2.1.3 Eigenvectors of R

Let us now look at eigenvectors of R and let's first investigate the situation when all eigenvalues of R are real.

$$p(\lambda) = |(\lambda\,\mathtt{I} - \mathtt{R})| = \left|\left(\begin{bmatrix} \lambda - r_{11} & -r_{12} & -r_{13} \\ -r_{21} & \lambda - r_{22} & -r_{23} \\ -r_{31} & -r_{32} & \lambda - r_{33} \end{bmatrix}\right)\right| \tag{2.12}$$

$$= \lambda^3 - (r_{11} + r_{22} + r_{33})\,\lambda^2 + (r_{11}\,r_{22} - r_{21}\,r_{12} + r_{11}\,r_{33} - r_{31}\,r_{13} + r_{22}\,r_{33} - r_{23}\,r_{32})\,\lambda + r_{11}\,(r_{23}\,r_{32} - r_{22}\,r_{33}) - r_{21}\,(r_{32}\,r_{13} - r_{12}\,r_{33}) + r_{31}\,(r_{13}\,r_{22} - r_{12}\,r_{23}) \tag{2.13}$$

$$= \lambda^3 - (r_{11} + r_{22} + r_{33})\,\lambda^2 + (r_{33} + r_{22} + r_{11})\,\lambda - |\mathtt{R}| \tag{2.14}$$

$$= \lambda^3 - \operatorname{trace}\mathtt{R}\,(\lambda^2 - \lambda) - 1 \tag{2.15}$$

$$= (\lambda - 1)(\lambda^2 + (1 - \operatorname{trace}\mathtt{R})\,\lambda + 1) \tag{2.16}$$

§ 1 $\lambda_1 = \lambda_2 = \lambda_3 = 1$**:** Let $\lambda_1 = \lambda_2 = \lambda_3 = 1$. Then $p(\lambda) = (\lambda - 1)^3 = \lambda^3 - 3\,\lambda^2 + 3\,\lambda - 1$. It means that $r_{11} + r_{22} + r_{33} = 3$ and since $r_{11} \leqslant 1$, $r_{22} \leqslant 1$, $r_{33} \leqslant 1$, it leads to $r_{11} = r_{22} = r_{33} = 1$, which implies $\mathtt{R} = \mathtt{I}$. Then $\mathtt{I} - \mathtt{R} = 0$ and all non-zero vectors of $\mathbb{R}^3$ are eigenvectors of R. Notice that rank of $\mathtt{R} - \mathtt{I}$ is zero in this case.

Next, consider $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = -1$. The eigenvectors $\vec{v}$ corresponding to $\lambda_2 = \lambda_3 = -1$ are solutions to

$$\mathtt{R}\,\vec{v} = -\vec{v} \tag{2.17}$$

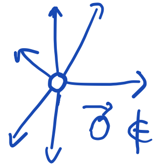
There is always at least one one-dimensional space of such vectors. We also see that there is a rotation matrix

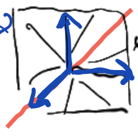
$$\mathtt{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{2.18}$$

with real eigenvectors

$$r \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ r \neq 0, \quad \text{and} \quad s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ s^2 + t^2 \neq 0, \tag{2.19}$$

which means that there is a one-dimensional space of real eigenvectors corresponding to 1 and a two-dimensional space of real eigenvectors corresponding to -1. Notice that rank of $\mathtt{R} - \mathtt{I}$ is two here.





§ 2 $\lambda_1 = 1, \lambda_2 = \lambda_3 = -1$**:** How does the situation look for a general R with eigenvalues $1, -1, -1$? Consider an eigenvector $\vec{v}_1$ corresponding to 1 and an eigenvector $\vec{v}_2$ corresponding to -1. They are linearly independent. Otherwise there has to be $s \in \mathbb{R}$ such that $\vec{v}_2 = s\,\vec{v}_1 \neq 0$ and then

$$\vec{v}_2 = s\,\vec{v}_1 \tag{2.20}$$

$$\mathtt{R}\,\vec{v}_2 = s\,\mathtt{R}\,\vec{v}_1 \tag{2.21}$$

$$-\vec{v}_2 = s\,\vec{v}_1 \tag{2.22}$$

leading to $s = -s$ and therefore $s = 0$ which contradicts $\vec{v}_2 \neq 0$. Now, let us look at vectors $\vec{v}_3 \in \mathbb{R}^3$ defined by

$$\begin{bmatrix} \vec{v}_1^\top \\ \vec{v}_2^\top \end{bmatrix} \vec{v}_3 = 0 \tag{2.23}$$

The above linear system has a one-dimensional space of solutions since the rows of its matrix are independent. Chose a fixed solution $\vec{v}_3 \neq 0$. Then

$$\begin{bmatrix} \vec{v}_1^\top \\ \vec{v}_2^\top \end{bmatrix} \mathtt{R}^\top \vec{v}_3 = \begin{bmatrix} \vec{v}_1^\top \mathtt{R}^\top \\ \vec{v}_2^\top \mathtt{R}^\top \end{bmatrix} \vec{v}_3 = \begin{bmatrix} \vec{v}_1^\top \\ -\vec{v}_2^\top \end{bmatrix} \vec{v}_3 = 0 \tag{2.24}$$

We see that $\mathtt{R}^\top \vec{v}_3$ and $\vec{v}_3$ are in the same one-dimensional space, i.e. they are linearly dependent and we can write

$$\mathtt{R}^\top \vec{v}_3 = s\,\vec{v}_3 \tag{2.25}$$

for some non-zero $s \in \mathbb{C}$. Multiplying equation 2.25 by R from the left and dividing both sides by s gives

$$\frac{1}{s}\vec{v}_3 = \mathtt{R}\,\vec{v}_3 \tag{2.26}$$

Clearly, $\vec{v}_3$ is an eigenvector of R. Since it is not a multiple of $\vec{v}_1$, it must correspond to eigenvalue -1. Moreover, $\vec{v}_2^\top \vec{v}_3 = 0$ and hence they are linearly independent. We have shown that if -1 is an eigenvalue of $\mathtt{R}$, then there are always at least two linearly independent vectors corresponding to the eigenvalue -1, and therefore there is a two-dimensional space of eigenvectors corresponding to -1. Notice that the rank of $\mathtt{R}-\mathtt{I}$ is two in this case since the two-dimensional subspace corresponding to -1 can be complemented only by a one-dimensional subspace corresponding to 1 to avoid intersecting the subspaces in a non-zero vector.

§ 3 General $\lambda_1, \lambda_2, \lambda_3$: Finally, let us look at arbitrary (even non-real) eigenvalues. Assume $\lambda = x + yi$ for real x, y. Then we have

$$\mathtt{R}\,\vec{v} = (x+yi)\,\vec{v} \tag{2.27}$$

If $y \neq 0$, vector $\vec{v}$ must be non-real since otherwise we would have a real vector on the left and a non-real vector on the right. Furthermore, the eigenvalues are pairwise distinct and hence there are three one-dimensional subspaces of eigenvectors (we now understand the space as $\mathbb{C}^3$ over $\mathbb{C}$). In particular, there is exactly one one-dimensional subspace corresponding to eigenvalue 1. The rank of $\mathtt{R}-\mathtt{I}$ is two.

Let $\vec{v}$ be an eigenvector of a rotation matrix $\mathtt{R}$. Then

$$\mathtt{R}\,\vec{v} = (x+yi)\,\vec{v} \tag{2.28}$$

$$\mathtt{R}^\top\mathtt{R}\,\vec{v} = (x+yi)\,\mathtt{R}^\top\vec{v} \tag{2.29}$$

$$\vec{v} = (x+yi)\,\mathtt{R}^\top\vec{v} \tag{2.30}$$

$$\frac{1}{(x+yi)}\,\vec{v} = \mathtt{R}^\top\vec{v} \tag{2.31}$$

$$(x-yi)\,\vec{v} = \mathtt{R}^\top\vec{v} \tag{2.32}$$

We see that the eigenvector $\vec{v}$ of $\mathtt{R}$ corresponding to eigenvalue $x+yi$ is the eigenvector of $\mathtt{R}^\top$ corresponding to eigenvalue $x-yi$ and vice versa. Thus, there is the following interesting correspondence between eigenvalues and eigenvectors of $\mathtt{R}$ and $\mathtt{R}^\top$. Considering eigenvalue-eigenvector pairs

$(1, \vec{v}_1)$, $(x + yi, \vec{v}_2)$, $(x - yi, \vec{v}_3)$ of R we have $(1, \vec{v}_1)$, $(x - yi, \vec{v}_2)$, $(x + yi, \vec{v}_3)$ pairs of $\mathtt{R}^\top$, respectively.

§ 4 Orthogonality of eigenvectors The next question to ask is what are the angles between eignevectors of R? We will considers pairs $(\lambda_1 = 1, \vec{v}_1)$, $(\lambda_2 = x + yi, \vec{v}_2)$, $(\lambda_3 = x - yi, \vec{v}_3)$ of eigenvectors associated with their respective eigenvalues. For instance, vector $\vec{v}_1$ denotes an eigenvector associated with egenvalue 1.

If all eigenvalues are equal to 1, i.e. $\mathtt{R} = \mathtt{I}$, then all non-zero vectors of $\mathbb{R}^3$ are eigenvectors of R and hence we can alway find two eignevectors containing a given angle. In particular, we can choose three mutually orthogonal eignevectors.

If $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = -1$, then we have seen that every $\vec{v}_1$ is perpendicular to $\vec{v}_2$ and $\vec{v}_3$ and that $\vec{v}_2$ and $\vec{v}_3$ can be any two non-zero vectors in a two-dimensional subspace of $\mathbb{R}^3$, which is orthogonal to $\vec{v}_1$. Therefore, for every angle, there are $\vec{v}_2$ and $\vec{v}_3$ which contain it. In particular, it is possible to choose $\vec{v}_2$ to be orthogonal to $\vec{v}_3$ and hence there are three mutually orthogonal eigenvectors.

Finally, if λ_2, λ_3 are non-real, i.e. $y \neq 0$, we have three mutually distinct eigenvalues and hence there are exactly three one-dimensional subspaces (each without the zero vector) of eigenvectors. If two eigenvectors are from the same subspace, then they are linearly dependent and hence they contain the zero angle.

Let us now evaluate $\vec{v}_1^\dagger \vec{v}_2$

$$\vec{v}_1^\dagger \vec{v}_2 = \vec{v}_1^\top \vec{v}_2 = \vec{v}_1^\top \mathtt{R}^\top \mathtt{R} \, \vec{v}_2 = \vec{v}_1^\top (x + yi) \, \vec{v}_2 = (x + yi) \, \vec{v}_1^\top \vec{v}_2 \tag{2.33}$$

We conclude that either $(x + yi) = 1$ or $\vec{v}_1^\top \vec{v}_2 = 0$. Since the latter can't be the case as $y \neq 0$, the former must hold true. We see that $\vec{v}_1$ is orthogonal to $\vec{v}_2$. We can show that $\vec{v}_1$ is orthogonal to $\vec{v}_3$ exactly in the same way.

Let us next consider the angle between eigenvectors $\vec{v}_2$ and $\vec{v}_3$

$$\begin{aligned}\vec{v}_3^\dagger\vec{v}_2 = \vec{v}_3^\dagger \mathtt{R}^\top \mathtt{R}\,\vec{v}_2 = (\mathtt{R}\,\vec{v}_3)^\dagger \mathtt{R}\,\vec{v}_2 &= ((x-yi)\,\vec{v}_3)^\dagger (x+yi)\,\vec{v}_2 && (2.34)\\ &= \vec{v}_3^\dagger\,(x+yi)\,(x+yi)\,\vec{v}_2 && (2.35)\\ \vec{v}_3^\dagger\vec{v}_2 &= (x^2+2\,xyi-y^2)\,\vec{v}_3^\dagger\vec{v}_2 && (2.36)\end{aligned}$$

We conclude that either $(x^2+2xyi-y^2) = 1$ or $\vec{v}_3^\dagger\vec{v}_2 = 0$. The former implies $xy = 0$ and threfore $x = 0$ since $y \neq 0$ but then $-y^2 = 1$, which is, for a real y, impossible. We see that $\vec{v}_3^\dagger\vec{v}_2 = 0$, i.e. vectors $\vec{v}_2$ are orthogonal to vectors $\vec{v}_3$.

Clearly, it is always possible to choose three mutually orhogonal eigenvectors. We can further normalize them to unit legth and thus obtain an orthonormal basis as non-zero orthogonal vectors are linearly independent. Therefore

$$\mathtt{R}\begin{bmatrix}\vec{v}_1 & \vec{v}_2 & \vec{v}_3\end{bmatrix} = \begin{bmatrix}\vec{v}_1 & \vec{v}_2 & \vec{v}_3\end{bmatrix}\begin{bmatrix}\lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3\end{bmatrix} \quad (2.37)$$

$$\begin{bmatrix}\vec{v}_1 & \vec{v}_2 & \vec{v}_3\end{bmatrix}^\dagger \mathtt{R}\begin{bmatrix}\vec{v}_1 & \vec{v}_2 & \vec{v}_3\end{bmatrix} = \begin{bmatrix}\lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3\end{bmatrix} \quad (2.38)$$

Let us further investigate the structure of eigenvectors $\vec{v}_2, \vec{v}_3$. We shall show that they are "conjugated". Let's write $\vec{v}_2 = \vec{u} + \vec{w}i$ with real vectors $\vec{u}, \vec{w}$. There holds true

$$\begin{aligned}\mathtt{R}\,\vec{v}_2 &= \mathtt{R}\,(\vec{u}+\vec{w}i) = \mathtt{R}\,\vec{u} + \mathtt{R}\,\vec{w}\,i && (2.39)\\ (x+yi)\,\vec{v}_2 &= (x+yi)\,(\vec{u}+\vec{w}i) = x\,\vec{u} - y\,\vec{w} + (x\vec{w}+y\vec{u})i && (2.40)\end{aligned}$$

which implies

$$\mathtt{R}\,\vec{u} = x\,\vec{u} - y\,\vec{w} \quad \text{and} \quad \mathtt{R}\,\vec{w} = x\,\vec{w} + y\,\vec{u} \quad (2.41)$$

Now, let us compare two expressions: $\mathtt{R}\,(\vec{u}-\vec{w}i)$ and $(x-yi)\,(\vec{u}-\vec{w}i)$

$$\begin{aligned}\mathtt{R}\,(\vec{u}-\vec{w}i) &= \mathtt{R}\,\vec{u}-\mathtt{R}\,\vec{w}i = x\,\vec{u}-y\,\vec{w}-(x\,\vec{w}+y\,\vec{u})\,i &(2.42)\\ (x-yi)\,(\vec{u}-\vec{w}i) &= x\,\vec{u}-y\,\vec{w}-(x\,\vec{w}+y\,\vec{u})\,i &(2.43)\end{aligned}$$

We see that

$$\mathtt{R}\,(\vec{u}-\vec{w}i) = (x-yi)\,(\vec{u}-\vec{w}i) \quad (2.44)$$

which means that $(x-yi, \vec{u}-\vec{w}i)$ are an eigenvalue-eigenvector pair of $\mathtt{R}$. It is importatnt to understand what has been shown. We have shown that if $\vec{u}+\vec{w}i$ is an eigenvector of $\mathtt{R}$ corresponding to an eigenvalue λ, then the conjugated vector $\vec{u}-\vec{w}i$ is an eignevector of $\mathtt{R}$ corresponding to eigenvalue, which is conjugated to λ (This does not mean that every two eigenvectors corresponding to $x+yi$ and $x-yi$ must be conjugated).

The conclusion from the previous analysis is that the both non-real eigenvectors of $\mathtt{R}$ are generated by the same two real vectors $\vec{u}$ and $\vec{w}$. Let us look at the angle between $\vec{u}$ and $\vec{w}$. Consider that

$$\begin{aligned}0 = \vec{v}_3^\dagger\vec{v}_2 &= (\vec{u}-\vec{w}i)^\dagger(\vec{u}+\vec{w}i) = (\vec{u}^\top+\vec{w}^\top i)(\vec{u}+\vec{w}i) &(2.45)\\ &= (\vec{u}^\top\vec{u}-\vec{w}^\top\vec{w})+(\vec{u}^\top\vec{w}+\vec{w}^\top\vec{u})\,i &(2.46)\\ &= (\vec{u}^\top\vec{u}-\vec{w}^\top\vec{w})+2\,\vec{w}^\top\vec{u}\,i &(2.47)\end{aligned}$$

and therefore

$$\vec{u}^\top\vec{u} = \vec{w}^\top\vec{w} \quad \text{and} \quad \vec{w}^\top\vec{u} = 0 \quad (2.48)$$

which means that vectors $\vec{u}$ and $\vec{w}$ are orthogonal.

Finally, let us consider

$$0 = \vec{v}_1^\top\vec{v}_2 = \vec{v}_1^\top\vec{u}+\vec{v}_1^\top\vec{w}i \quad (2.49)$$

and hence

$$\vec{v}_1^\top\vec{u} = 0 \quad \text{and} \quad \vec{v}_1^\top\vec{w} = 0 \quad (2.50)$$

which means that $\vec{u}$ and $\vec{w}$ are also orthogonal to $\vec{v}_1$.

2.1.4 Rotation axis

A one-dimensional subspace generated by an eigenvector $\vec{v}_1$ of $\mathtt{R}$ corresponding to $\lambda = 1$, is called the *rotation axis* (or axis of rotation) of $\mathtt{R}$. If $\mathtt{R} = \mathtt{I}$, then there is an infinite number of rotation axes, otherwise there is exactly one. Vectors $\vec{v}$, which are in a rotation axis of rotation $\mathtt{R}$, remain unchanged by $\mathtt{R}$, i.e. $\mathtt{R}\,\vec{v} = \vec{v}$.

Consider that the eigenvector of $\mathtt{R}$ corresponding to 1 is also an eigenvector of $\mathtt{R}^\top$ since

$$\mathtt{R}\,\vec{v}_1 = \vec{v}_1 \tag{2.51}$$

$$\mathtt{R}^\top\mathtt{R}\,\vec{v}_1 = \mathtt{R}^\top\vec{v}_1 \tag{2.52}$$

$$\vec{v}_1 = \mathtt{R}^\top\vec{v}_1 \tag{2.53}$$

It implies

$$(\mathtt{R} - \mathtt{R}^\top)\,\vec{v}_1 = 0 \tag{2.54}$$

$$\begin{bmatrix} 0 & r_{12} - r_{21} & r_{13} - r_{31} \\ r_{21} - r_{12} & 0 & r_{23} - r_{32} \\ r_{31} - r_{13} & r_{32} - r_{23} & 0 \end{bmatrix} \vec{v}_1 = 0 \tag{2.55}$$

and we see that

$$\begin{bmatrix} 0 & r_{12} - r_{21} & r_{13} - r_{31} \\ r_{21} - r_{12} & 0 & r_{23} - r_{32} \\ r_{31} - r_{13} & r_{32} - r_{23} & 0 \end{bmatrix} \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{2.56}$$

Clearly, we have a nice formula for an eigenvector corresponding to $\lambda_1 = 1$, when vector $\begin{bmatrix} r_{32} - r_{23} & r_{13} - r_{31} & r_{21} - r_{12} \end{bmatrix}^\top$ is non-zero. That is when $\mathtt{R} - \mathtt{R}^\top$ is a non-zero matrix, which is exactly when $\mathtt{R}$ is not symmetric.

Let us now investigate the situation when $\mathtt{R}$ is symmetric. Then, $\mathtt{R} = \mathtt{R}^\top = \mathtt{R}^{-1}$ and therefore

$$\mathtt{R}\,(\mathtt{R} + \mathtt{I}) = \mathtt{R}\,\mathtt{R} + \mathtt{R} = \mathtt{I} + \mathtt{R} = \mathtt{R} + \mathtt{I} \tag{2.57}$$

which shows that the non-zero columns of the matrix $\mathtt{R}+\mathtt{I}$ are eigenvectors corresponding to the unit eigenvalue. Clearly, at least one of the columns must be non-zero since otherwise, $\mathtt{R} = -\mathtt{I}$ and $|\mathtt{R}|$ would be minus one, which is impossible for a rotation.

2.1.5 Rotation angle

Rotation angle θ of rotation $\mathtt{R}$ is the angle between a non-zero real vector $\vec{x}$ which is orthogonal to $\vec{v}_1$ and its image $\mathtt{R}\,\vec{x}$. There holds

$$\cos\theta = \frac{\vec{x}^\top \mathtt{R}\,\vec{x}}{\vec{x}^\top \vec{x}} \tag{2.58}$$

Let us set

$$\vec{x} = \vec{u} + \vec{w} \tag{2.59}$$

Clearly, $\vec{x}$ is a real vector which is orthogonal to $\vec{v}_1$ since both $\vec{u}$ and $\vec{w}$ are. Let's see that it is non-zero. Vector $\vec{v}_2$ is an eigenvector and thus

$$0 \neq \vec{v}_2^\top \vec{v}_2 = \vec{u}^\top \vec{u} + \vec{w}^\top \vec{w} \tag{2.60}$$

and therefore $\vec{u} \neq \vec{0}$ or $\vec{w} \neq \vec{0}$. Vectors $\vec{u}$, $\vec{w}$ are orthogonal and therefore their sum can be zero only if they both are zero since otherwise for, e.g., a non-zero $\vec{u}$ we get the following contradiction

$$0 = \vec{u}^\top \vec{0} = \vec{u}^\top (\vec{u} + \vec{v}) = \vec{u}^\top \vec{u} + \vec{u}^\top \vec{v} = \vec{u}^\top \vec{u} \neq 0 \tag{2.61}$$

Let us now evaluate

$$\begin{aligned}
\cos\theta &= \frac{\vec{x}^\top \mathtt{R}\,\vec{x}}{\vec{x}^\top \vec{x}} = \frac{(\vec{u}+\vec{w})^\top \mathtt{R}\,(\vec{u}+\vec{w})}{(\vec{u}+\vec{w})^\top (\vec{u}+\vec{w})} = \frac{(\vec{u}+\vec{w})^\top (x\,\vec{u} - y\,\vec{w} + x\,\vec{w} + y\,\vec{u})}{\vec{u}^\top \vec{u} + \vec{w}^\top \vec{w}} \\
&= \frac{x\,(\vec{u}^\top \vec{u} + \vec{w}^\top \vec{w}) + y\,(\vec{u}^\top \vec{u} - \vec{w}^\top \vec{w})}{\vec{u}^\top \vec{u} + \vec{w}^\top \vec{w}} && (2.62) \\
&= x && (2.63)
\end{aligned}$$

We have used equation 2.41 and equation 2.48. We see that the rotation angle is given by the real part of λ_2 (or λ_3). Consider the characteristic equation of R, Equation 2.13

$$\begin{aligned} 0 &= \lambda^3 - \operatorname{trace} \mathtt{R}\, \lambda^2 + (\mathtt{R}_{11} + \mathtt{R}_{22} + \mathtt{R}_{33})\, \lambda - |\mathtt{R}| & (2.64) \\ &= (\lambda - 1)(\lambda - x - yi)(\lambda - x + yi) & (2.65) \\ &= \lambda^3 - (2\,x + 1)\, \lambda^2 + (x^2 + 2\,x + y^2)\, \lambda - (x^2 + y^2) & (2.66) \end{aligned}$$

We see that $\operatorname{trace} \mathtt{R} = 2\,x + 1$ and thus

$$\cos\theta = \frac{1}{2}(\operatorname{trace} \mathtt{R} - 1) \tag{2.67}$$

2.1.6 Matrix $(\mathtt{R} - \mathtt{I})$.

We have seen that $\operatorname{rank}(\mathtt{R} - \mathtt{I}) = 0$ for $\mathtt{R} = \mathtt{I}$ and $\operatorname{rank}(\mathtt{R} - \mathtt{I}) = 2$ for all rotation matrices $\mathtt{R} \neq \mathtt{I}$.

Let us next investigate the relationship between the range and the null space of $(\mathtt{R} - \mathtt{I})$. The null space of $(\mathtt{R} - \mathtt{I})$ is generated by eigenvectors corresponding to 1 since $(\mathtt{R} - \mathtt{I})\,\vec{v} = 0$ implies $\mathtt{R}\,\vec{v} = \vec{v}$.

Now assume that vector $\vec{v}$ is also in the range of $(\mathtt{R} - \mathtt{I})$. Then, there is a vector $\vec{a} \in \mathbb{R}^3$ such that $\vec{v} = (\mathtt{R} - \mathtt{I})\,\vec{a}$. Let us evaluate the square of the length of $\vec{v}$

$$\vec{v}^\top \vec{v} \;=\; \vec{v}^\top (\mathtt{R} - \mathtt{I})\,\vec{a} = (\vec{v}^\top \mathtt{R} - \vec{v}^\top)\,\vec{a} = (\vec{v}^\top - \vec{v}^\top)\,\vec{a} = 0 \tag{2.68}$$

which implies $\vec{v} = \vec{0}$. We have used result 2.32 with $x = 1$ and $y = 0$. Hence, the range of $\mathtt{R} - \mathtt{I}$ intersects the null space of $\mathtt{R} - \mathtt{I}$ in the zero vector.

2.1.7 Tangent space to rotations

The set of rotation matrices

$$\mathcal{R} = \left\{\mathtt{R} \in \mathbb{R}^{3\times 3} \,|\, \mathtt{R}^\top \mathtt{R} = \mathtt{I},\;\; |\mathtt{R}| = 1\right\} \tag{2.69}$$

can be understood as a subset of $\mathbb{R}^9$ with

$$\mathtt{r} = \begin{bmatrix} r_{11} & r_{21} & r_{31} & r_{12} & r_{22} & r_{32} & r_{12} & r_{23} & r_3 \end{bmatrix}^\top \text{ representing } \mathtt{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \tag{2.70}$$

Rotation constraints in definition 2.69 are algebraic and thus $\mathcal{R}$ is a *an affine variety*.[3]. Let us investigate how does look the tangent space to $\mathcal{R}$.

To get the tangent space to $\mathcal{R}$, we will first find the normal $N_\mathtt{R}$ to $\mathcal{R}$ at rotation $\mathtt{R}$ and then take its orthogonal complement $T_\mathtt{R}$, which is tangent to $\mathcal{R}$ at $\mathtt{R}$. In the end, we will write it all down in a convenient matrix form.

The space $N_\mathtt{R}$, normal to $\mathcal{R}$, is generated by columns of the *Jacobian matrix* [**?**] of constraints in 2.69, written in a matrix form as

$$\mathtt{C} = \begin{bmatrix} r_{11}\,r_{12} + r_{21}\,r_{22} + r_{31}\,r_{32} \\ r_{11}\,r_{13} + r_{21}\,r_{23} + r_{31}\,r_{33} \\ r_{12}\,r_{13} + r_{22}\,r_{23} + r_{32}\,r_{33} \\ r_{11}^2 + r_{21}^2 + r_{31}^2 - 1 \\ r_{12}^2 + r_{22}^2 + r_{32}^2 - 1 \\ r_{13}^2 + r_{23}^2 + r_{33}^2 - 1 \\ r_{11}\,r_{22}\,r_{33} - r_{11}\,r_{23}\,r_{32} - r_{12}\,r_{21}\,r_{33} + r_{12}\,r_{23}\,r_{31} + r_{13}\,r_{21}\,r_{32} - r_{13}\,r_{22}\,r_{31} - 1 \end{bmatrix} \tag{2.71}$$

[3]Affine variety is a subset of a linear space defined by algebraic constraints

The Jacobian matrix of $\mathtt{C}$ is obtained as

$$\mathtt{J}_{ij} = \frac{\partial \mathtt{C}_i}{\partial \mathtt{r}_j}, \qquad \mathtt{J} = \begin{bmatrix} r_{12} & r_{22} & r_{32} & r_{11} & r_{21} & r_{31} & 0 & 0 & 0 \\ r_{13} & r_{23} & r_{33} & 0 & 0 & 0 & r_{11} & r_{21} & r_{31} \\ 0 & 0 & 0 & r_{13} & r_{23} & r_{33} & r_{12} & r_{22} & r_{32} \\ 2\,r_{11} & 2\,r_{21} & 2\,r_{31} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\,r_{12} & 2\,r_{22} & 2\,r_{32} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2\,r_{13} & 2\,r_{23} & 2\,r_{33} \\ J_{71} & J_{72} & J_{73} & J_{74} & J_{75} & J_{76} & J_{77} & J_{78} & J_{79} \end{bmatrix}$$

with

$$\begin{aligned}
J_{71} &= r_{22}\,r_{33} - r_{23}\,r_{32} \\
J_{72} &= -r_{12}\,r_{33} + r_{13}\,r_{32} \\
J_{73} &= r_{12}\,r_{23} - r_{13}\,r_{22} \\
J_{74} &= -r_{21}\,r_{33} + r_{23}\,r_{31} \\
J_{75} &= r_{11}\,r_{33} - r_{13}\,r_{31} \\
J_{76} &= -r_{11}\,r_{23} + r_{13}\,r_{21} \\
J_{77} &= r_{21}\,r_{32} - r_{22}\,r_{31} \\
J_{78} &= -r_{11}\,r_{32} + r_{12}\,r_{31} \\
J_{79} &= r_{11}\,r_{22} - r_{12}\,r_{21}
\end{aligned}$$

Jacobian matrix $\mathtt{J}$ is a 7×9 matrix. The first three rows of $\mathtt{J}$ contain the elements of two columns of $\mathtt{R}$. The next three rows contain one column of $\mathtt{R}$. It suggests to construct a basis $\mathtt{T}$ of the tangent space $T_{\mathtt{R}}$ to $\mathcal{R}$ from columns of R. We can check that

$$\mathtt{J}\,\mathtt{T} = 0 \quad \text{with} \quad \mathtt{T} = \begin{bmatrix} 0 & -r_{13} & r_{12} \\ 0 & -r_{23} & r_{22} \\ 0 & -r_{33} & r_{32} \\ r_{13} & 0 & -r_{11} \\ r_{23} & 0 & -r_{21} \\ r_{33} & 0 & -r_{31} \\ -r_{12} & r_{11} & 0 \\ -r_{22} & r_{21} & 0 \\ -r_{32} & r_{31} & 0 \end{bmatrix}. \tag{2.72}$$

Next, we can see that each column of T contains two different columns of R and hence $\mathtt{T}\,\mathbf{x} = 0$ for a non-zero $\mathbf{x}$ implies that every two columns of R are linearly dependent, which is impossible. Therefore, T has rank equal to three at least.

Finally, the first six rows of J contain columns of R. We see that $\begin{bmatrix} \mathbf{x}^\top & 0 \end{bmatrix} \mathtt{J} = 0$ for a non-zero $\mathbf{x}$ implies that columns of R are linearly dependent, which is impossible. Therefore, the rank of $N_\mathtt{R}$ is not smaller than six. Hence, the dimension of the tangent space $T_\mathtt{R}$ is exactly three at every $\mathtt{R} \in \mathcal{R}$ and T is indeed a basis of $T_\mathtt{R}$.

Let us now rewrite the above back into a matrix form by inverting the matrix vectorization used in 2.70. We rewrite columns of T into three matrices

$$\mathtt{T}_1 = \begin{bmatrix} 0 & r_{13} & -r_{12} \\ 0 & r_{23} & -r_{22} \\ 0 & r_{33} & -r_{32} \end{bmatrix}, \mathtt{T}_2 = \begin{bmatrix} -r_{13} & 0 & r_{11} \\ -r_{23} & 0 & r_{21} \\ -r_{33} & 0 & r_{31} \end{bmatrix}, \mathtt{T}_3 = \begin{bmatrix} r_{12} & -r_{11} & 0 \\ r_{22} & -r_{21} & 0 \\ r_{32} & -r_{31} & 0 \end{bmatrix} \tag{2.73}$$

and then can write the reformated tangent space of rotations at $\mathtt{R}$ for some real vector $\mathtt{s} = \begin{bmatrix} s_1 & s_2 & s_3 \end{bmatrix}$ as

$$
\begin{aligned}
\mathtt{T}_\mathtt{R}(\mathtt{s}) &= \mathtt{T}_1\, s_1 + \mathtt{T}_2\, s_2 + \mathtt{T}_3\, s_3 && (2.74)\\
&= \left[-s_2 \begin{bmatrix} r_{13} \\ r_{23} \\ r_{33} \end{bmatrix} + s_3 \begin{bmatrix} r_{12} \\ r_{22} \\ r_{32} \end{bmatrix}, \quad s_1 \begin{bmatrix} r_{13} \\ r_{23} \\ r_{33} \end{bmatrix} - s_3 \begin{bmatrix} r_{11} \\ r_{21} \\ r_{31} \end{bmatrix}, \quad -s_1 \begin{bmatrix} r_{12} \\ r_{22} \\ r_{32} \end{bmatrix} + s_2 \begin{bmatrix} r_{11} \\ r_{21} \\ r_{31} \end{bmatrix} \right] \\
&= \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} 0 & -s_3 & s_2 \\ s_3 & 0 & -s_1 \\ -s_2 & s_1 & 0 \end{bmatrix} && (2.75)\\
&= \mathtt{R}\,[\mathtt{s}]_\times && (2.76)
\end{aligned}
$$

The first order approximation of rotations around $\mathtt{R}$ is then obtained as

$$\mathtt{R} + \mathtt{T}_\mathtt{R}(\mathtt{s}) = \mathtt{R} + \mathtt{R}\,[\mathtt{s}]_\times = \mathtt{R}\,(\mathtt{I} + [\mathtt{s}]_\times) \tag{2.77}$$

In particular, vectors in the tangent spaces to the space of rotations at the identity, which are called *infinitesimal rotations*, are

$$\mathtt{T}_\mathtt{I}(\mathtt{s}) = [\mathtt{s}]_\times \tag{2.78}$$

and the first order approximation of rotations at identity is

$$\mathtt{I} + \mathtt{T}_\mathtt{I}(\mathtt{s}) = \mathtt{I} + [\mathtt{s}]_\times \tag{2.79}$$

END