1 Rotation representation and parameterization

We have seen Chapter **??** that rotation can be represented by an orthonormal matrix $\mathtt{R}$. Matrix $\mathtt{R}$ has nine elements and there are six constraints $\mathtt{R}^\top\mathtt{R} = \mathtt{I}$ and one constratint $|\mathtt{R}| = 1$. Hence, we can view the space of all rotation matrices as a subset of $\mathbb{R}^9$. This subset[1] is determined by seven polynomial equations in nine variables. We will next investigate how to describe, i.e. *parameterize*, this set with fewer parameters and fewer constraints.

1.1 Angle-axis representation of rotation

We know, Paragraph **??**, that every rotation is etermined by a rotation axis and a rotation angle. Let us next give a classical construction of the rotation matrix from an axis and angle.

Figure 1.1 shows how the vector $\vec{x}$ rotates by angle θ around an axis given by a unit vector $\vec{v}$ into vector $\vec{y}$. To find the relationship between $\vec{x}$ and $\vec{y}$, we shall construct a special basis of $\mathbb{R}^3$. Vector $\vec{x}$ either is, or it is not a multiple of $\vec{v}$. If it is, than $\vec{y} = \vec{x}$ and $\mathtt{R} = \mathtt{I}$. Let us alternatively consider $\vec{x}$, which is not a multiple of $\vec{v}$ (an hence is not the zero vector!). Futher, let us consider the standard basis σ of $\mathbb{R}^3$ and coordinates of vectors $\vec{x}_\sigma$ and $\vec{v}_\sigma$. We construct three non-zero vectors

$$\vec{x}_{\|\sigma} = (\vec{v}_\sigma^\top \vec{x}_\sigma)\, \vec{v}_\sigma \tag{1.1}$$

$$\vec{x}_{\perp\sigma} = \vec{x} - (\vec{v}_\sigma^\top \vec{x}_\sigma)\, \vec{v}_\sigma \tag{1.2}$$

$$\vec{x}_{\times\sigma} = \vec{v}_\sigma \times \vec{x}_\sigma \tag{1.3}$$

which are mutually orthogonal and hence form a basis of $\mathbb{R}^3$. We may notice that cooridate vectors $\vec{x} \in \mathbb{R}^3$, are actually equal to their coordinates w.r.t. the standard basis σ. Hence we can drop σ index and write

$$\vec{x}_\| = (\vec{v}^\top \vec{x})\, \vec{v} = \vec{v}\,(\vec{v}^\top \vec{x}) = (\vec{v}\,\vec{v}^\top)\, \vec{x} = [\vec{v}]_\|\, \vec{x} \tag{1.4}$$

$$\vec{x}_\perp = \vec{x} - (\vec{v}^\top \vec{x})\, \vec{v} = \vec{x} - (\vec{v}\,\vec{v}^\top)\, \vec{x} = (\mathtt{I} - \vec{v}\,\vec{v}^\top)\, \vec{x} = [\vec{v}]_\perp\, \vec{x} \tag{1.5}$$

$$\vec{x}_\times = \vec{v} \times \vec{x} = [\vec{v}]_\times\, \vec{x} \tag{1.6}$$

[1] It is often called algebraic variaty in specialized literature [1].

We have introduced two new matrices

$$[\vec{v}]_{\|} = \vec{v}\,\vec{v}^\top \quad \text{and} \quad [\vec{v}]_{\perp} = \mathtt{I} - \vec{v}\,\vec{v}^\top \tag{1.7}$$

Let us next study how the three matrices $[\vec{v}]_{\|}$, $[\vec{v}]_{\perp}$, $[\vec{v}]_{\times}$ behave under the transposition and mutual multiplication. We see that the following indentities

$$\begin{array}{llll}
[\vec{v}]_{\|}^\top = [\vec{v}]_{\|}, & [\vec{v}]_{\|}\,[\vec{v}]_{\|} = [\vec{v}]_{\|}, & [\vec{v}]_{\|}\,[\vec{v}]_{\perp} = \mathtt{0}, & [\vec{v}]_{\|}\,[\vec{v}]_{\times} = \mathtt{0}, \\
[\vec{v}]_{\perp}^\top = [\vec{v}]_{\perp}, & [\vec{v}]_{\perp}\,[\vec{v}]_{\|} = \mathtt{0}, & [\vec{v}]_{\perp}\,[\vec{v}]_{\perp} = [\vec{v}]_{\perp}, & [\vec{v}]_{\perp}\,[\vec{v}]_{\times} = [\vec{v}]_{\times}, \\
[\vec{v}]_{\times}^\top = -[\vec{v}]_{\times}, & [\vec{v}]_{\times}\,[\vec{v}]_{\|} = \mathtt{0}, & [\vec{v}]_{\times}\,[\vec{v}]_{\perp} = [\vec{v}]_{\times}, & [\vec{v}]_{\times}\,[\vec{v}]_{\times} = -[\vec{v}]_{\perp}
\end{array} \tag{1.8}$$

hold true. The last identity is obtained as follows

$$[\vec{v}]_{\times}\,[\vec{v}]_{\times} = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix} \tag{1.9}$$

$$= \begin{bmatrix} -v_2^2 - v_3^2 & v_1 v_2 & v_1 v_3 \\ v_1 v_2 & -v_1^2 - v_3^2 & v_2 v_3 \\ v_1 v_3 & v_2 v_3 & -v_1^2 - v_2^2 \end{bmatrix} \tag{1.10}$$

$$= \begin{bmatrix} v_1^2 - 1 & v_1 v_2 & v_1 v_3 \\ v_1 v_2 & v_2^2 - 1 & v_2 v_3 \\ v_1 v_3 & v_2 v_3 & v_3^2 - 1 \end{bmatrix} = [\vec{v}]_{\|} - \mathtt{I} = -[\vec{v}]_{\perp} \tag{1.11}$$

It is also interesting to investigate the norms of vectors $\vec{x}_\perp$ and $\vec{x}_\times$. Consider

$$\|\vec{x}_\times\|^2 = \vec{x}_\times^\top \vec{x}_\times = \vec{x}^\top\,[\vec{v}]_{\times}^\top\,[\vec{v}]_{\times}\,\vec{x} = \vec{x}^\top(-[\vec{v}]_{\times}^2)\vec{x} = \vec{x}^\top\,[\vec{v}]_{\perp}\,\vec{x} \tag{1.12}$$

$$\|\vec{x}_\perp\|^2 = \vec{x}_\perp^\top \vec{x}_\perp = \vec{x}^\top\,[\vec{v}]_{\perp}^\top\,[\vec{v}]_{\perp}\,\vec{x} = \vec{x}^\top\,[\vec{v}]_{\perp}^2\,\vec{x} = \vec{x}^\top\,[\vec{v}]_{\perp}\,\vec{x} \tag{1.13}$$

Since norms are non-negaive, we conclude that $\|\vec{x}_\perp\| = \|\vec{x}_\times\|$.

We can now write $\vec{y}$ in the basis $[\vec{x}_\|, \vec{x}_\perp, \vec{x}_\times]$ as

$$\vec{y} = \vec{x}_\| + ||\vec{x}_\perp||\,\cos\theta\,\frac{\vec{x}_\perp}{||\vec{x}_\perp||} + ||\vec{x}_\perp||\,\sin\theta\,\frac{\vec{x}_\times}{||\vec{x}_\times||} \tag{1.14}$$

$$= \vec{x}_\| + \cos\theta\,\vec{x}_\perp + \sin\theta\,\vec{x}_\times \tag{1.15}$$

$$= [\vec{v}]_{\|}\,\vec{x} + \cos\theta\,[\vec{v}]_{\perp}\,\vec{x} + \sin\theta\,[\vec{v}]_{\times}\,\vec{x} \tag{1.16}$$

$$= ([\vec{v}]_{\|} + \cos\theta\,[\vec{v}]_{\perp} + \sin\theta\,[\vec{v}]_{\times})\,\vec{x} = \mathtt{R}\,\vec{x} \tag{1.17}$$

We obtained matrix

$$\mathtt{R} = [\vec{v}]_{\|} + \cos\theta\,[\vec{v}]_{\perp} + \sin\theta\,[\vec{v}]_{\times} \tag{1.18}$$

Let us check that this indeed is a rotation matrix

$$
\begin{aligned}
\mathtt{R}^\top\mathtt{R} &= \left([\vec{v}]_{||} + \cos\theta\,[\vec{v}]_\perp + \sin\theta\,[\vec{v}]_\times\right)^\top \left([\vec{v}]_{||} + \cos\theta\,[\vec{v}]_\perp + \sin\theta\,[\vec{v}]_\times\right) \\
&= \left([\vec{v}]_{||} + \cos\theta\,[\vec{v}]_\perp - \sin\theta\,[\vec{v}]_\times\right) \left([\vec{v}]_{||} + \cos\theta\,[\vec{v}]_\perp + \sin\theta\,[\vec{v}]_\times\right) \\
&= [\vec{v}]_{||} + \cos^2\theta\,[\vec{v}]_\perp + \sin\theta\cos\theta\,[\vec{v}]_\times - \sin\theta\cos\theta\,[\vec{v}]_\times + \sin^2\theta\,[\vec{v}]_\perp \\
&= [\vec{v}]_{||} + [\vec{v}]_\perp = \mathtt{I}
\end{aligned} \tag{1.19}
$$

$\mathtt{R}$ can be wrtten in many variations, which are useful in different situations when simplifying formulas. Let us provide the most common of them using $[\vec{v}]_{||} = \vec{v}\,\vec{v}^\top$, $[\vec{v}]_\perp = \mathtt{I} - [\vec{v}]_{||} = \mathtt{I} - \vec{v}\,\vec{v}^\top$ and $[\vec{v}]_\times$

$$
\begin{aligned}
\mathtt{R} &= [\vec{v}]_{||} + \cos\theta\,[\vec{v}]_\perp + \sin\theta\,[\vec{v}]_\times && (1.20)\\
&= \vec{v}\,\vec{v}^\top + \cos\theta\,(\mathtt{I} - \vec{v}\,\vec{v}^\top) + \sin\theta\,[\vec{v}]_\times && (1.21)\\
&= \cos\theta\,\mathtt{I} + (1-\cos\theta)\,\vec{v}\,\vec{v}^\top + \sin\theta\,[\vec{v}]_\times && (1.22)\\
&= \cos\theta\,\mathtt{I} + (1-\cos\theta)\,[\vec{v}]_{||} + \sin\theta\,[\vec{v}]_\times && (1.23)\\
&= \cos\theta\,\mathtt{I} + (1-\cos\theta)\,(\mathtt{I} + [\vec{v}]_\times^2) + \sin\theta\,[\vec{v}]_\times && (1.24)\\
&= \mathtt{I} + (1-\cos\theta)\,[\vec{v}]_\times^2 + \sin\theta\,[\vec{v}]_\times && (1.25)
\end{aligned}
$$

1.1.1 Angle-axis parameterization

Let us write $\mathtt{R}$ in more detail

$$
\begin{aligned}
\mathtt{R} &= \cos\theta\,\mathtt{I} + (1-\cos\theta)\,\vec{v}\,\vec{v}^\top + \sin\theta\,[\vec{v}]_\times && (1.26)\\
&= (1-\cos\theta)\,\vec{v}\,\vec{v}^\top + \cos\theta\,\mathtt{I} + \sin\theta\,[\vec{v}]_\times && (1.27)\\
&= (1-\cos\theta)\begin{bmatrix} v_1v_1 & v_1v_2 & v_1v_3 \\ v_2v_1 & v_2v_2 & v_2v_3 \\ v_3v_1 & v_3v_2 & v_3v_3 \end{bmatrix} + \cos\theta \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \sin\theta \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} v_1v_1(1-\cos\theta) + \cos\theta & v_1v_2(1-\cos\theta) - v_3\sin\theta & v_1v_3(1-\cos\theta) + v_2\sin\theta \\ v_2v_1(1-\cos\theta) + v_3\sin\theta & v_2v_2(1-\cos\theta) + \cos\theta & v_2v_3(1-\cos\theta) - v_1\sin\theta \\ v_3v_1(1-\cos\theta) - v_2\sin\theta & v_3v_2(1-\cos\theta) + v_1\sin\theta & v_3v_3(1-\cos\theta) + \cos\theta \end{bmatrix} && (1.28)
\end{aligned}
$$

which allows us to parameterize rotation by four numbers

$$
\begin{bmatrix} \theta & v_1 & v_2 & v_3 \end{bmatrix}^\top \quad \text{with} \quad v_1^2 + v_2^2 + v_3^2 = 1 \tag{1.29}
$$

The parameterization uses goniometric functions.

1.1.2 Computing the axis and the angle of rotation from $\mathtt{R}$

Let us now discuss how to get a unit vector $\vec{v}$ of the axis and the corresponding angle θ of rotation from a rotation matrix $\mathtt{R}$, such that the pair $[\theta, \vec{v}]$ gives $\mathtt{R}$ by Equation 1.28. To avoid multiple representations due to periodicity of θ, we will confine θ to real interval $(-\pi, \pi]$.

We can get $\cos(\theta)$ from Equation **??**.

If $\cos\theta = 1$, then $\sin\theta = 0$, and thus $\theta = 0$. Then, $\mathtt{R} = \mathtt{I}$ and any unit vector can be taken as $\vec{v}$, i.e. all paris $[0, \vec{v}]$ for unit vector $\vec{v} \in \mathbb{R}^3$ represent $\mathtt{I}$.

If $\cos\theta = -1$, then $\sin\theta = 0$, and thus $\theta = \pi$. Then $\mathtt{R}$ is a symmetrical matrix and we use Equation **??** to get $\vec{v}_1$, a non-zero multiple of $\vec{v}$, i.e. $\vec{v} = \alpha\,\vec{v}_1$, with real non-zero α, and therefore $\vec{v}_1/||\vec{v}_1|| = s\,\vec{v}$ with $s = \pm 1$. We are getting

$$
\begin{aligned}
\mathtt{R} &= 2\,[\vec{v}]_{||} - \mathtt{I} = 2\,\vec{v}\,\vec{v}^\top - \mathtt{I} = 2\,s^2\vec{v}\,\vec{v}^\top - \mathtt{I} = 2\,(s\,\vec{v})\,(s\,\vec{v})^\top - \mathtt{I} & (1.30)\\
&= 2\left(\frac{\vec{v}_1}{\|\vec{v}_1\|}\right)\left(\frac{\vec{v}_1}{\|\vec{v}_1\|}\right)^\top - \mathtt{I} = 2\left(-\frac{\vec{v}_1}{\|\vec{v}_1\|}\right)\left(-\frac{\vec{v}_1}{\|\vec{v}_1\|}\right)^\top - \mathtt{I} & (1.31)
\end{aligned}
$$

from Equation 1.27 and hence we can form two pairs

$$
\left[\pi, +\frac{\vec{v}_1}{\|\vec{v}_1\|}\right], \quad \left[\pi, -\frac{\vec{v}_1}{\|\vec{v}_1\|}\right] \tag{1.32}
$$

representing this rotation.

Let's now move to $-1 < \cos\theta < 1$. We construct matrix

$$
\begin{aligned}
\mathtt{R} - \mathtt{R}^\top &= (1-\cos\theta)\,[\vec{v}]_{||} + \cos\theta\,\mathtt{I} + \sin\theta\,[\vec{v}]_\times \\
&\qquad - \left((1-\cos\theta)\,[\vec{v}]_{||} + \cos\theta\,\mathtt{I} + \sin\theta\,[\vec{v}]_\times\right)^\top & (1.33)\\
&= (1-\cos\theta)\,[\vec{v}]_{||} + \cos\theta\,\mathtt{I} + \sin\theta\,[\vec{v}]_\times \\
&\qquad - \left((1-\cos\theta)\,[\vec{v}]_{||} + \cos\theta\,\mathtt{I} - \sin\theta\,[\vec{v}]_\times\right) & (1.34)\\
&= 2\,\sin\theta\,[\vec{v}]_\times & (1.35)
\end{aligned}
$$

which gives

$$
\begin{bmatrix} 0 & r_{12}-r_{21} & r_{13}-r_{31} \\ r_{21}-r_{12} & 0 & r_{23}-r_{32} \\ r_{31}-r_{13} & r_{32}-r_{23} & 0 \end{bmatrix} = 2\,\sin\theta \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix} \tag{1.36}
$$

and thus

$$
\sin\theta\,\vec{v} = \frac{1}{2}\begin{bmatrix} r_{32}-r_{23} \\ r_{13}-r_{31} \\ r_{21}-r_{12} \end{bmatrix} \tag{1.37}
$$

We thus get

$$|\sin\theta|\,||\vec{v}|| = |\sin\theta| = \frac{1}{2}\sqrt{(r_{23}-r_{32})^2+(r_{31}-r_{13})^2+(r_{12}-r_{21})^2} \tag{1.38}$$

There holds

$$\sin\theta\,\vec{v} = \sin(-\theta)\,(-\vec{v}) \tag{1.39}$$

true and hence we define

$$\theta = \arccos\left(\frac{1}{2}(\operatorname{trace}(\mathtt{R})-1)\right), \quad \vec{r} = \frac{1}{2}\begin{bmatrix} r_{32}-r_{23} \\ r_{13}-r_{31} \\ r_{21}-r_{12} \end{bmatrix} \tag{1.40}$$

and write two pairs

$$\left[+\theta, +\frac{\vec{r}}{\sin\theta}\right], \quad \left[-\theta, -\frac{\vec{r}}{\sin\theta}\right] \tag{1.41}$$

representing rotation $\mathtt{R}$.

We see that all rotations are represented by two pairs of $[\theta, \vec{v}]$ except for the identity, which is represented by an infinite number of pairs.

1.2 Euler vector representation and the exponential map

Let us now discuss another classical and natural representation of rotations. It may seem as only a slight variation of the angle-axis representation but it leads to several interesting connections and properties.

Let us consider the *euler vector* defined as

$$\vec{e} = \theta\,\vec{v} \tag{1.42}$$

where θ is the rotation angle and $\vec{v}$ is the unit vector representing the rotation axis in the angle-axis representation as in Equation 1.27.

Next, let us recall the very fundamental real functions [2] and their related power series

$$\exp x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{1.43}$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \tag{1.44}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \tag{1.45}$$

It makes sense to define the exponential function of an $m \times m$ real matrix $\mathtt{A} \in \mathbb{R}^{m \times m}$ as

$$\exp \mathtt{A} = \sum_{n=0}^{\infty} \frac{\mathtt{A}^n}{n!} \tag{1.46}$$

We will now show that the rotation matrix $\mathtt{R}$ corresponding to the angle-axis parameterization $[\theta, \vec{v}]$ can be obtained as

$$\mathtt{R}([\theta, \vec{v}]) = \exp\, [\vec{e}]_\times = \exp\, [\theta\, \vec{v}]_\times \tag{1.47}$$

The basic tool we have to employ is the relationship between $[\vec{e}]_\times^3$ and $[\vec{e}]_\times$. It will allow us to pass form the ifinite summantion of matrix powers to the infinite summation of the powers of the θ and hence to $\sin\theta$ and $\cos\theta$, which will, at the end, give the rodrigues formula. We write, Equation 1.11,

$$\begin{aligned}
[\theta\, \vec{v}]_\times^2 &= \theta^2\, (\vec{v}\, \vec{v}^\top - \mathtt{I}) \\
[\theta\, \vec{v}]_\times^3 &= -\theta^2\, [\theta \vec{v}]_\times \\
[\theta\, \vec{v}]_\times^4 &= -\theta^2\, [\theta \vec{v}]_\times^2 \\
[\theta\, \vec{v}]_\times^5 &= \theta^4\, [\theta \vec{v}]_\times \\
[\theta\, \vec{v}]_\times^6 &= \theta^4\, [\theta \vec{v}]_\times^2 \\
&\vdots
\end{aligned} \tag{1.48}$$

and substitute into Equation 1.46 to get

$$\exp\, [\theta\, \vec{v}]_\times = \sum_{n=0}^{\infty} \frac{[\theta\, \vec{v}]_\times^n}{n!} \tag{1.49}$$

$$= \sum_{n=0}^{\infty} \frac{[\theta\, \vec{v}]_\times^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{[\theta\, \vec{v}]_\times^{2n+1}}{(2n+1)!} \tag{1.50}$$

Let us notice the identities, which are obtained by generalizing Equations 1.48 to an arbitrary power n

$$\begin{aligned}
[\theta\,\vec{v}]_\times^0 &= \mathtt{I} && (1.51)\\
[\theta\,\vec{v}]_\times^{2n} &= (-1)^{n-1}\,\theta^{2(n-1)}\,[\theta\,\vec{v}]_\times^2 \;\text{ for } n = 1,\ldots && (1.52)\\
[\theta\,\vec{v}]_\times^{2n+1} &= (-1)^n\,\theta^{2n}\,[\theta\,\vec{v}]_\times \;\text{ for } n = 0,\ldots && (1.53)
\end{aligned}$$

and substitute them into Equation 1.50 to get

$$\begin{aligned}
\exp\,[\theta\,\vec{v}]_\times &= \mathtt{I} + \left(\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\theta^{2(n-1)}}{(2n)!}\right)[\theta\,\vec{v}]_\times^2 + \left(\sum_{n=0}^{\infty}\frac{(-1)^{n}\theta^{2n}}{(2n+1)!}\right)[\theta\,\vec{v}]_\times\\
&= \mathtt{I} + \left(\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\theta^{2n}}{(2n)!}\right)[\vec{v}]_\times^2 + \left(\sum_{n=0}^{\infty}\frac{(-1)^{n}\theta^{2n+1}}{(2n+1)!}\right)[\vec{v}]_\times\\
&= \mathtt{I} - \left(\sum_{n=0}^{\infty}\frac{(-1)^{n}\theta^{2n}}{(2n)!} - 1\right)[\vec{v}]_\times^2 + \sin\theta\,[\vec{v}]_\times\\
&= \mathtt{I} - (\cos\theta - 1)\,[\vec{v}]_\times^2 + \sin\theta\,[\vec{v}]_\times\\
&= \mathtt{I} + \sin\theta\,[\vec{v}]_\times + (1-\cos\theta)\,[\vec{v}]_\times^2\\
&= \mathtt{I} + \sin\|\vec{e}\|\left[\frac{\vec{e}}{\|\vec{e}\|}\right]_\times + (1-\cos\|\vec{e}\|)\left[\frac{\vec{e}}{\|\vec{e}\|}\right]_\times^2\\
&= \mathtt{R}([\theta,\vec{v}]) && (1.54)
\end{aligned}$$

by the comparison with Equation 1.25.

1.3 Quaternion representation of rotation

1.3.1 Quaternion parameterization

We shall now introdude another parameterization of $\mathtt{R}$ by four numbers but this time we will not use goniometric functions but polynomials only. We shall see later that this parameterization has other useful properties.

This paramterization is known as *unit quaternion* parameterization of rotations since rotations are represented by unit vectors from $\mathbb{R}^4$. In general, it may sense to talk even about non-unit quaternions and we will see how to use them later when applying rotations represented by unit quaternions on points represented by non-unit quaternions. To simplify our notation, we will often write "quaternions" insted of more correct "unit quaternions".

Let us do a seemingly unnecessary trick. We will pass from θ to $\frac{\theta}{2}$ and introduce

$$\vec{q} = \begin{bmatrix} \cos\frac{\theta}{2} \\ \vec{v}\sin\frac{\theta}{2} \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} \cos\frac{\theta}{2} \\ v_1\sin\frac{\theta}{2} \\ v_2\sin\frac{\theta}{2} \\ v_3\sin\frac{\theta}{2} \end{bmatrix} \tag{1.55}$$

There still holds

$$\|\vec{q}\| = q_1^2+q_2^2+q_3^2+q_4^2 = \cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}\,v_1^2+\sin^2\frac{\theta}{2}\,v_2^2+\sin^2\frac{\theta}{2}\,v_3^2 = \cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2} = 1 \tag{1.56}$$

true. We can verify that the following identities

$$\begin{aligned}
\cos\theta &= 2\cos^2\frac{\theta}{2} - 1 = 2\,q_1^2 - 1 && (1.57)\\
\sin\theta &= 2\cos\frac{\theta}{2}\,\sin\frac{\theta}{2} && (1.58)\\
\sin\theta\,\vec{v} &= 2\cos\frac{\theta}{2}\,\sin\frac{\theta}{2}\,\vec{v} = 2\,q_1\,\begin{bmatrix} q_2 & q_3 & q_4 \end{bmatrix}^\top && (1.59)\\
\cos\theta &= 1 - 2\sin^2\frac{\theta}{2} = 1 - 2\,(q_2^2+q_3^2+q_4^2) = q_1^2 - q_2^2 - q_3^2 - q_4^2 && (1.60)\\
1-\cos\theta &= 2\sin^2\frac{\theta}{2} = 2\,(q_2^2+q_3^2+q_4^2) && (1.61)
\end{aligned}$$

hold true. We can now substitute the above into Equation 1.23 to get

$$
\begin{aligned}
\mathtt{R} &= \mathtt{I} + \sin\theta\,[\vec{v}]_\times + (1-\cos\theta)\,[\vec{v}]_\times^2 && (1.62)\\
&= \mathtt{I} + 2\,\cos\frac{\theta}{2}\,\sin\frac{\theta}{2}\,[\vec{v}]_\times + 2\,\sin^2\frac{\theta}{2}\,[\vec{v}]_\times^2 && (1.63)\\
&= \mathtt{I} + 2\,\cos\frac{\theta}{2}\left[\sin\frac{\theta}{2}\vec{v}\right]_\times + 2\left[\sin\frac{\theta}{2}\vec{v}\right]_\times^2 && (1.64)\\
&= \mathtt{I} + 2\,\cos\frac{\theta}{2}\left[\sin\frac{\theta}{2}\vec{v}\right]_\times + 2\left(\left[\sin\frac{\theta}{2}\vec{v}\right]_{\|} - \mathtt{I}\right) && (1.65)\\
&= \mathtt{I} + 2\,q_1\left[\begin{bmatrix} q_2\\ q_3\\ q_4\end{bmatrix}\right]_\times + 2\left(\left[\begin{bmatrix} q_2\\ q_3\\ q_4\end{bmatrix}\right]_{\|} - \mathtt{I}\right) && (1.66)\\
&= \begin{bmatrix} 1 & -2\,q_1q_4 & 2\,q_1q_3\\ 2\,q_1q_4 & 1 & -2\,q_1q_2\\ -2\,q_1q_3 & 2\,q_1q_2 & 1\end{bmatrix} + \begin{bmatrix} 2\,q_2q_2-2 & 2\,q_2q_3 & 2\,q_2q_4\\ 2\,q_3q_2 & 2\,q_3q_3-2 & 2\,q_3q_4\\ 2\,q_4q_2 & 2\,q_4q_3 & 2\,q_4q_4-2\end{bmatrix}\\
&= \begin{bmatrix} q_1^2+q_2^2-q_3^2-q_4^2 & 2\,(q_2q_3-q_1q_4) & 2\,(q_2q_4+q_1q_3)\\ 2\,(q_2q_3+q_1q_4) & q_1^2-q_2^2+q_3^2-q_4^2 & 2\,(q_3q_4-q_1q_2)\\ 2\,(q_2q_4-q_1q_3) & 2\,(q_3q_4+q_1q_2) & q_1^2-q_2^2-q_3^2+q_4^2\end{bmatrix} && (1.67)
\end{aligned}
$$

which uses only second order polynomials in elements of $\vec{q}$.

1.3.2 Computing quaternions from $\mathtt{R}$

To get the quaternions representing a rotation matrix $\mathtt{R}$, we start with Equation 1.64. Let us first confine θ to the real interval $(-\pi,\pi]$ as we did for the angle-axis parameterization.

Matrix $\mathtt{R}$ either is or it is not symmetric.

If $\mathtt{R}$ is symmetric, then either $\sin\theta/2\,\vec{v} = \vec{0}$ or $\cos\theta/2 = 0$. If $\sin\theta/2\,\vec{v} = \vec{0}$, then $\sin\theta/2 = 0$ since $\|\vec{v}\| = 1$ and thus $\cos\theta/2 = \pm 1$. However, $\cos\theta/2 = -1$ for no $\theta \in (-\pi,\,\pi]$ and hence $\cos\theta/2 = 1$. This corresponds to $\theta = 0$ and hence to $\mathtt{R} = \mathtt{I}$ which is thus represented by quaternion

$$
\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^\top \tag{1.68}
$$

If $\cos\theta/2 = 0$, then $\sin\theta/2 = \pm 1$ but $\sin\theta/2 = -1$ for no $\theta \in (-\pi,\,\pi]$ and hence $\sin\theta/2 = 1$. This corresponds to the rotation the by $\theta = \pi$ around the axis given by unit $\vec{v} = [v_1, v_2, v_3]^\top$. This rotation is thus represented by quaternion

$$\begin{bmatrix} 0 & v_1 & v_2 & v_3 \end{bmatrix}^\top \tag{1.69}$$

Notice that $\vec{v}$ and $-\vec{v}$ generate the same rotation matrix $\mathtt{R}$ and hence every rotation by $\theta = \pi$ is represented by two quaternions.

If $\mathtt{R}$ is not symmetric, then $\mathtt{R} - \mathtt{R}^\top \neq \mathtt{0}$ and hence we are geting a useful relationship

$$\mathtt{R} - \mathtt{R}^\top = 4\cos\frac{\theta}{2}\left[\sin\frac{\theta}{2}\vec{v}\right]_\times \tag{1.70}$$

and next continue with writing

$$\cos^2\frac{\theta}{2} = 1 - \sin^2\frac{\theta}{2} = 1 - \frac{1}{2}(1-\cos\theta) = 1 - \frac{1}{2}\left(1 - \frac{1}{2}(\operatorname{trace}\mathtt{R} - 1)\right) = \frac{1}{4}(1 + \operatorname{trace}\mathtt{R}) \tag{1.71}$$

using $\operatorname{trace}\mathtt{R}$, and thus

$$q_1 = \cos\frac{\theta}{2} = \frac{s}{2}\sqrt{\operatorname{trace}\mathtt{R} + 1} \tag{1.72}$$

with $s = \pm 1$. We can form equation

$$\begin{bmatrix} 0 & r_{12} - r_{21} & r_{13} - r_{31} \\ r_{21} - r_{12} & 0 & r_{23} - r_{32} \\ r_{31} - r_{13} & r_{32} - r_{23} & 0 \end{bmatrix} = \left[\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}\right]_\times = s\sqrt{\operatorname{trace}\mathtt{R} + 1}\left[\begin{bmatrix} q_2 \\ q_3 \\ q_4 \end{bmatrix}\right]_\times \tag{1.73}$$

which gives the following two quaternions

$$\frac{+1}{2\sqrt{\operatorname{trace}\mathtt{R} + 1}}\begin{bmatrix} \operatorname{trace}\mathtt{R} + 1 \\ r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}, \quad \frac{-1}{2\sqrt{\operatorname{trace}\mathtt{R} + 1}}\begin{bmatrix} \operatorname{trace}\mathtt{R} + 1 \\ r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix} \tag{1.74}$$

which represent the same rotation as $\mathtt{R}$.

We see that all rotations are represented by the above by two quaternions $\vec{q}$ and $-\vec{q}$ except for the identity, which is represented by exactly one quaternion.

The quaternion representation of rotation presented above represents every rotation by a finite number of quaternions whereas angle-axis repesentation allowed for an infinite number of angle-axis pairs to correspond to the indentity.

Yet, even this still has an "aesthetic flaw" at the identity, which has only one quaternion whereas all other rotations have two quaternions. The "flaw" can be removed by realizing that $\vec{q} = [-1, 0, 0, 0]^\top$ also maps to the identity. However, if we look for θ that corresponds to $\cos\theta/2 = -1$ we see that such $\theta/2 = \pm k\,\pi$ and hence $\theta = \pm 2\,k\,\pi$ for $k = 1, 2, \ldots$, which are points isolated from $(-\pi, \pi]$. Now, if we allow θ to be in interval $(-2\,\pi, +2\,\pi]$, then the set

$$\left\{ \begin{bmatrix} \cos\theta/2 \\ \vec{v}\,\sin\theta/2 \end{bmatrix} \middle|\; \theta \in [-2\,\pi, +2\,\pi],\ \vec{v} \in \mathbb{R}^3,\ \|\vec{v}\| = 1 \right\} \tag{1.75}$$

of quaternions contains exactly two quaternions for every rotation matrix $\mathtt{R}$ and is obtained by a continuous mapping of a closed interval of angles, which is boundend, times a sphere in $\mathbb{R}^3$, which is also closed and bounded.

1.3.3 Quaternion composition

Consider two rotations represented by $\vec{q}_1$ and $\vec{q}_2$. The respective rotation matrices $\mathtt{R}_1$, $\mathtt{R}_2$ can be composed into rotation matrix $\mathtt{R}_{21} = \mathtt{R}_2\,\mathtt{R}_1$, which can be represented by $\vec{q}_{21}$. Let us investigate how to obtain $\vec{q}_{21}$ from $\vec{q}_1$ and $\vec{q}_2$. We shall use Equation 1.76 to relate $\mathtt{R}_1$ to $\vec{q}_1$ and $\mathtt{R}_2$ to $\vec{q}_1$, then evaluate $\mathtt{R}_{21} = \mathtt{R}_2\,\mathtt{R}_1$ and recover $\vec{q}_{21}$ from $\mathtt{R}_{21}$. We use Equation 1.23 to write

$$\mathtt{R} = 2\,\sin^2\frac{\theta}{2}\,\vec{v}\,\vec{v}^\top + (2\,\cos^2\frac{\theta}{2} - 1)\,\mathtt{I} + 2\,\cos\frac{\theta}{2}\,\sin\frac{\theta}{2}\,[\vec{v}]_\times \tag{1.76}$$

and

$$\begin{aligned}
\mathtt{R}_1 &= 2\,(s_1\vec{v}_1)\,(s_1\vec{v}_1)^\top + (2\,c_1^2 - 1)\,\mathtt{I} + 2\,c_1\,[s_1\vec{v}_1]_\times && (1.77)\\
\mathtt{R}_2 &= 2\,(s_2\vec{v}_2)\,(s_2\vec{v}_2)^\top + (2\,c_2^2 - 1)\,\mathtt{I} + 2\,c_2\,[s_2\vec{v}_2]_\times && (1.78)\\
\mathtt{R}_{21} &= 2\,(s_{21}\vec{v}_{21})\,(s_{21}\vec{v}_{21})^\top + (2\,c_{21}^2 - 1)\,\mathtt{I} + 2\,c_{21}\,[s_{21}\vec{v}_{21}]_\times
\end{aligned}$$

with shortcuts

$$c_1 = \cos\frac{\theta_1}{2},\ s_1 = \sin\frac{\theta_1}{2},\ c_2 = \cos\frac{\theta_2}{2},\ s_2 = \sin\frac{\theta_2}{2},\ c_{21} = \cos\frac{\theta_{21}}{2},\ s_{21} = \sin\frac{\theta_{21}}{2}$$

Let us next assume that both $\mathtt{R}_1$, $\mathtt{R}_2$ are not identities. Then $\theta_1 \neq 0$ and $\theta_2 \neq 0$ and rotation axes $\vec{v}_1 \neq \vec{0}$, $\vec{v}_2 \neq \vec{0}$ are well defined. We can now distinguish two cases. Either $\vec{v}_1 = \pm\vec{v}_2$, and then $\vec{v}_{21} = \vec{v}_1 = \pm\vec{v}_2$, or $\vec{v}_1 \neq \pm\vec{v}_2$, and then

$$[\vec{v}_1,\ \vec{v}_2,\ \vec{v}_2 \times \vec{v}_1] \tag{1.79}$$

forms a basis of $\mathbb{R}^3$. We also notice that $\vec{v}_1$, $\vec{v}_2$ always appear in $\mathtt{R}_1$, $\mathtt{R}_2$ in the product with s_1, s_2.

We can thus write

$$\sin\frac{\theta_{21}}{2}\,\vec{v}_{21} = a_1 \sin\frac{\theta_1}{2}\,\vec{v}_1 + a_2 \sin\frac{\theta_2}{2}\,\vec{v}_2 + a_3\,(\sin\frac{\theta_2}{2}\,\vec{v}_2 \times \sin\frac{\theta_1}{2}\,\vec{v}_1) \tag{1.80}$$

with coefficients $a_1, a_2, a_3 \in \mathbb{R}$. To find coefficients a_1, a_2, a_3, we will consider the following special situations:

1. $\vec{v}_1 = \pm\vec{v}_2$ implies $\vec{v}_{21} = \vec{v}_1 = \pm\vec{v}_2$ and $\theta_{21} = \theta_1 \pm \theta_2$ for all real θ_1 and θ_2.
2. $\vec{v}_2^\top \vec{v}_1 = 0$ and $\theta_1 = \theta_2 = \pi$ implies

$$\begin{aligned}
\mathtt{R}_1 &= 2\,\vec{v}_1\vec{v}_1^\top - \mathtt{I} && (1.81)\\
\mathtt{R}_2 &= 2\,\vec{v}_2\vec{v}_2^\top - \mathtt{I} && (1.82)\\
\mathtt{R}_{21} &= (2\,\vec{v}_2\vec{v}_2^\top - \mathtt{I})(2\,\vec{v}_1\vec{v}_1^\top - \mathtt{I}) = \mathtt{I} - 2\,(\vec{v}_2\vec{v}_2^\top + \vec{v}_1\vec{v}_1^\top) && (1.83)
\end{aligned}$$

We see that in the former case we are getting

$$\sin\frac{\theta_{21}}{2}\,\vec{v}_1 = (a_1 \sin\frac{\theta_1}{2} + a_2 \sin\frac{\theta_2}{2})\,\vec{v}_1 \quad \text{for all } \theta_1, \theta_2 \in \mathbb{R} \tag{1.84}$$

which for $\vec{v}_1 \neq \vec{0}$ leads to

$$\begin{aligned}
\sin\frac{\theta_{21}}{2} &= a_1 \sin\frac{\theta_1}{2} + a_2 \sin\frac{\theta_2}{2} && (1.85)\\
\sin\frac{\theta_1 + \theta_2}{2} &= a_1 \sin\frac{\theta_1}{2} + a_2 \sin\frac{\theta_2}{2} && (1.86)\\
\sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2} + \cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2} &= a_1 \sin\frac{\theta_1}{2} + a_2 \sin\frac{\theta_2}{2} && (1.87)
\end{aligned}$$

for all $\theta_1, \theta_2 \in \mathbb{R}$. But that means that

$$a_1 = \cos\frac{\theta_2}{2} \quad \text{and} \quad a_2 = \cos\frac{\theta_1}{2} \tag{1.88}$$

In the latter case we find that $\vec{v}_{21}$ is a non-zero multiple of $\vec{v}_2 \times \vec{v}_1$ since

$$\begin{aligned}
\mathtt{R}_{21}\,(\vec{v}_2 \times \vec{v}_1) &= (\mathtt{I} - 2\,(\vec{v}_2\vec{v}_2^\top + \vec{v}_1\vec{v}_1^\top))\,(\vec{v}_2 \times \vec{v}_1) && (1.89)\\
&= \vec{v}_2 \times \vec{v}_1 - 2\,\vec{v}_2\vec{v}_2^\top\,(\vec{v}_2 \times \vec{v}_1) - 2\,\vec{v}_1\vec{v}_1^\top\,(\vec{v}_2 \times \vec{v}_1) && (1.90)\\
&= \vec{v}_2 \times \vec{v}_1 && (1.91)
\end{aligned}$$

But that means that

$$\sin\frac{\theta_{21}}{2}\,\vec{v}_{21} = a_3\,(\sin\frac{\theta_2}{2}\vec{v}_2 \times \vec{v}_1 \sin\frac{\theta_1}{2}) \tag{1.92}$$

We next get θ_{21} using Equation **??** as

$$\cos\theta_{21} = \frac{1}{2}(\operatorname{trace}\mathtt{R} - 1) = \frac{1}{2}(3 - 2\,(\|\vec{v}_2\|^2 + \|\vec{v}_1\|^2) - 1) = \frac{1}{2}(3 - 4 - 1) = -1 \tag{1.93}$$

and hence $\theta_{21} = \pm\pi$ and thus

$$\vec{v}_{21} = a_3\,(\vec{v}_1 \times \vec{v}_2) \tag{1.94}$$

but since $\vec{v}_1$ is perpendicular to $\vec{v}_2$, $\vec{v}_1 \times \vec{v}_2$ is a unit vector and thus $a_3 = 1$. We can thus hypothesize that in general

$$\sin\frac{\theta_{21}}{2}\,\vec{v}_{21} = \cos\frac{\theta_2}{2}\left(\sin\frac{\theta_1}{2}\,\vec{v}_1\right) + \cos\frac{\theta_1}{2}\left(\sin\frac{\theta_2}{2}\,\vec{v}_2\right) + \left(\sin\frac{\theta_2}{2}\,\vec{v}_2\right) \times \left(\sin\frac{\theta_1}{2}\,\vec{v}_1\right) \tag{1.95}$$

Let's next find $\cos\frac{\theta_{21}}{2}$ consistent with the above hypothesis. We see that

$$\cos^2\frac{\theta_{21}}{2} = 1 - \sin^2\frac{\theta_{21}}{2} \tag{1.96}$$

and hence we evaluate

$$\begin{aligned}
\sin^2\frac{\theta_{21}}{2} &= \sin^2\frac{\theta_{21}}{2}\,\vec{v}_{21}^\top\vec{v}_{21} = \left(\sin\frac{\theta_{21}}{2}\,\vec{v}_{21}\right)^\top\left(\sin\frac{\theta_{21}}{2}\,\vec{v}_{21}\right) && (1.97)\\
&= \cos^2\frac{\theta_2}{2}\,\sin^2\frac{\theta_1}{2} + \cos^2\frac{\theta_1}{2}\,\sin^2\frac{\theta_2}{2} && (1.98)\\
&+ 2\cos\frac{\theta_2}{2}\,\cos\frac{\theta_1}{2}\left(\sin\frac{\theta_2}{2}\,\vec{v}_2\right)^\top\left(\sin\frac{\theta_1}{2}\,\vec{v}_1\right) && (1.99)\\
&+ \left[\left(\sin\frac{\theta_2}{2}\,\vec{v}_2\right) \times \left(\sin\frac{\theta_1}{2}\,\vec{v}_1\right)\right]^\top\left[\left(\sin\frac{\theta_2}{2}\,\vec{v}_2\right) \times \left(\sin\frac{\theta_1}{2}\,\vec{v}_1\right)\right] && (1.100)
\end{aligned}$$

We used the fact that $\vec{v}_1$, $\vec{v}_2$ are perpendicular to their vector product.

To move further, we will use that for every two unit vectors $\vec{u}$, $\vec{v}$ in $\mathbb{R}^3$ there holds

$$\begin{aligned}
(\vec{u} \times \vec{v})^\top(\vec{u} \times \vec{v}) &= \|(\vec{u} \times \vec{v})\|^2 = \|\vec{u}\|^2\|\vec{v}\|^2\sin^2\angle(\vec{u},\vec{v}) && (1.101)\\
&= \|\vec{u}\|^2\|\vec{v}\|^2(1 - \cos^2\angle(\vec{u},\vec{v})) = \|\vec{u}\|^2\|\vec{v}\|^2 - (\vec{u}^\top\vec{v})^2 && (1.102)
\end{aligned}$$

true.

Applying this to the last summand in Equation 1.100, we get

$$\begin{aligned}
\sin^2\frac{\theta_{21}}{2} &= \cos^2\frac{\theta_2}{2}\,\sin^2\frac{\theta_1}{2} + \cos^2\frac{\theta_1}{2}\,\sin^2\frac{\theta_2}{2} && (1.103)\\
&+ 2\cos\frac{\theta_2}{2}\,\cos\frac{\theta_1}{2}\,\left(\sin\frac{\theta_2}{2}\,\vec{v}_2\right)^\top\!\left(\sin\frac{\theta_1}{2}\,\vec{v}_1\right) && (1.104)\\
&+ \sin^2\frac{\theta_2}{2}\,\sin^2\frac{\theta_1}{2} - \left[\left(\sin\frac{\theta_2}{2}\,\vec{v}_2\right)^\top\!\left(\sin\frac{\theta_1}{2}\,\vec{v}_1\right)\right]^2 && (1.105)\\
&= \sin^2\frac{\theta_1}{2} + \cos^2\frac{\theta_1}{2}\,\sin^2\frac{\theta_2}{2} && (1.106)\\
&+ 2\cos\frac{\theta_2}{2}\,\cos\frac{\theta_1}{2}\,\left(\sin\frac{\theta_2}{2}\,\vec{v}_2\right)^\top\!\left(\sin\frac{\theta_1}{2}\,\vec{v}_1\right) - \left[\left(\sin\frac{\theta_2}{2}\,\vec{v}_2\right)^\top\!\left(\sin\frac{\theta_1}{2}\,\vec{v}_1\right)\right]^2 &&\\
&= 1 - \cos^2\frac{\theta_1}{2}\,\cos^2\frac{\theta_2}{2} && (1.107)\\
&+ 2\cos\frac{\theta_2}{2}\,\cos\frac{\theta_1}{2}\,\left(\sin\frac{\theta_2}{2}\,\vec{v}_2\right)^\top\!\left(\sin\frac{\theta_1}{2}\,\vec{v}_1\right) - \left[\left(\sin\frac{\theta_2}{2}\,\vec{v}_2\right)^\top\!\left(\sin\frac{\theta_1}{2}\,\vec{v}_1\right)\right]^2 &&
\end{aligned}$$

where we used the fact that

$$\begin{aligned}
\sin^2\frac{\theta_1}{2} + \cos^2\frac{\theta_1}{2}\,\sin^2\frac{\theta_2}{2} &= 1 - \cos^2\frac{\theta_1}{2} + \cos^2\frac{\theta_1}{2}\,\sin^2\frac{\theta_2}{2} && (1.108)\\
&= 1 + \cos^2\frac{\theta_1}{2}\left(\sin^2\frac{\theta_2}{2} - 1\right) = 1 - \cos^2\frac{\theta_1}{2}\,\cos^2\frac{\theta_2}{2} &&
\end{aligned}$$

We are thus obtaining

$$\begin{aligned}
\cos^2\frac{\theta_{21}}{2} &= 1 - \sin^2\frac{\theta_{21}}{2} && (1.109)\\
&= \cos^2\frac{\theta_1}{2}\,\cos^2\frac{\theta_2}{2} && (1.110)\\
&- 2\cos\frac{\theta_2}{2}\,\cos\frac{\theta_1}{2}\,\left(\sin\frac{\theta_2}{2}\,\vec{v}_2\right)^\top\!\left(\sin\frac{\theta_1}{2}\,\vec{v}_1\right) + \left[\left(\sin\frac{\theta_2}{2}\,\vec{v}_2\right)^\top\!\left(\sin\frac{\theta_1}{2}\,\vec{v}_1\right)\right]^2 &&\\
&= \left(\cos\frac{\theta_1}{2}\,\cos\frac{\theta_2}{2} - \left(\sin\frac{\theta_2}{2}\,\vec{v}_2\right)^\top\!\left(\sin\frac{\theta_1}{2}\,\vec{v}_1\right)\right)^2 && (1.111)
\end{aligned}$$

Our complete hypothesis will be

$$
\begin{aligned}
\sin\frac{\theta_{21}}{2}\,\vec{v}_{21} &= \cos\frac{\theta_2}{2}\left(\sin\frac{\theta_1}{2}\,\vec{v}_1\right) + \cos\frac{\theta_1}{2}\left(\sin\frac{\theta_2}{2}\,\vec{v}_2\right) + \left(\sin\frac{\theta_2}{2}\,\vec{v}_2\right)\times\left(\sin\frac{\theta_1}{2}\,\vec{v}_1\right)\\
\cos\frac{\theta_{21}}{2} &= \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} - \left(\sin\frac{\theta_2}{2}\,\vec{v}_2\right)^{\top}\left(\sin\frac{\theta_1}{2}\,\vec{v}_1\right)
\end{aligned}
\tag{1.112}
$$

To verify this, we will run the following Maple [3] program

```
> restart:
> with(LinearAlgebra):
> E:=IdentityMatrix(3):
> X_:=proc(u) <<0|-u[3]|u[2]>,<u[3]|0|-u[1]>,<-u[2]|u[1]|0>> end proc:

> v1:=<x1,y1,z1>:
> v2:=<x2,y2,z2>:
> R1:=2*(s1*v1).Transpose(s1*v1)+(2*c1^2-1)*E+2*c1*X_(s1*v1):
> R2:=2*(s2*v2).Transpose(s2*v2)+(2*c2^2-1)*E+2*c2*X_(s2*v2):
> R21:=expand~(R2.R1):
> c21:=c2*c1-Transpose(s2*v2).(s1*v1);
```

$$c21 := c2\,c1 - s1\,x1\,s2\,x2 - s1\,y1\,s2\,y2 - s1\,z1\,s2\,z2$$

```
> s21v21:=c2*s1*v1+s2*c1*v2+X_(s2*v2).(s1*v1);
```

$$s21v21 := \begin{bmatrix} c2\,s1\,x1 + s2\,c1\,x2 - s2\,z2\,s1\,y1 + s2\,y2\,s1\,z1 \\ c2\,s1\,y1 + s2\,c1\,y2 + s2\,z2\,s1\,x1 - s2\,x2\,s1\,z1 \\ c2\,s1\,z1 + s2\,c1\,z2 - s2\,y2\,s1\,x1 + s2\,x2\,s1\,y1 \end{bmatrix}$$

```
> RR21:=2*s21v21.Transpose(s21v21)+(2*c21^2-1)*E+2*c21*X_(s21v21):

> simplify(expand~(RR21-R21),[x1^2+y1^2+z1^2=1,x2^2+y2^2+z2^2=1,
                             c1^2+s1^2=1,c2^2+s2^2=1]);
```

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which verifies that our hypothesis was correct.

Considering two unit quaternions

$$\vec{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix}, \quad \text{and} \quad \vec{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} \tag{1.113}$$

we can now give their composition as

$$\vec{q}_{21} = \vec{q}\,\vec{p} \quad = \begin{bmatrix} q_1\,p_1 - q_2\,p_2 - q_3\,p_3 - q_4\,p_4 \\ q_1\,p_2 + q_2\,p_1 + q_3\,p_4 - q_4\,p_3 \\ q_1\,p_3 + q_3\,p_1 + q_4\,p_2 - q_2\,p_4 \\ q_1\,p_4 + q_4\,p_1 + q_2\,p_3 - q_3\,p_2 \end{bmatrix} \tag{1.114}$$

$$= \begin{bmatrix} q_1\,p_1 - q_2\,p_2 - q_3\,p_3 - q_4\,p_4 \\ q_2\,p_1 + q_1\,p_2 - q_4\,p_3 + q_3\,p_4 \\ q_3\,p_1 + q_4\,p_2 + q_1\,p_3 - q_2\,p_4 \\ q_4\,p_1 - q_3\,p_2 + q_2\,p_3 + q_1\,p_4 \end{bmatrix} \tag{1.115}$$

$$= \begin{bmatrix} q_1 & -q_2 & -q_3 & -q_4 \\ q_2 & q_1 & -q_4 & q_3 \\ q_3 & q_4 & q_1 & -q_2 \\ q_4 & -q_3 & q_2 & q_1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} \tag{1.116}$$

1.3.4 Application of quaternions to vectors

Consider a rotation by angle θ around an axis with direection $\vec{v}$ represented by a unit quaternion $\vec{q} = \begin{bmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2}\,\vec{v} \end{bmatrix}$ and a vector $\vec{x} \in \mathbb{R}^3$. To rotate the vector, we may construct the rotation matrix $\mathtt{R}(\vec{q})$ and apply it to the vector $\vec{x}$ as $\mathtt{R}(\vec{q})\,\vec{x}$.

Interestingly enough, it is possible to accomplish this in somewhat different and more efficient way by first "embedding" vector $\vec{x}$ into a (non-unit!) quaternion

$$\vec{p}(\vec{x}) = \begin{bmatrix} 0 \\ \vec{x} \end{bmatrix} = \begin{bmatrix} 0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{1.117}$$

and then composing it with quaternion $\vec{q}$ from both sides

$$\vec{q}\,\vec{p}(\vec{x})\,\vec{q}^{-1} \quad = \begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\,\vec{v} \end{bmatrix} \begin{bmatrix} 0 \\ \vec{x} \end{bmatrix} \begin{bmatrix} \cos\frac{\theta}{2} \\ -\sin\frac{\theta}{2}\,\vec{v} \end{bmatrix} \tag{1.118}$$

One can verify that the following

$$\begin{bmatrix} 0 \\ \mathtt{R}(\vec{q})\,\vec{x} \end{bmatrix} = \vec{q}\,\vec{p}(\vec{x})\,\vec{q}^{-1} \tag{1.119}$$

holds true.

1.4 "Cayley transform" parameterization

We see that unit quaternions provide a nice parameterization. It is given as a matrix with polynomial entries of four parameters. However, unit quaternions still are somewhat redundant since every rotation is represented twice.

Let us now mention yet another classical rotation parameterization, which is known as "Cayley transform". This parameterization uses only three parameters to represent three-dimensional rotations. In a sense, it is as ecconomic as it can be. On the other hand, it can't represent rotations by 180°.

Actually, it can be proven [4] that there is no mapping (parameterization), which could be (i) continuous, (ii) one-to-one, (iii) onto, and (iv) three-dimensional (i.e. mapping a "three-dimensional box" onto all three-dimensional rotations).

Axis-angle parameterization is continuous and onto but not one-to-one and not three-dimensional. Euler vector parameterization is continuous, onto, three-dimensional but not one-to one. Unit quaternions are continuous, onto but not three-dimensional and not one-to one (although they are close to that by being two-to-one). Finally, Cayley transform parameterization is continuous, one-to-one, three-dimensional but it not onto.

In addition, unit quaternions and Cayley transform parameterizations are "finite" in the sense that they are polynomial rational functions of their parameters while other above mentioned representations require some "infinite" process for computing goniometric functions. This may be no probelem if approximate evaluation of functions is acceptable but, as we will see, it is a findamental obstackle to solving interestign engineering problems using computational algebra.

1.4.1 Cayley transform parameterization of two-dimensional rotations

Let us first look at two-dimesional roations. Figure 1.2 shows an illustartion of the relationship between parameter c and $\cos\theta$, $\sin\theta$ on the unit circle.

We see that, using the similarity of triangles, $\frac{\sin\theta}{\cos\theta+1} = \frac{c}{1}$. Considering that $(\cos\theta)^2 + (\sin\theta)^2 = 1$ we are getting

$$1 - (\cos\theta)^2 = (\sin\theta)^2 = c^2(\cos\theta + 1)^2 = c^2((\cos\theta)^2 + 2\cos\theta + 1) \tag{1.120}$$

$$0 = (c^2+1)(\cos\theta)^2 + 2c^2\cos\theta + c^2 - 1 \tag{1.121}$$

and thus

$$\cos\theta = \frac{-2c^2 \pm \sqrt{4c^4 - 4(c^2+1)(c^2-1)}}{2(c^2+1)} = \frac{-c^2 \pm \sqrt{c^4 - (c^4-1)}}{c^2+1} = \frac{\pm 1 - c^2}{1+c^2} \tag{1.122}$$

gives either $\cos\theta = -1$ or

$$\cos\theta = \frac{1-c^2}{1+c^2} \tag{1.123}$$

The former case corresponds to point $[-1\ 0]^\top$. In the latter case, we have

$$(\sin\theta)^2 = 1 - (\cos\theta)^2 = 1 - (\frac{1-c^2}{1+c^2})^2 = \frac{(1+c^2)^2 - (1-c^2)^2}{(1+c^2)^2} \tag{1.124}$$

$$= \frac{(1+2c^2+c^4) - (1-2c^2+c^4)}{(1+c^2)^2} = \frac{4c^2}{(1+c^2)^2} = \left(\frac{2c}{1+c^2}\right)^2 \tag{1.125}$$

and thus $\sin\theta = \pm\frac{2c}{1+c^2}$. Now, we see from Figure 1.2 that we want $\sin\theta$ to be positive for positive c. Therefore, we conclude that

$$\sin\theta = \frac{2c}{1+c^2} \tag{1.126}$$

It is impotant to notice that with the parameterization given by Equation 1.123, we can never get $\cos\theta = -1$ for a real c since if that was true, we would get $-1 - c^2 = 1 - c^2$ and hence $-1 = 1$. On the other hand, we see that Cayley transform maps every $c \in \mathbb{R}$ into a point on the unit circle $[\cos\theta\ \ \sin\theta]^\top$, and hence to the corresponding rotation

$$\mathtt{R}(c) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \frac{1-c^2}{1+c^2} & -\frac{2c}{1+c^2} \\ \frac{2c}{1+c^2} & \frac{1-c^2}{1+c^2} \end{bmatrix} \tag{1.127}$$

The mapping $\mathtt{R}(c)\colon \mathbb{R} \rightarrow \mathtt{R}$ is one-to-one since when two c_1, c_2 map into the same point, then

$$\frac{2c_1}{1+c_1^2} = \frac{2c_2}{1+c_2^2} \tag{1.128}$$

$$c_1(1+c_2^2) = c_2(1+c_1^2) \tag{1.129}$$

$$c_1 - c_2 = c_1c_2(c_1 - c_2) \tag{1.130}$$

implies that either $c_1 c_2 \neq 0$, and then $c_1 = c_2$, or $c_1 c_2 = 0$, and then $c_1 = 0 = c_2$ because both $1 + c_1^2, 1 + c_2^2$ are positive. Next, let us see that the mapping is also onto $\mathbb{R} \backslash \{[-1\ 0]^\top\}$. Consider a point $[\cos\theta\ \ \sin\theta]^\top \neq [-1\ 0]^\top$. Its preimage c, is obtained as

$$c = \frac{\sin\theta}{1+\cos\theta} \tag{1.131}$$

which is clearly defined for $\cos\theta \neq -1$.

1.4.1.1 Two-dimensional rational rotations

It is also important to notice that the $\mathtt{R}(c)$ is a rational function of c as well as c is a rational function or $\mathtt{R}$ (e.g. of the two elements in its first column). Hence, every rational number c gives a rational point $[a\ b]^\top$ on the unit circle as well as every rational point $[a\ b]^\top$ provides a rational c. This way, we can obtain all rational two-dimensional rotations by going over all rational c's plus the rotation $-\mathtt{I}_{2\times 2}$.

1.4.2 Cayley transform parameterization of three-dimensional rotations

We saw that we have obtained a bijective (one-to-one and onto) mapping between all real numbers and all two-dimensional rotations other than the rotation by 180° degrees. Now, since every three-dimensional rotation can be actually seen as a two-dimensional rotation after aligning the z-axis with the rotation axis, we may hint on having an analogous situation in three dimensons after removing all rotations by 180°. Let us investigate this further and see that we can indeed establish a bijective mapping between $\mathbb{R}^3$ and all three-dimesnional rotations by other than 180° angle.

Let us consider that all rotations by 180° are represented by unit quaternons in the form $\begin{bmatrix} 0 & q_2 & q_3 & q_4 \end{bmatrix}$. Hence, to remove them, it is enough to remove from all cases when $c_1 = 0$. One way to do it, is to write down the rotation matrix in tems of (non-unit) quaternions $\vec{q}$

$$\mathtt{R}(\vec{q}) = \frac{1}{q_1^2+q_2^2+q_3^2+q_4^2} \begin{bmatrix} q_1^2+q_2^2-q_3^2-q_4^2 & 2\,(q_2q_3-q_1q_4) & 2\,(q_2q_4+q_1q_3) \\ 2\,(q_2q_3+q_1q_4) & q_1^2-q_2^2+q_3^2-q_4^2 & 2\,(q_3q_4-q_1q_2) \\ 2\,(q_2q_4-q_1q_3) & 2\,(q_3q_4+q_1q_2) & q_1^2-q_2^2-q_3^2+q_4^2 \end{bmatrix} \tag{1.132}$$

and then set $q_1 = 1$, $q_2 = c_1$, $q_3 = c_2$, $q_4 = c_3$, to get

$$\mathtt{R}(\vec{c}) = \frac{1}{1+c_1^2+c_2^2+c_3^2}\begin{bmatrix} 1+c_1^2-c_2^2-c_3^2 & 2\,(c_1c_2-c_3) & 2\,(c_1c_3+c_2) \\ 2\,(c_1c_2+c_3) & 1-c_1^2+c_2^2-c_3^2 & 2\,(c_2c_3-c_1) \\ 2\,(c_1c_3-c_2) & 2\,(c_2c_3+c_1) & 1-c_1^2-c_2^2+c_3^2 \end{bmatrix} \tag{1.133}$$

with $\vec{c} = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix}^\top \in \mathbb{R}^3$.

It can be verified that $\mathtt{R}(\vec{c})^\top\mathtt{R}(\vec{c}) = \mathtt{I}$ for all $\vec{c} \in \mathbb{R}^3$ and hence the mapping $\mathtt{R}(\vec{c})\colon \mathbb{R}^3 \to \mathtt{R}$ maps the space $\mathbb{R}^3$ into rotation matrices $\mathtt{R}$. Let us next see that the mapping is also one-to-one.

First, notice that by setting $c_1 = c_2 = 0$, we are getting

$$\mathtt{R}(c_3) = \frac{1}{1+c_3^2}\begin{bmatrix} 1-c_3^2 & -2\,c_3 & 0 \\ 2\,c_3 & 1-c_3^2 & 0 \\ 0 & 0 & 1+c_3^2 \end{bmatrix} = \begin{bmatrix} \frac{1-c_3^2}{1+c_3^2} & \frac{-2\,c_3}{1+c_3^2} & 0 \\ \frac{2\,c_3}{1+c_3^2} & \frac{1-c_3^2}{1+c_3^2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.134}$$

which is exactly the Cayley parameterization for two-dimensional rotation aroung the z-axis. In the same way, we get that $\mathtt{R}(c_1)$ are rotations around the x-axis and $\mathtt{R}(c_2)$ are rotations around the y-axis.

We have seen in Paragraph 1.3.2 that the mapping between the unit quaternions $\vec{q}$ and rotation matrices $\mathtt{R}(\vec{q})$ was "two-to-one" in the way that there were exactly two quaternions $\vec{q}$, $-\vec{q}$ mapping into one $\mathtt{R}$, i.e. $\mathtt{R}(\vec{q}) = \mathtt{R}(-\vec{q})$. Now, we are forcing the first coordinate of the unit quaternion $\vec{q} = \frac{\begin{bmatrix} 1 & c_1 & c_2 & c_3 \end{bmatrix}^\top}{1+c_1^2+c_2^3+c_3}$ be positive. Therefore, the mapping $\mathtt{R}(\vec{c})$ becomes one-to-one.

Now, let us see that by $\mathtt{R}(\vec{c})$ we can represent all rotations that are not by 180°.

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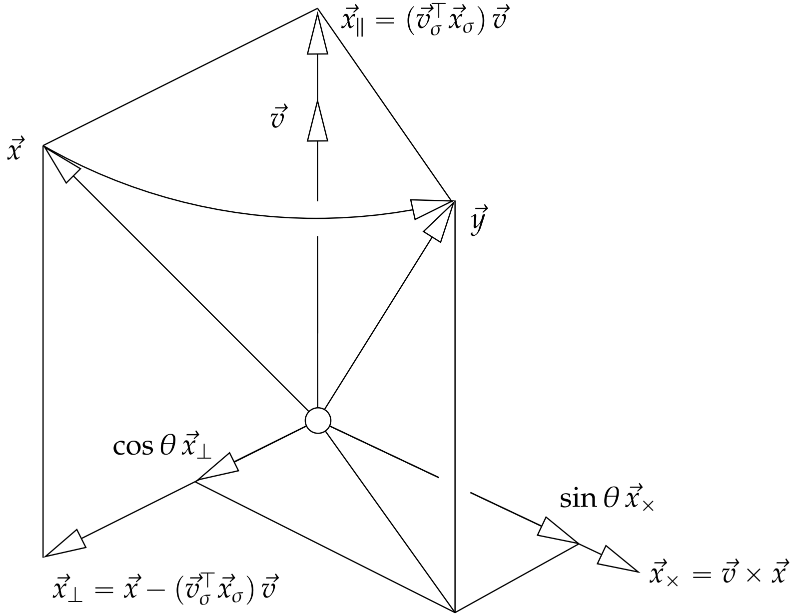

Figure 1.1: Vector $\vec{y}$ is obtained by rotating vector $\vec{x}$ by angle θ around the rotation axis given by unit vector $\vec{v}$. Vector $\vec{y}$ can be written as a linear combination of an orthogonal basis $[\vec{x} - (\vec{v}_\sigma^\top \vec{x}_\sigma)\,\vec{v}, \vec{v} \times \vec{x}, (\vec{v}_\sigma^\top \vec{x}_\sigma)\,\vec{v}]$.

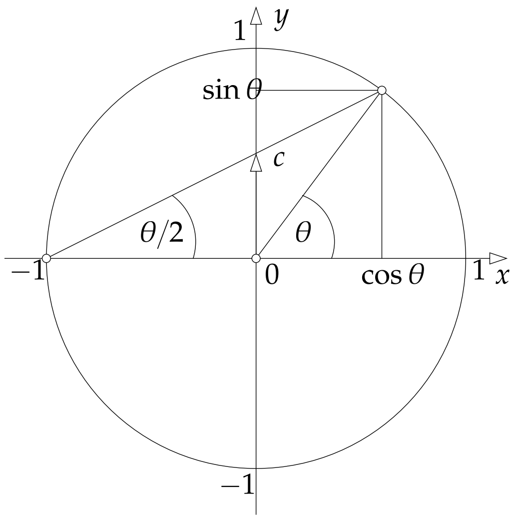

Figure 1.2: Cayley transform parameterization of two-dimensional rotations.

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