since if that was true, we would get $-1-c^2 = 1-c^2$ and hence $-1 = 1$. On the other hand, we see that Cayley transform maps every $c \in \mathbb{R}$ into a point on the unit circle $[\cos\theta \;\; \sin\theta]^\top$, and hence to the corresponding rotation

$$\boxed{\mathtt{R}(c) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \frac{1-c^2}{1+c^2} & -\frac{2c}{1+c^2} \\ \frac{2c}{1+c^2} & \frac{1-c^2}{1+c^2} \end{bmatrix}} \tag{1.127}$$



~~180°~~

$\boxed{c} \in \mathbb{R}$

$c = \cos\theta$
$s = \sin\theta$ $\Big\}$ $c^2+s^2=1$

The mapping $\mathtt{R}(c)\colon \mathbb{R} \to \mathtt{R}$ is one-to-one since when two c_1, c_2 map into the same point, then

$$\begin{aligned} \frac{2\,c_1}{1+c_1^2} &= \frac{2\,c_2}{1+c_2^2} && (1.128)\\ c_1(1+c_2^2) &= c_2(1+c_1^2) && (1.129)\\ c_1 - c_2 &= c_1c_2(c_1-c_2) && (1.130) \end{aligned}$$

implies that either $c_1c_2 \neq 0$, and then $c_1 = c_2$, or $c_1c_2 = 0$, and then $c_1 = 0 = c_2$ because both $1+c_1^2$, $1+c_2^2$ are positive. Next, let us see that the mapping is also onto $\mathbb{R}\backslash\{[-1\;0]^\top\}$. Consider a point $[\cos\theta \;\; \sin\theta]^\top \neq [-1\;0]^\top$. Its preimage c, is obtained as

$$c = \frac{\sin\theta}{1+\cos\theta} \tag{1.131}$$

which is clearly defined for $\cos\theta \neq -1$.

1.4.1.1 Two-dimensional rational rotations

It is also important to notice that the $\mathtt{R}(c)$ is a rational function of c as well as c is a rational function or $\mathtt{R}$ (e.g. of the two elements in its first column). Hence, every rational number c gives a rational point $[a\ b]^\top$ on the unit circle as well as every rational point $[a\ b]^\top$ provides a rational c. This way, we can obtain all rational two-dimensional rotations by going over all rational c's plus the rotation $-\mathtt{I}_{2\times 2}$.

1.4.2 Cayley transform parameterization of three-dimensional rotations

We saw that we have obtained a bijective (one-to-one and onto) mapping between all real numbers and all two-dimensional rotations other than the rotation by 180° degrees. Now, since every three-dimensional rotation can be actually seen as a two-dimensional rotation after aligning the z-axis with the rotation axis, we may hint on having an analogous situation in three dimensons after removing all rotations by 180°. Let us investigate this further and see that we can indeed establish a bijective mapping between $\mathbb{R}^3$ and all three-dimesnional rotations by other than 180° angle.

Let us consider that all rotations by 180° are represented by unit quaternons in the form $\begin{bmatrix} 0 & q_2 & q_3 & q_4 \end{bmatrix}$. Hence, to remove them, it is enough to remove from all cases when $c_1 = 0$. One way to do it, is to write down the rotation matrix in tems of (non-unit) quaternions $\vec{q}$

$$\mathtt{R}(\vec{q}) = \frac{1}{q_1^2+q_2^2+q_3^2+q_4^2}\begin{bmatrix} q_1^2+q_2^2-q_3^2-q_4^2 & 2\,(q_2q_3-q_1q_4) & 2\,(q_2q_4+q_1q_3) \\ 2\,(q_2q_3+q_1q_4) & q_1^2-q_2^2+q_3^2-q_4^2 & 2\,(q_3q_4-q_1q_2) \\ 2\,(q_2q_4-q_1q_3) & 2\,(q_3q_4+q_1q_2) & q_1^2-q_2^2-q_3^2+q_4^2 \end{bmatrix} \tag{1.132}$$

$\vec{q}_1^{\,2} \rightarrow \dfrac{q_1^2}{\|q_1\|^2}$

$R(\vec{q}) \longrightarrow R\left(\dfrac{\vec{q}}{\|\vec{q}\|}\right)$

$\downarrow$

$\sqrt{q_1^2+q_2^2+q_3^2+q_4^2}$

$q_1 q_1 \hookrightarrow \dfrac{q_1}{\|\vec{q}\|} \cdot \dfrac{q_2}{\|\vec{q}_1\|} = \dfrac{q_1 q_2}{\|q\|^2}$

$(\ \ )^2 \rightarrow$ poly in elem

rational f.

and then set $q_1 = 1$, $q_2 = c_1$, $q_3 = c_2$, $q_4 = c_3$, to get

$$\mathtt{R}(\vec{c}) = \frac{1}{1+c_1^2+c_2^2+c_3^2}\begin{bmatrix} 1+c_1^2-c_2^2-c_3^2 & 2\,(c_1c_2-c_3) & 2\,(c_1c_3+c_2) \\ 2\,(c_1c_2+c_3) & 1-c_1^2+c_2^2-c_3^2 & 2\,(c_2c_3-c_1) \\ 2\,(c_1c_3-c_2) & 2\,(c_2c_3+c_1) & 1-c_1^2-c_2^2+c_3^2 \end{bmatrix} \tag{1.133}$$

$q_1 = 1$

$c_1 \equiv q_2$

$c_2 \equiv q_3$

$c_3 \equiv q_4$

not possible

$q_1 = \cos\frac{\theta}{2} = 0$

with $\vec{c} = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix}^\top \in \mathbb{R}^3$.

It can be verified that $\mathtt{R}(\vec{c})^\top \mathtt{R}(\vec{c}) = \mathtt{I}$ for all $\vec{c} \in \mathbb{R}^3$ and hence the mapping $\mathtt{R}(\vec{c})\colon \mathbb{R}^3 \to \mathtt{R}$ maps the space $\mathbb{R}^3$ into rotation matrices $\mathtt{R}$. Let us next see that the mapping is also one-to-one.

First, notice that by setting $c_1 = c_2 = 0$, we are getting

$$\mathtt{R}(c_3) = \frac{1}{1+c_3^2}\begin{bmatrix} 1-c_3^2 & -2\,c_3 & 0 \\ 2\,c_3 & 1-c_3^2 & 0 \\ 0 & 0 & 1+c_3^2 \end{bmatrix} = \begin{bmatrix} \frac{1-c_3^2}{1+c_3^2} & \frac{-2\,c_3}{1+c_3^2} & 0 \\ \frac{2\,c_3}{1+c_3^2} & \frac{1-c_3^2}{1+c_3^2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.134}$$

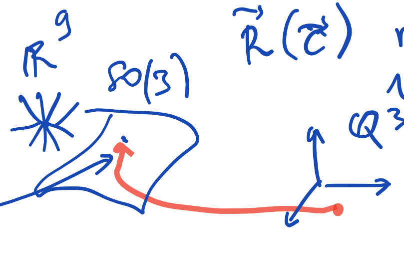
which is exactly the Cayley parameterization for two-dimensional rotation aroung the z-axis. In the same way, we get that $\mathtt{R}(c_1)$ are rotations around the x-axis and $\mathtt{R}(c_2)$ are rotations around the y-axis.

We have seen in Paragraph 1.3.2 that the mapping between the unit quaternions $\vec{q}$ and rotation matrices $\mathtt{R}(\vec{q})$ was "two-to-one" in the way that there were exactly two quaternions $\vec{q}$, $-\vec{q}$ mapping into one $\mathtt{R}$, i.e. $\mathtt{R}(\vec{q}) = \mathtt{R}(-\vec{q})$. Now, we are forcing the first coordinate of the unit quaternion $\vec{q} = \frac{\begin{bmatrix} 1 & c_1 & c_2 & c_3 \end{bmatrix}^\top}{1+c_1^2+c_2^3+c_3}$ be positive. Therefore, the mapping $\mathtt{R}(\vec{c})$ becomes one-to-one.

$\vec{c} \in \mathbb{Q}^3$ rational

$\downarrow$

$\vec{R}(\vec{c})$ rational numbers



Now, let us see that by $\mathtt{R}(\vec{c})$ we can represent all rotations that are not by 180°. ...

Rational rotations are dense in SO(3) ≡ arbitrarily close