1 Rotation representation and parameterization

$R \in \mathbb{R}^{3\times 3}$

$\|\vec{v}\| = 1$

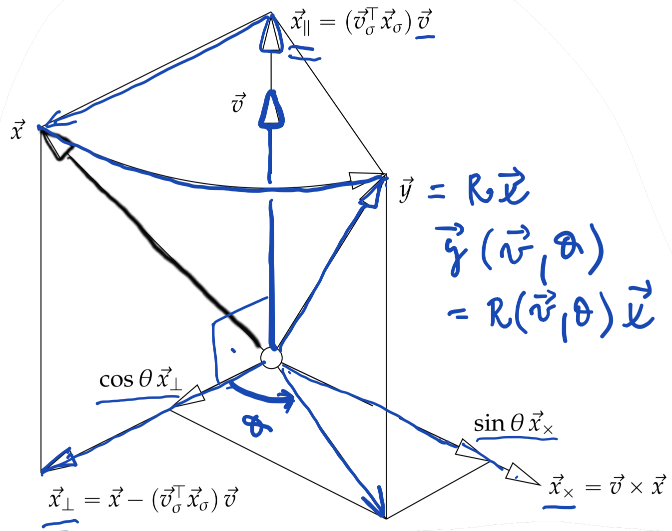
We have seen Chapter **??** that rotation can be represented by an orthonormal matrix $\mathtt{R}$. Matrix $\mathtt{R}$ has nine elements and there are six constraints $\mathtt{R}^\top\mathtt{R} = \mathtt{I}$ and one constraint $|\mathtt{R}| = 1$. Hence, we can view the space of all rotation matrices as a subset of $\mathbb{R}^9$. This subset[1] is determined by seven polynomial equations in nine variables. We will next investigate how to describe, i.e. *parameterize*, this set with fewer parameters and fewer constraints.

1.1 Angle-axis representation of rotation

We know, Paragraph **??**, that every rotation is etermined by a rotation axis and a rotation angle. Let us next give a classical construction of the rotation matrix from an axis and angle.

Figure 1.1 shows how the vector $\vec{x}$ rotates by angle θ around an axis given by a unit vector $\vec{v}$ into vector $\vec{y}$. To find the relationship between $\vec{x}$ and $\vec{y}$, we shall construct a special basis of $\mathbb{R}^3$. Vector $\vec{x}$ either is, or it is not a multiple of $\vec{v}$. If it is, than $\vec{y} = \vec{x}$ and $\mathtt{R} = \mathtt{I}$. Let us alternatively consider $\vec{x}$, which is not a multiple of $\vec{v}$ (an hence is not the zero vector!). Futher, let us consider the standard basis σ of $\mathbb{R}^3$ and coordinates of vectors $\vec{x}_\sigma$ and

the length (of the) projection of $\vec{x}$ on $\vec{v}$



$\vec{y} = R\vec{x}$

$\vec{y}(\vec{v},\theta) = R(\vec{v},\theta)\,\vec{x}$

$\vec{x} \parallel \vec{v}$, $\exists\, \alpha \in \mathbb{R}$ $\vec{x} = \alpha\vec{v}$

$R = I$

$\vec{x}$ not a multiple of $\vec{v}$

[1]It is often called algebraic variaty in specialized literature [1].

$\vec{v}_\sigma$. We construct three non-zero vectors

$$\boxed{\begin{aligned}\vec{x}_{\|\sigma} &= (\vec{v}_\sigma^\top \vec{x}_\sigma)\,\vec{v}_\sigma \quad (1.1)\\ \vec{x}_{\perp\sigma} &= \vec{x} - (\vec{v}_\sigma^\top \vec{x}_\sigma)\,\vec{v}_\sigma \quad (1.2)\\ \vec{x}_{\times\sigma} &= \vec{v}_\sigma \times \vec{x}_\sigma \quad (1.3)\end{aligned}}$$

which are mutually orthogonal and hence form a basis of $\mathbb{R}^3$. We may notice that cooridate vectors $\vec{x} \in \mathbb{R}^3$, are actually equal to their coordinates w.r.t. the standard basis σ. Hence we can drop σ index and write

$$\underline{\vec{x}_\|} = (\vec{v}^\top \vec{x})\,\vec{v} = \vec{v}\,(\vec{v}^\top \vec{x}) = (\vec{v}\,\vec{v}^\top)\,\vec{x} = \underline{[\vec{v}]_\|}\,\vec{x} \quad (1.4)$$

$$\underline{\vec{x}_\perp} = \underline{\vec{x} - (\vec{v}^\top \vec{x})\,\vec{v}} = \vec{x} - (\vec{v}\,\vec{v}^\top)\,\vec{x} = (\mathtt{I} - \vec{v}\,\vec{v}^\top)\,\vec{x} = \underline{[\vec{v}]_\perp}\,\vec{x} \quad (1.5)$$

$$\underline{\vec{x}_\times} = \vec{v} \times \vec{x} = \underline{[\vec{v}]_\times}\,\vec{x} \quad (1.6)$$

$[\,|\,|\,|\,] =$ rank $= 1$

$[v_1\vec{v} \quad v_2\vec{v} \quad v_3\vec{v}]$

We have introduced two new matrices

symmetric $\rightarrow$

$$\underline{[\vec{v}]_\|} = \vec{v}\,\vec{v}^\top \quad \text{and} \quad \underline{[\vec{v}]_\perp} = \mathtt{I} - \vec{v}\,\vec{v}^\top \quad (1.7)$$

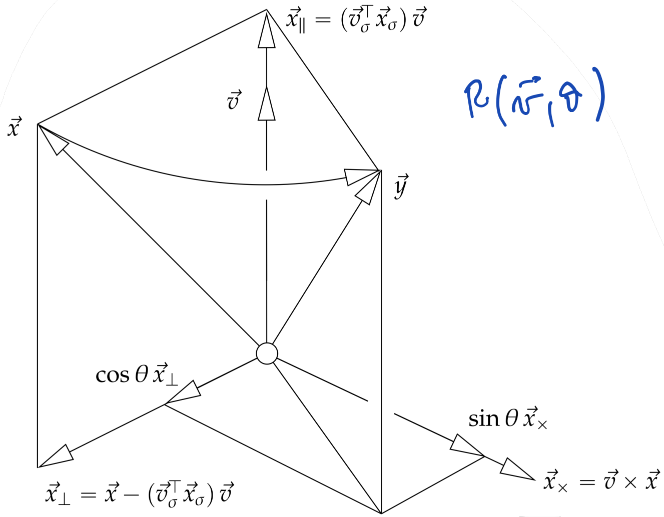
Let us next study how the three matrices $[\vec{v}]_\|$, $[\vec{v}]_\perp$, $[\vec{v}]_\times$ behave under the transposition and mutual multiplication. We see that the following indentities

transp. multiplic.

$$\begin{array}{llll} [\vec{v}]_\|^\top = [\vec{v}]_\|, & [\vec{v}]_\|\,[\vec{v}]_\| = [\vec{v}]_\|, & [\vec{v}]_\|\,[\vec{v}]_\perp = \mathbf{0}, & [\vec{v}]_\|\,[\vec{v}]_\times = \mathbf{0},\\ [\vec{v}]_\perp^\top = [\vec{v}]_\perp, & [\vec{v}]_\perp\,[\vec{v}]_\| = \mathbf{0}, & [\vec{v}]_\perp\,[\vec{v}]_\perp = [\vec{v}]_\perp, & [\vec{v}]_\perp\,[\vec{v}]_\times = [\vec{v}]_\times,\\ [\vec{v}]_\times^\top = -[\vec{v}]_\times, & [\vec{v}]_\times\,[\vec{v}]_\| = \mathbf{0}, & [\vec{v}]_\times\,[\vec{v}]_\perp = [\vec{v}]_\times, & [\vec{v}]_\times\,[\vec{v}]_\times = -[\vec{v}]_\perp \end{array} \quad (1.8)$$

relationships on $[\;]_\|, \ldots$

$$= (\mathtt{I} - \vec{v}\,\vec{v}^\top)(\mathtt{I} - \vec{v}\,\vec{v}^\top) = \mathtt{I} - \underline{2\vec{v}\,\vec{v}^\top} + \overbrace{\vec{v}\,\underbrace{\vec{v}^\top\vec{v}}_{\|\vec{v}\|^2 = 1}\,\vec{v}^\top}^{1\,\vec{v}\,\vec{v}^\top} = \underline{\mathtt{I} - \vec{v}\,\vec{v}^\top}$$



$R(\vec{v}, \theta)$

$$\vec{v} \times \vec{x} = \underbrace{\begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}}_{[\vec{v}]_\times} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} -v_3x_2 + v_2x_3 \\ v_3x_1 - v_1x_3 \\ -v_2x_1 + v_1x_2 \end{bmatrix}$$

$\underbrace{12}3$ $(-1)^2$

anti symmetric

hold true. The last identity is obtained as follows

$$\rightarrow [\vec{v}]_\times [\vec{v}]_\times = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix} \quad (1.9)$$

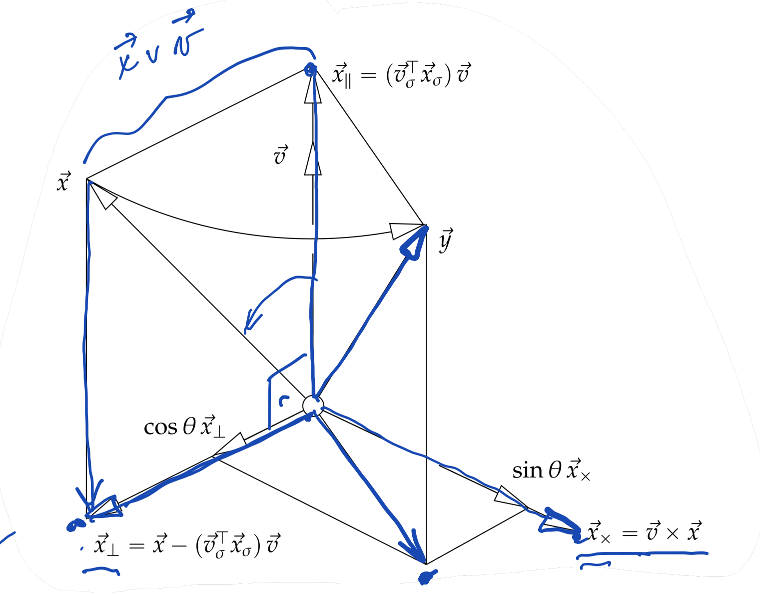
$$= \begin{bmatrix} -v_2^2 - v_3^2 & v_1 v_2 & v_1 v_3 \\ v_1 v_2 & -v_1^2 - v_3^2 & v_2 v_3 \\ v_1 v_3 & v_2 v_3 & -v_1^2 - v_2^2 \end{bmatrix} \quad (1.10)$$

$\|\vec{v}\|^2 = 1$

$v_1^2 + v_2^2 + v_3^2 = 1$

$$= \begin{bmatrix} v_1^2 - 1 & v_1 v_2 & v_1 v_3 \\ v_1 v_2 & v_2^2 - 1 & v_2 v_3 \\ v_1 v_3 & v_2 v_3 & v_3^2 - 1 \end{bmatrix} = [\vec{v}]_{||} - \mathtt{I} = -[\vec{v}]_\perp \quad (1.11)$$

$\vec{v}\vec{v}^\top$



It is also interesting to investigate the norms of vectors $\vec{x}_\perp$ and $\vec{x}_\times$. Consider

$$1 \rightarrow \|\vec{x}_\times\|^2 = \vec{x}_\times^\top \vec{x}_\times = \vec{x}^\top [\vec{v}]_\times^\top [\vec{v}]_\times \vec{x} = \vec{x}^\top (-[\vec{v}]_\times^2)\vec{x} = \vec{x}^\top [\vec{v}]_\perp \vec{x} \quad (1.12)$$

$$2 \rightarrow \|\vec{x}_\perp\|^2 = \vec{x}_\perp^\top \vec{x}_\perp = \vec{x}^\top [\vec{v}]_\perp^\top [\vec{v}]_\perp \vec{x} = \vec{x}^\top [\vec{v}]_\perp^2 \vec{x} = \vec{x}^\top [\vec{v}]_\perp \vec{x} \quad // \quad (1.13)$$

Since norms are non-negaive, we conclude that $\|\vec{x}_\perp\| = \|\vec{x}_\times\|$.

We can now write $\vec{y}$ in the basis $[\vec{x}_{||}, \vec{x}_\perp, \vec{x}_\times]$ as

$$\vec{y} = \vec{x}_{||} + ||\vec{x}_\perp|| \cos\theta \frac{\vec{x}_\perp}{||\vec{x}_\perp||} + ||\vec{x}_\perp|| \sin\theta \frac{\vec{x}_\times}{||\vec{x}_\times||} \quad (1.14)$$

$$= \vec{x}_{||} + \cos\theta\,\vec{x}_\perp + \sin\theta\,\vec{x}_\times \quad (1.15)$$

$$= [\vec{v}]_{||}\,\vec{x} + \cos\theta\,[\vec{v}]_\perp\,\vec{x} + \sin\theta\,[\vec{v}]_\times\,\vec{x} \quad (1.16)$$

$$= ([\vec{v}]_{||} + \cos\theta\,[\vec{v}]_\perp + \sin\theta\,[\vec{v}]_\times)\,\vec{x} = \mathtt{R}\,\vec{x} \quad (1.17)$$

We obtained matrix

R

$$\boxed{\mathtt{R} = [\vec{v}]_{||} + \cos\theta\,[\vec{v}]_\perp + \sin\theta\,[\vec{v}]_\times} \quad (1.18)$$

Let us check that this indeed is a rotation matrix

$$
\begin{aligned}
\mathtt{R}^\top \mathtt{R} &= \left([\vec{v}]_{\|} + \cos\theta\, [\vec{v}]_{\perp} + \sin\theta\, [\vec{v}]_{\times}\right)^\top \left([\vec{v}]_{\|} + \cos\theta\, [\vec{v}]_{\perp} + \sin\theta\, [\vec{v}]_{\times}\right) \\
&= \left([\vec{v}]_{\|} + \cos\theta\, [\vec{v}]_{\perp} - \sin\theta\, [\vec{v}]_{\times}\right) \left([\vec{v}]_{\|} + \cos\theta\, [\vec{v}]_{\perp} + \sin\theta\, [\vec{v}]_{\times}\right) \\
&= [\vec{v}]_{\|} + \cos^2\theta\, [\vec{v}]_{\perp} + \sin\theta\cos\theta\, [\vec{v}]_{\times} - \sin\theta\cos\theta\, [\vec{v}]_{\times} + \sin^2\theta\, [\vec{v}]_{\perp} \\
&= [\vec{v}]_{\|} + [\vec{v}]_{\perp} = \mathtt{I} \qquad (1.19)
\end{aligned}
$$

orthogonal matrix.

R can be wrtten in many variations, which are useful in different situations when simplifying formulas. Let us provide the most common of them using $[\vec{v}]_{\|} = \vec{v}\vec{v}^\top$, $[\vec{v}]_{\perp} = \mathtt{I} - [\vec{v}]_{\|} = \mathtt{I} - \vec{v}\vec{v}^\top$ and $[\vec{v}]_{\times}$

$$
\begin{aligned}
\mathtt{R} &= [\vec{v}]_{\|} + \cos\theta\, [\vec{v}]_{\perp} + \sin\theta\, [\vec{v}]_{\times} && (1.20)\\
&= \vec{v}\vec{v}^\top + \cos\theta\,(\mathtt{I} - \vec{v}\vec{v}^\top) + \sin\theta\, [\vec{v}]_{\times} && (1.21)\\
&= \underline{\cos\theta\, \mathtt{I} + (1-\cos\theta)\, \vec{v}\vec{v}^\top + \sin\theta\, [\vec{v}]_{\times}} && (1.22)\\
&= \cos\theta\, \mathtt{I} + (1-\cos\theta)\, [\vec{v}]_{\|} + \sin\theta\, [\vec{v}]_{\times} && (1.23)\\
&= \cos\theta\, \mathtt{I} + (1-\cos\theta)\, (\mathtt{I} + [\vec{v}]_{\times}^2) + \sin\theta\, [\vec{v}]_{\times} && (1.24)\\
&= \underline{\mathtt{I} + (1-\cos\theta)\, [\vec{v}]_{\times}^2 + \sin\theta\, [\vec{v}]_{\times}} && (1.25)
\end{aligned}
$$

variations

1.1.1 Angle-axis parameterization

Let us write R in more detail

$$
\begin{aligned}
\mathtt{R} &= \cos\theta\, \mathtt{I} + (1-\cos\theta)\, \vec{v}\vec{v}^\top + \sin\theta\, [\vec{v}]_{\times} && (1.26)\\
&= \underline{(1-\cos\theta)\, \vec{v}\vec{v}^\top + \cos\theta\, \mathtt{I} + \sin\theta\, [\vec{v}]_{\times}} && (1.27)\\
&= (1-\cos\theta)\begin{bmatrix} v_1 v_1 & v_1 v_2 & v_1 v_3 \\ v_2 v_1 & v_2 v_2 & v_2 v_3 \\ v_3 v_1 & v_3 v_2 & v_3 v_3 \end{bmatrix} + \cos\theta \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \sin\theta \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} v_1 v_1 (1-\cos\theta) + \cos\theta & v_1 v_2 (1-\cos\theta) - v_3 \sin\theta & v_1 v_3 (1-\cos\theta) + v_2 \sin\theta \\ v_2 v_1 (1-\cos\theta) + v_3 \sin\theta & v_2 v_2 (1-\cos\theta) + \cos\theta & v_2 v_3 (1-\cos\theta) - v_1 \sin\theta \\ v_3 v_1 (1-\cos\theta) - v_2 \sin\theta & v_3 v_2 (1-\cos\theta) + v_1 \sin\theta & v_3 v_3 (1-\cos\theta) + \cos\theta \end{bmatrix} && (1.28)
\end{aligned}
$$

explicit in coordinates

which allows us to parameterize rotation by four numbers

$$\begin{bmatrix} \theta & v_1 & v_2 & v_3 \end{bmatrix}^\top \quad \text{with} \quad v_1^2 + v_2^2 + v_3^2 = 1 \tag{1.29}$$

The parameterization uses goniometric functions.

1.1.2 Computing the axis and the angle of rotation from R

Let us now discuss how to get a unit vector $\vec{v}$ of the axis and the corresponding angle θ of rotation from a rotation matrix R, such that the pair $[\theta, \vec{v}]$ gives R by Equation 1.28. To avoid multiple representations due to periodicity of θ, we will confine θ to real interval $(-\pi, \pi]$.

We can get $\cos(\theta)$ from Equation **??**.

If $\cos\theta = 1$, then $\sin\theta = 0$, and thus $\theta = 0$. Then, $\mathtt{R} = \mathtt{I}$ and any unit vector can be taken as $\vec{v}$, i.e. all paris $[0, \vec{v}]$ for unit vector $\vec{v} \in \mathbb{R}^3$ represent $\mathtt{I}$.

If $\cos\theta = -1$, then $\sin\theta = 0$, and thus $\theta = \pi$. Then R is a symmetrical matrix and we use Equation **??** to get $\vec{v}_1$, a non-zero multiple of $\vec{v}$, i.e. $\vec{v} = \alpha\,\vec{v}_1$, with real non-zero α, and therefore $\vec{v}_1/||\vec{v}_1|| = s\,\vec{v}$ with $s = \pm 1$. We are getting

$$\mathtt{R} = 2\,[\vec{v}]_{\|} - \mathtt{I} = 2\,\vec{v}\,\vec{v}^\top - \mathtt{I} = 2\,s^2\vec{v}\,\vec{v}^\top - \mathtt{I} = 2\,(s\,\vec{v})\,(s\,\vec{v})^\top - \mathtt{I} \tag{1.30}$$

$$= 2\left(\frac{\vec{v}_1}{\|\vec{v}_1\|}\right)\left(\frac{\vec{v}_1}{\|\vec{v}_1\|}\right)^\top - \mathtt{I} = 2\left(-\frac{\vec{v}_1}{\|\vec{v}_1\|}\right)\left(-\frac{\vec{v}_1}{\|\vec{v}_1\|}\right)^\top - \mathtt{I} \tag{1.31}$$

from Equation 1.27 and hence we can form two pairs

$$\left[\pi, +\frac{\vec{v}_1}{\|\vec{v}_1\|}\right], \quad \left[\pi, -\frac{\vec{v}_1}{\|\vec{v}_1\|}\right] \tag{1.32}$$

representing this rotation.

SO(3).. special orthogonal group

det R = 1

$\in \mathbb{R}^{3\times 3}$ $\underbrace{R^T R = I}_{6}$

O(3)

$\underbrace{\mathbb{R}^9 + 7 \text{ relations}}$

"3 Dim = 9−6"

3D det = 1

det = −1

Alg: 9 dim + 7 rels (alg.)

Goniom: 4 dim + 1 rel (alg.)

$I \longrightarrow \infty\ [0, \vec{v}],\ \|\vec{v}\| = 1$

180° → 2 reps.

Let's now move to $-1 < \cos\theta < 1$. We construct matrix

$$\begin{aligned}
\mathtt{R} - \mathtt{R}^\top &= (1-\cos\theta)\,[\vec{v}]_{\|} + \cos\theta\,\mathtt{I} + \sin\theta\,[\vec{v}]_\times \\
&\quad - \left((1-\cos\theta)\,[\vec{v}]_{\|} + \cos\theta\,\mathtt{I} + \sin\theta\,[\vec{v}]_\times\right)^\top && (1.33)\\
&= (1-\cos\theta)\,[\vec{v}]_{\|} + \cos\theta\,\mathtt{I} + \sin\theta\,[\vec{v}]_\times \\
&\quad - \left((1-\cos\theta)\,[\vec{v}]_{\|} + \cos\theta\,\mathtt{I} - \sin\theta\,[\vec{v}]_\times\right) && (1.34)\\
&= 2\,\sin\theta\,[\vec{v}]_\times && (1.35)
\end{aligned}$$

$[\vec{v}]_{\|} = \vec{v}\,\vec{v}^\top = \vec{v}\,\vec{v}^\top$

Generic case

$R \rightarrow 2$ reps.

which gives

$$\begin{bmatrix} 0 & r_{12}-r_{21} & r_{13}-r_{31} \\ r_{21}-r_{12} & 0 & r_{23}-r_{32} \\ r_{31}-r_{13} & r_{32}-r_{23} & 0 \end{bmatrix} = 2\,\sin\theta \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix} \tag{1.36}$$

and thus

$$\boxed{\sin\theta\,\vec{v} = \frac{1}{2}\begin{bmatrix} r_{32}-r_{23} \\ r_{13}-r_{31} \\ r_{21}-r_{12} \end{bmatrix}} \tag{1.37}$$

We thus get

$$|\sin\theta|\,||\vec{v}|| = |\sin\theta| = \frac{1}{2}\sqrt{(r_{23}-r_{32})^2 + (r_{31}-r_{13})^2 + (r_{12}-r_{21})^2} \tag{1.38}$$

There holds

$$\sin\theta\,\vec{v} = \sin(-\theta)\,(-\vec{v}) \tag{1.39}$$

true and hence we define

$$\theta = \arccos\left(\frac{1}{2}(\operatorname{trace}(\mathtt{R}) - 1)\right), \quad \vec{r} = \frac{1}{2}\begin{bmatrix} r_{32}-r_{23} \\ r_{13}-r_{31} \\ r_{21}-r_{12} \end{bmatrix} \tag{1.40}$$

and write two pairs

$$\left[+\theta, +\frac{\vec{r}}{\sin\theta}\right], \quad \left[-\theta, -\frac{\vec{r}}{\sin\theta}\right] \tag{1.41}$$

representing rotation R.

We see that all rotations are represented by two pairs of $[\theta, \vec{v}]$ except for the identity, which is represented by an infinite number of pairs.

1.2 Euler vector representation and the exponential map

Let us now discuss another classical and natural representation of rotations. It may seem as only a slight variation of the angle-axis representation but it leads to several interesting connections and properties.

Let us consider the *euler vector* defined as

$$\vec{e} = \theta\,\vec{v} \tag{1.42}$$

where θ is the rotation angle and $\vec{v}$ is the unit vector representing the rotation axis in the angle-axis representation as in Equation 1.27.

Next, let us recall the very fundamental real functions [2] and their related power series

$$\exp x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{1.43}$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \tag{1.44}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \tag{1.45}$$

$\mathbb{R} \to \mathbb{R}$

It makes sense to define the exponential function of an $m \times m$ real matrix $\mathtt{A} \in \mathbb{R}^{m\times m}$ as

$$\exp \mathtt{A} \overset{\text{def}}{=} \sum_{n=0}^{\infty} \frac{\mathtt{A}^n}{n!} \tag{1.46}$$

We will now show that the rotation matrix R corresponding to the angle-axis parameterization $[\theta, \vec{v}]$ can be obtained as

$$\boxed{\mathtt{R}([\theta, \vec{v}]) = \exp{[\vec{e}]_\times} = \exp{[\theta\, \vec{v}]_\times}} \quad ✓ \tag{1.47}$$

$\vec{e} = \theta\, \vec{v} \longrightarrow R = I$

$\theta = 0$

$\| \vec{e} \| = \theta$

$0 \cdot \vec{v} = \vec{0}$

$\vec{v} = \dfrac{\vec{e}}{\|\vec{e}\|}$

angle-axis rep.

Rodriguez formula

The basic tool we have to employ is the relationship between $[\vec{e}]^3_\times$ and $[\vec{e}]_\times$. It will allow us to pass form the ifinite summantion of matrix powers to the infinite summation of the powers of the θ and hence to $\sin\theta$ and $\cos\theta$, which will, at the end, give the rodrigues formula. We write, Equation 1.11,

$$\begin{aligned}
\longrightarrow [\theta\, \vec{v}]^2_\times &= \theta^2\, (\vec{v}\,\vec{v}^\top - \mathtt{I}) \\
\longrightarrow [\theta\, \vec{v}]^3_\times &= -\theta^2\; [\theta \vec{v}]^1_\times \\
\longrightarrow [\theta\, \vec{v}]^4_\times &= -\theta^2\; [\theta \vec{v}]^2_\times \\
[\theta\, \vec{v}]^5_\times &= \theta^4\; [\theta \vec{v}]^1_\times \\
[\theta\, \vec{v}]^6_\times &= \theta^4\; [\theta \vec{v}]^2_\times \\
&\vdots
\end{aligned} \tag{1.48}$$

and substitute into Equation 1.46 to get

$$\longrightarrow \exp{[\theta\, \vec{v}]_\times} = \sum_{n=0}^{\infty} \frac{[\theta\, \vec{v}]^n_\times}{n!} \tag{1.49}$$

$$= \sum_{n=0}^{\infty} \frac{[\theta\, \vec{v}]^{2n}_\times}{(2n)!} + \sum_{n=0}^{\infty} \frac{[\theta\, \vec{v}]^{2n+1}_\times}{(2n+1)!} \tag{1.50}$$

Let us notice the identities, which are obtained by generalizing Equations 1.48 to an arbitrary power n

$$[\theta\, \vec{v}]^0_\times = \mathtt{I} \tag{1.51}$$

$$[\theta\, \vec{v}]^{2n}_\times = (-1)^{n-1}\, \theta^{2(n-1)}\; [\theta\, \vec{v}]^2_\times \quad \text{for } n = 1, \ldots \tag{1.52}$$

$$[\theta\, \vec{v}]^{2n+1}_\times = (-1)^n\, \theta^{2n}\; [\theta\, \vec{v}]^1_\times \quad \text{for } n = 0, \ldots \tag{1.53}$$

and substitute them into Equation 1.50 to get → manipulate to get sin, cos

$$
\begin{aligned}
\exp\,[\theta\,\vec{v}]_\times &= \mathtt{I} + \left(\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\theta^{2(n-1)}}{(2n)!}\right)[\theta\,\vec{v}]_\times^2 + \left(\sum_{n=0}^{\infty}\frac{(-1)^{n}\theta^{2n}}{(2n+1)!}\right)[\theta\,\vec{v}]_\times \\
&= \mathtt{I} + \left(\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\theta^{2n}}{(2n)!}\right)[\vec{v}]_\times^2 + \left(\sum_{n=0}^{\infty}\frac{(-1)^{n}\theta^{2n+1}}{(2n+1)!}\right)[\vec{v}]_\times \\
&= \mathtt{I} - \left(\sum_{n=0}^{\infty}\frac{(-1)^{n}\theta^{2n}}{(2n)!} - 1\right)[\vec{v}]_\times^2 + \sin\theta\,[\vec{v}]_\times \\
&= \mathtt{I} - (\cos\theta - 1)\,[\vec{v}]_\times^2 + \sin\theta\,[\vec{v}]_\times \\
&= \mathtt{I} + \sin\theta\,[\vec{v}]_\times + (1-\cos\theta)\,[\vec{v}]_\times^2 \\
&= \mathtt{I} + \sin\|\vec{e}\|\left[\frac{\vec{e}}{\|\vec{e}\|}\right]_\times + (1-\cos\|\vec{e}\|)\left[\frac{\vec{e}}{\|\vec{e}\|}\right]_\times^2 \\
&= \mathtt{R}([\theta,\vec{v}])
\end{aligned}
\tag{1.54}
$$

by the comparison with Equation 1.25.

ang. ax   euler vec.

$[\theta, \vec{v}] \leftrightarrow [\theta \cdot \vec{v}]$

$\mathtt{I} \leftrightarrow \vec{0}$

$+\theta, \vec{v}$
$-\theta, -\vec{v}$ $\rightarrow \theta\vec{v}$

$\theta\vec{v} \in \mathbb{R}$

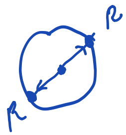
1.3 Quaternion representation of rotation

1.3.1 Quaternion parameterization

We shall now introduce another parameterization of $\mathtt{R}$ by four numbers but this time we will not use goniometric functions but polynomials only. We shall see later that this parameterization has other useful properties.

This paramterization is known as *unit quaternion* parameterization of rotations since rotations are represented by unit vectors from $\mathbb{R}^4$. In general, it may sense to talk even about non-unit quaternions and we will see how to use them later when applying rotations represented by unit quaternions on points represented by non-unit quaternions. To simplify

$q_1 \in \mathbb{R}^4$, $q_2 \in \mathbb{R}^4$ $\Big\} \rightarrow R$



$q_2 = -q_1$, $\|q\| = 1$