ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Sampling and interpolation for biomedical imaging: Part I

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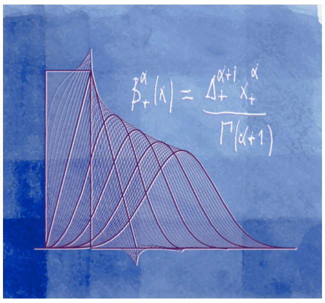


ISBI 2006, Tutorial, Washington DC, April 2006

INTRODUCTION

- **Fundamental issue in biomedical imaging**

  Linking the *discrete* and the *continuous*

- **Mismatch between theory and practice**
  - Theory : Shannon's sampling theorem
  - Practice: nearest neighbor, linear interpolation

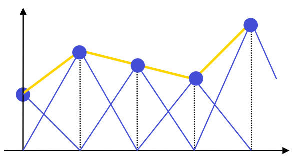
- **Limitations of Shannon sampling theory**
  - Ideal lowpass filters do not exist
  - Incompatible with finite support signals
  - Gibbs oscillations
  - Slow decay of $\mathrm{sinc}(x)$

- **Basic problem**

  How do you interpolate a signal ?

Acquisition

Algorithm design

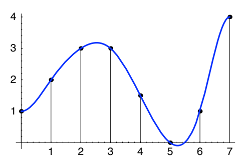


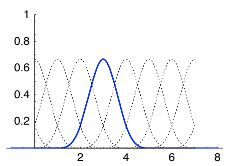
Interpolation and biomedical imaging

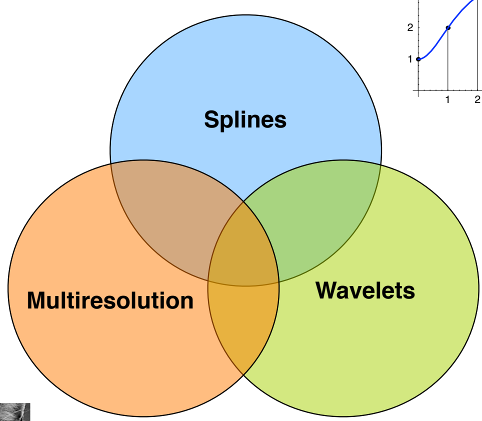
Image processing task	Specific operation	Imaging modality
Tomographic reconstruction	• Filtered backprojection • Fourier reconstruction • Iterative techniques • 3D + time	Commercial CT (X-rays) EM PET, SPECT Dynamic CT, SPECT, PET
Sampling grid conversion	• Polar-to-cartesian coordinates • Spiral sampling • k-space sampling • Scan conversion	Ultrasound (endovascular) Spiral CT, MRI MRI
Visualization	*2D operations* • Zooming, panning, rotation • Re-sizing, scaling	All
	• Stereo imaging • Range, topography	Fundus camera OCT
	3D operations • Re-slicing • Max. intensity projection • Simulated X-ray projection	CT, MRI, MRA
	Surface/volume rendering • Iso-surface ray tracing • Gradient-based shading • Stereogram	CT MRI
Geometrical correction	• Wide-angle lenses • Projective mapping • Aspect ratio, tilt • Magnetic field distortions	Endoscopy C-Arm fluoroscopy Dental X-rays MRI
Registration	• Motion compensation • Image subtraction • Mosaicking • Correlation-averaging • Patient positioning • Retrospective comparisons • Multi-modality imaging • Stereotactic normalization • Brain warping	fMRI, fundus camera DSA Endoscopy, fundus camera, EM microscopy Surgery, radiotherapy CT/PET/MRI
Feature detection	• Contours • Ridges • Differential geometry	All
	Contour extraction • Snakes and active contours	MRI, Microscopy (cytology)

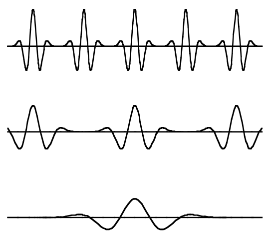
Splines: a unifying framework

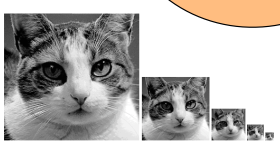
Linking the discrete and the continuous …..











Splines: bad press phenomenon

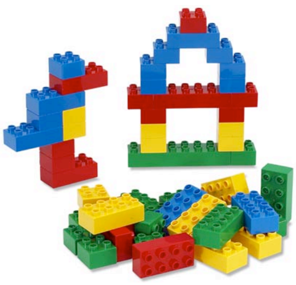
- Classical review article on interpolation, IEEE TMI, 1983
  Comparison of four interpolators:
  "The cubic B-spline provides the most smoothing."
- Classical book on Digital Image Processing, 1991 (2^{nd} ed)
  About high-order B-splines:
  "[out-of-band] interpolation error reduces significantly for higher-order interpolation functions, but at the expense of resolution error [i.e., distortion]"

⋮

- Recent book on Volume Rendering, 1998
  "The results of scaling the original image using [cubic] B-spline interpolation are shown in Figure 5.20. You can see the blurring effects ….."

CONTINUOUS/DISCRETE REPRESENTATION

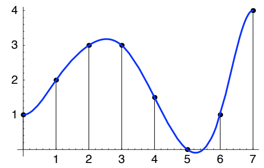
- Splines: definition
- Basic atoms: B-splines
- Riesz bases



Splines: definition

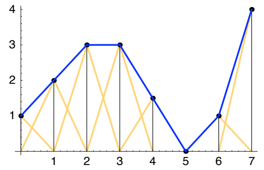
Definition: A function $s(x)$ is a polynomial spline of degree n with knots $\cdots < x_k < x_{k+1} < \cdots$ iff. it satisfies the following two properties:

- Piecewise polynomial:
  $s(x)$ is a polynomial of degree n within each interval $[x_k, x_{k+1})$;
- Higher-order continuity:
  $s(x), s^{(1)}(x), \cdots, s^{(n-1)}(x)$ are continuous at the knots x_k.



- Effective degrees of freedom per segment:

$$\underset{\text{(polynomial coefficients)}}{(n+1)} \quad - \quad \underset{\text{(constraints)}}{n} \quad = \quad 1$$



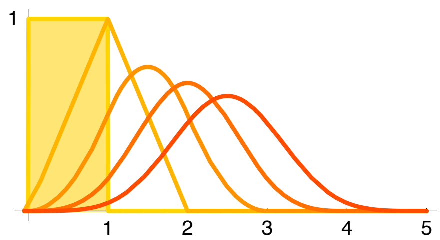
- **Cardinal splines** = unit spacing and infinite number of knots

The right framework for signal processing !

Polynomial B-splines

- B-spline of degree n

$$\beta_+^n(x) = \underbrace{\beta_+^0 * \beta_+^0 * \cdots * \beta_+^0(x)}_{(n+1)\text{ times}}$$

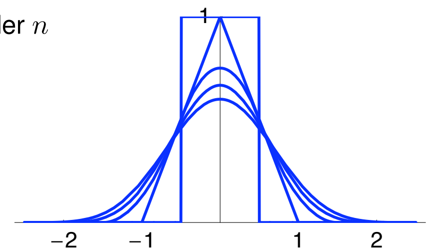


$$\beta_+^0(x) = \begin{cases} 1, & x \in [0,1) \\ 0, & \text{otherwise.} \end{cases}$$

- Key properties
  - Compact support: shortest polynomial spline of degree n
  - Positivity
  - Piecewise polynomial
  - Smoothness: Hölder-continuous of order n

- Symmetric B-spline

$$\beta^n(x) = \beta_+^n\left(x + \tfrac{n+1}{2}\right)$$



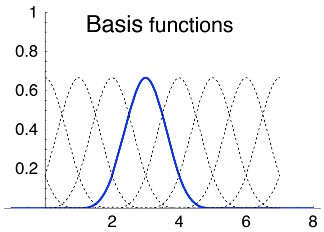
B-spline representation

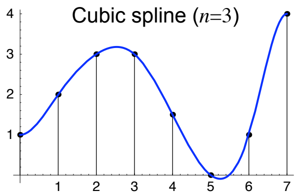
Theorem (Schoenberg, 1946)
Every cardinal polynomial spline $s(x)$ has a unique and stable representation in terms of its B-spline expansion

$$s(x) = \sum_{k\in\mathbb{Z}} c[k]\, \beta^n_+(x-k)$$

analog signal

discrete signal
(B-spline coefficients)



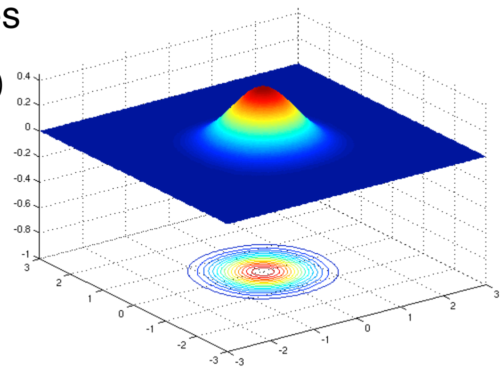


In modern terminology: $\{\beta^n_+(x-k)\}_{k\in\mathbb{Z}}$ forms a Riesz basis.

B-spline representation of images

- Symmetric, tensor-product B-splines

$$\beta^n(x_1,\cdots,x_d) = \beta^n(x_1)\times\cdots\times\beta^n(x_d)$$



- Multidimensional spline function

$$s(x_1,\cdots,x_d) = \sum_{(k_1,\cdots k_d)\in\mathbb{Z}^d} c[k_1,\cdots,k_d]\, \beta^n(x_1-k_1,\cdots,x_d-k_d)$$

continuous-space image

image array
(B-spline coefficients)

Compactly supported
basis functions

Riesz basis

Definition: Let $V = \text{span}\{\varphi_k\}_{k\in\mathbb{Z}}$ be a subspace of a Hilbert space H. Then, $\{\varphi_k\}_{k\in\mathbb{Z}}$ is a Riesz basis of V iff. there exist two constants $A > 0$ and $B < +\infty$ s.t.

$$\forall c \in \ell_2, \quad A \cdot \|c\|_{\ell_2} \leq \underbrace{\left\|\sum_{k\in\mathbb{Z}} c_k\varphi_k\right\|_H}_{\|f\|_H} \leq B \cdot \|c\|_{\ell_2}$$

Unique representation of a function $f \in V$: $\quad f = \sum_{k\in\mathbb{Z}} c_k\varphi_k$

- Properties
  - Linear independence
    Consequence of lower Riesz bound: $\quad f = 0 \Rightarrow c_k = 0$
  - Stability
    Perturbation: $\quad c + \Delta c \longrightarrow f + \Delta f$
    Consequence of upper Riesz bound: $\quad \|\Delta c\|_{\ell_2}$ bounded $\Rightarrow$ $\|\Delta f\|_H$ bounded
  - Norm equivalence
    The basis is orthonormal iff. $A = B = 1$, in which case, $\|c\|_{\ell_2} = \|f\|_H$

Shift-invariant spaces

Integer-shift-invariant subspace associated with a generating function φ (e.g. B-spline):

$$V(\varphi) = \left\{ f(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^p} c[\boldsymbol{k}]\varphi(\boldsymbol{x} - \boldsymbol{k}) : c \in \ell_2(\mathbb{Z}^p) \right\}$$

Generating function: $\quad \varphi(\boldsymbol{x}) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \hat{\varphi}(\boldsymbol{\omega}) = \int_{\boldsymbol{x}\in\mathbb{R}^p} \varphi(\boldsymbol{x}) e^{-j\langle\boldsymbol{\omega},\boldsymbol{x}\rangle} \mathrm{d}x_1 \cdots \mathrm{d}x_p$

Proposition. $V(\varphi)$ is a subspace of $L_2(\mathbb{R}^p)$ with $\{\varphi(\boldsymbol{x} - \boldsymbol{k})\}_{\boldsymbol{k}\in\mathbb{Z}^p}$ as its Riesz basis iff.

$$0 < A^2 \leq \sum_{\boldsymbol{n}\in\mathbb{Z}^p} |\hat{\varphi}(\boldsymbol{\omega} + 2\pi\boldsymbol{n})|^2 \leq B^2 < +\infty \quad \text{(almost everywhere)}$$

Hint for the proof (in 1D):

$$\|c\|^2_{\ell_2} = \frac{1}{2\pi}\int_0^{2\pi} |C(e^{j\omega})|^2 \mathrm{d}\omega \qquad \text{(Parseval)}$$

$$\begin{aligned}\|f\|^2_{L_2} &= \frac{1}{2\pi}\int_{\omega\in\mathbb{R}} |C(e^{j\omega})|^2 |\hat{\varphi}(\omega)|^2 \mathrm{d}\omega \\ &= \frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\int_0^{2\pi} |C(e^{j\omega})|^2 |\hat{\varphi}(\omega + 2\pi n)|^2 \mathrm{d}\omega = \frac{1}{2\pi}\int_0^{2\pi} |C(e^{j\omega})|^2 \sum_{n\in\mathbb{Z}} |\hat{\varphi}(\omega + 2\pi n)|^2 \mathrm{d}\omega\end{aligned}$$

INTERPOLATION REVISITED

- Classical interpolation
- Generalized interpolation
- Interpolation: filtering solution
- Application

Classical image interpolation

Discrete image data $\Longleftrightarrow$ Continuous image model

$f[\boldsymbol{k}], \quad \boldsymbol{k} = (k_1, \cdots, k_p) \in \mathbb{Z}^p$ $\qquad$ $f(\boldsymbol{x}), \quad \boldsymbol{x} = (x_1, \cdots, x_p) \in \mathbb{R}^p$

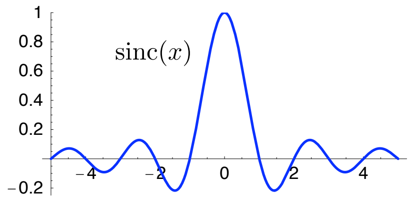
- Interpolation formula: $f(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in \mathbb{Z}^p} f[\boldsymbol{k}]\, \varphi_{\text{int}}(\boldsymbol{x} - \boldsymbol{k})$
  - $f[\boldsymbol{k}]$: pixel values at location $\boldsymbol{k}$
  - $\varphi_{\text{int}}(\boldsymbol{x})$: continuous-space interpolation function
  - $\varphi_{\text{int}}(\boldsymbol{x} - \boldsymbol{k})$: interpolation function translated to location $\boldsymbol{k}$
- Interpolation condition

At the grid points $\boldsymbol{x} = \boldsymbol{k}_0$: $f(\boldsymbol{k}_0) = \sum_{\boldsymbol{k} \in \mathbb{Z}^p} f[\boldsymbol{k}]\, \varphi_{\text{int}}(\boldsymbol{k}_0 - \boldsymbol{k})$

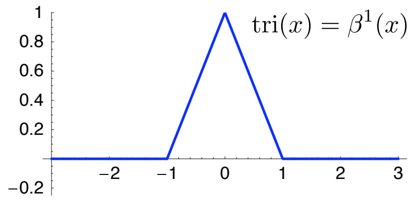
Only possible for all f iff. $\varphi_{\text{int}}(\boldsymbol{k}) = \begin{cases} 1, & \boldsymbol{k} = \boldsymbol{0} \\ 0, & \text{otherwise} \end{cases}$

Examples of popular interpolation functions

- Bandlimited



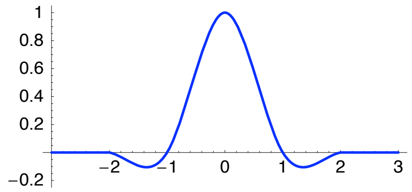
- Piecewise linear



Interpolation condition:

$$\varphi_{\text{int}}(k) = \delta_k = \begin{cases} 1, & k = 0 \\ 0, & \text{otherwise} \end{cases}$$

- Cubic convolution



[Keys, 1981; Karup-King 1899]

Generalized image interpolation

- Desired features for the interpolation kernel
  - short (to minimize computations)
  - simple expression (e.g., polynomial)
  - smooth (to avoid model discontinuities)
  - good approximation properties: reproduction of polynomials

- Generalized interpolation formula: $f(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in \mathbb{Z}^p} c[\boldsymbol{k}]\, \varphi(\boldsymbol{x} - \boldsymbol{k})$
  - Simple shift-invariant structure
  - simple expression (e.g., polynomial)
  - φ selected freely (not interpolating and much shorter)

$\Rightarrow$ Faster interpolation formulas!

but one new difficulty:

How to pre-compute the coefficients $c[\boldsymbol{k}]$?

- Separable basis functions: $\varphi(\boldsymbol{x}) = \varphi(x_1) \cdot \varphi(x_2) \cdots \varphi(x_p)$

$\Rightarrow$ Further acceleration

Interpolation: filtering solution

Interpolation problem: Given the samples $\{f[\boldsymbol{k}]\}$, find the (B-spline) expansion coefficients $\{c[\boldsymbol{k}]\}$

- Interpolation condition: $f(\boldsymbol{x})|_{\boldsymbol{x}=\boldsymbol{k}} = f[\boldsymbol{k}] = \sum_{\boldsymbol{k}_1 \in \mathbb{Z}^p} c[\boldsymbol{k}_1]\varphi(\boldsymbol{k}-\boldsymbol{k}_1)$

  ⇨ Discrete convolution equation: $f[\boldsymbol{k}] = (b * c)[\boldsymbol{k}]$

  with $b[\boldsymbol{k}] \stackrel{\Delta}{=} \varphi(\boldsymbol{k}) \quad \stackrel{z}{\longleftrightarrow} \quad B(\mathbf{z}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^p} b[\boldsymbol{k}]\mathbf{z}^{-\boldsymbol{k}}$

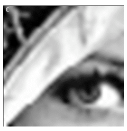
- Inverse filtering solution

$$f[\boldsymbol{k}] \longrightarrow \boxed{\text{Digital filter}} \longrightarrow c[\boldsymbol{k}] = (h_{\text{int}} * f)[\boldsymbol{k}]$$

with $H_{\text{int}}(\mathbf{z}) = \frac{1}{B(\mathbf{z})} = \frac{1}{\sum_{\boldsymbol{k}\in\mathbb{Z}^p}\varphi(\boldsymbol{k})\mathbf{z}^{-\boldsymbol{k}}}$

Note: $\varphi(\boldsymbol{x})$ separable $\Rightarrow$ $h_{\text{int}}[\boldsymbol{k}]$ separable

One-to-one continuous/discrete representation



Continuously defined signal

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^p} c[\boldsymbol{k}]\varphi(\boldsymbol{x}-\boldsymbol{k})$$

⇔ Riesz-basis property

B-spline coefficients

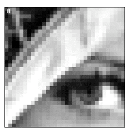
$c[\boldsymbol{k}]$

Digital filtering

$* \, b$ (FIR) ↓ ↑ $* \, h_{\text{int}}$ (IIR)

Sampling: $f(\boldsymbol{x})|_{\boldsymbol{x}=\boldsymbol{k}}$

$f[\boldsymbol{k}]$



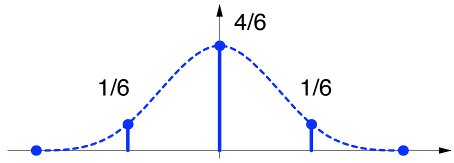
Discrete signal

In principle, all φ's are equally acceptable, but...

Example: cubic-spline interpolation

- Cubic B-spline

$$\varphi(x) = \beta^3(x) = \begin{cases} \frac{2}{3} - \frac{1}{2}|x|^2(2-|x|), & 0 \le |x| < 1 \\ \frac{1}{6}(2-|x|)^3, & 1 \le |x| < 2 \\ 0, & \text{otherwise} \end{cases}$$



- Discrete B-spline kernel: $B(z) = \dfrac{z + 4 + z^{-1}}{6}$

- Interpolation filter

$$\frac{6}{z+4+z^{-1}} = \frac{(1-\alpha)^2}{(1-\alpha z)(1-\alpha z^{-1})} \quad \stackrel{z}{\longleftrightarrow} \quad h_{\text{int}}[k] = \left(\frac{1-\alpha}{1+\alpha}\right)\alpha^{|k|}$$

(symmetric exponential)

$$\alpha = -2 + \sqrt{3} = -0.171573$$

⇨ Cascade of first-order recursive filters

$$\longrightarrow \boxed{\frac{1}{1-\alpha z^{-1}}} \longrightarrow \boxed{\frac{1}{1-\alpha z}} \longrightarrow$$

causal — anti-causal

Generic C-code (splines of any degree n)

- Main recursion

```
void   ConvertToInterpolationCoefficients (
          double c[ ],  long DataLength,   double  z[ ],  long NbPoles,  double  Tolerance)
{double Lambda = 1.0; long n, k;
   if (DataLength == 1L) return;
   for (k = 0L; k < NbPoles; k++) Lambda = Lambda * (1.0 - z[k]) * (1.0 - 1.0 / z[k]);
   for (n = 0L; n < DataLength; n++) c[n] *= Lambda;
   for (k = 0L; k < NbPoles; k++) {
      c[0] = InitialCausalCoefficient(c, DataLength, z[k], Tolerance);
      for (n = 1L; n < DataLength; n++) c[n] += z[k] * c[n - 1L];
      c[DataLength - 1L] = (z[k] / (z[k] * z[k] - 1.0))
           * (z[k] * c[DataLength - 2L] + c[DataLength - 1L]);
      for (n = DataLength - 2L; 0 <= n; n--) c[n] = z[k] * (c[n + 1L]- c[n]); }
}
```

- Initialization

```
double  InitialCausalCoefficient (
          double  c[ ],  long   DataLength,  double  z,  double  Tolerance)
{  double Sum, zn, z2n, iz; long n, Horizon;
   Horizon = (long)ceil(log(Tolerance) / log(fabs(z)));
   if (DataLength < Horizon) Horizon = DataLength;
   zn = z; Sum = c[0];
   for (n = 1L; n < Horizon; n++) {Sum += zn * c[n]; zn *= z;}
   return(Sum);
}
```

Interpolating basis function

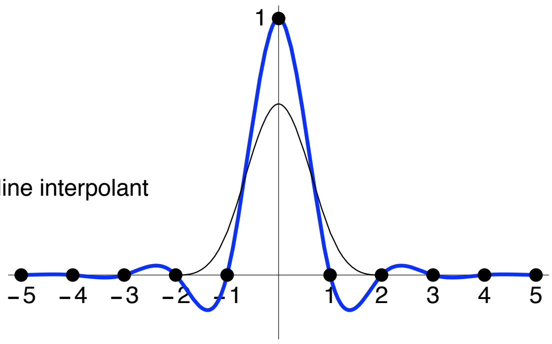
- Equivalent interpretation of generalized interpolationn

$$f(x) = \sum_{k\in\mathbb{Z}} c[k]\varphi(x-k) \quad = \quad \sum_{k\in\mathbb{Z}} (f[k] * h_{\text{int}}[k])\,\varphi(x-k)$$
$$= \sum_{k\in\mathbb{Z}} f[k]\ \varphi_{\text{int}}(x-k)$$

- Interpolation basis function

$$\varphi_{\text{int}}(x) = \sum_{k\in\mathbb{Z}} h_{\text{int}}[k]\ \varphi(x-k)$$

Example: cubic-spline interpolant



Finite-cost implementation of an infinite impulse response interpolator !

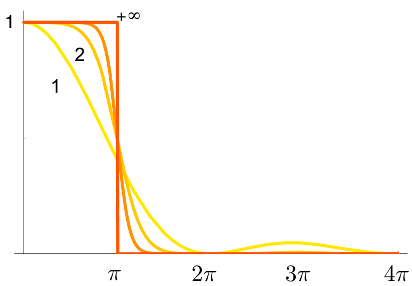
Limiting behavior (splines)

- Spline interpolator

Impulse response

Frequency response

$$\varphi^n_{\text{int}}(x) \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad \hat{\varphi}^n_{\text{int}}(\omega) = \left(\frac{\sin(\omega/2)}{\omega/2}\right)^{n+1} H^n_{\text{int}}(e^{j\omega})$$



- Asymptotic property

The cardinal spline interpolators converge to the sinc-interpolator (ideal filter) as the degree goes to infinity:

$$\lim_{n\to\infty} \varphi^n_{\text{int}}(x) = \text{sinc}(x), \qquad \lim_{n\to\infty} \hat{\varphi}^n_{\text{int}}(\omega) = \text{rect}\left(\frac{\omega}{2\pi}\right) \qquad \text{(in all } L_p\text{-norms)}$$

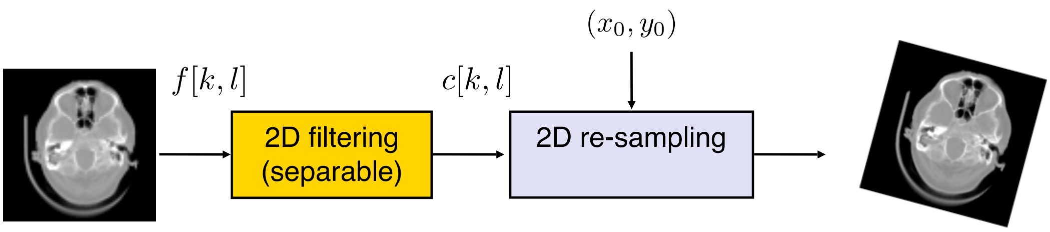
(Aldroubi et al., *Sig. Proc.*, 1992)

Includes Shannon's theory as a particular case !

Geometric transformation of images

- 2D separable model

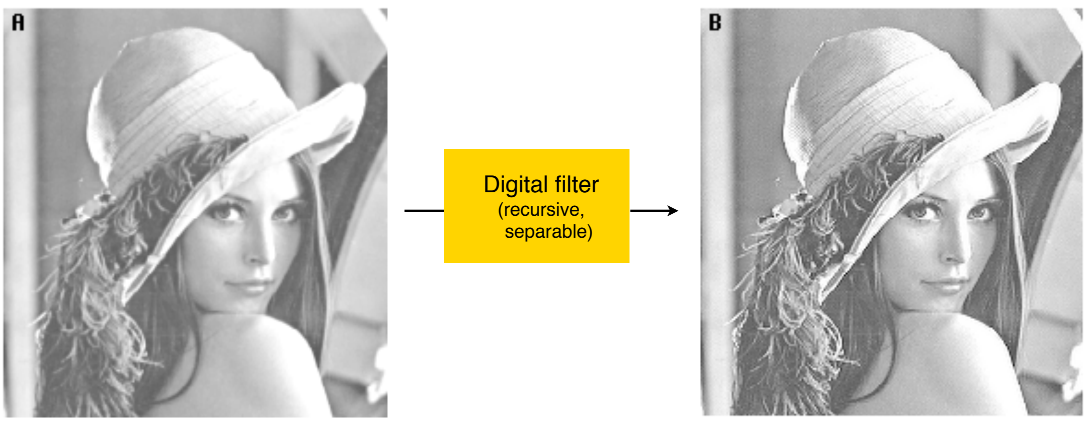
$$f(x_0, y_0) = \sum_{k=k_0(x_0)}^{k_0+n+1} \sum_{l=l_0(y_0)}^{l_0+n+1} c[k,l]\, \varphi(x_0 - l)\, \varphi(y_0 - l)$$



- Applications

zooming, rotation, re-sizing, re-formatting, warping

Cubic-spline coefficients in 2D



Pixel values $f[k,l]$

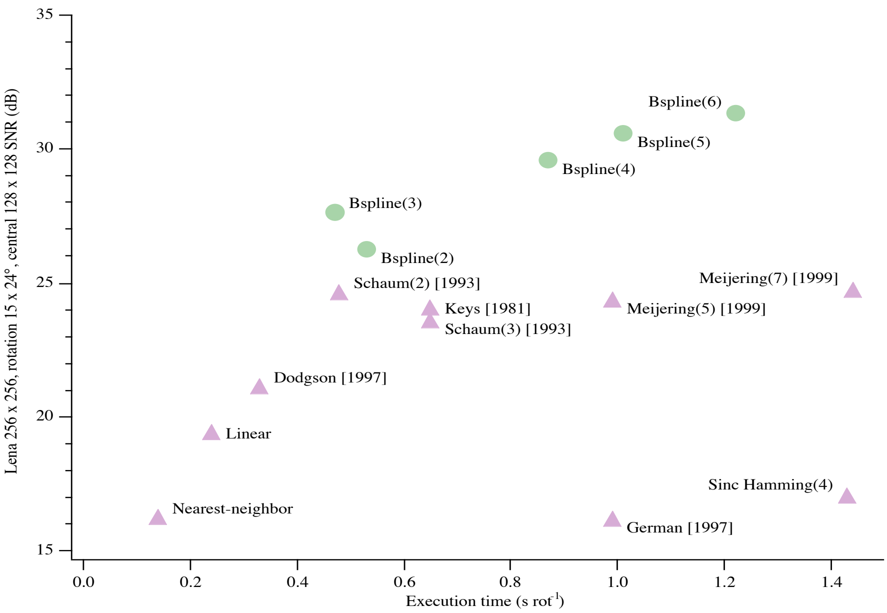
B-spline coefficients $c[k,l]$

Interpolation benchmark

Cumulative rotation experiment: the best algorithm wins !

Bilinear　　Windowed-sinc　　Cubic spline

High-quality image interpolation



Demo

Thévenaz et al., *Handbook of Medical Image Processing*, 2000

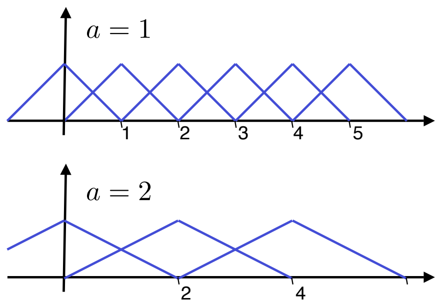
MINIMUM-ERROR SIGNAL APPROXIMATION

- Least-squares approximation
  - Orthogonal projection
- Image pyramids

Least-squares fit: multi-scale approximation

- Shift-invariant space at scale a

$$V_a(\varphi) = \left\{ s(x) = \sum_{k \in \mathbb{Z}} c[k] \varphi_a(x - ak) : c[k] \in \ell_2 \right\}$$



  - Rescaled basis function: $\varphi_a(x) \triangleq a^{-1/2} \varphi\left(\frac{x}{a}\right)$

- Minimum-error approximation at scale a

Continuous-space input $f(x)$ → Orthogonal projector → $c[k] = \langle f, \tilde{\varphi}_a(\cdot - ak) \rangle_{L_2}$

such that $\min\limits_{s \in V_a} \|f - s\|_{L_2}^2$

Biorthogonality condition: $\tilde{\varphi}_a \in V_a(\varphi)$ such that $\langle \varphi_a(\cdot), \tilde{\varphi}_a(\cdot - ak) \rangle_{L_2} = \delta_k$

Image pyramids

- Successive approximations at dyadic scales

$$V_{2^i}(\varphi) = \left\{ s(x) = \sum_{k\in\mathbb{Z}} c_i[k]\varphi_{2^i}(x - 2^i k) : c_i[k] \in \ell_2 \right\}$$

Rescaled basis function: $\varphi_{2^i}(x) \triangleq 2^{-i/2}\varphi\left(\frac{x}{2^i}\right)$

- Repeated application of REDUCE operator

$$c_{i-1}[k] \longrightarrow \boxed{\tilde{h}[k]} \longrightarrow \boxed{\downarrow 2} \longrightarrow c_i[k]$$

- Optimal prefilter

$$c_1[k] = \langle \sum_{l\in\mathbb{Z}} c_0[l]\varphi(\cdot - l),\ \tilde{\varphi}_2(\cdot - 2k)\rangle = (c_0 * \tilde{h})[2k]$$

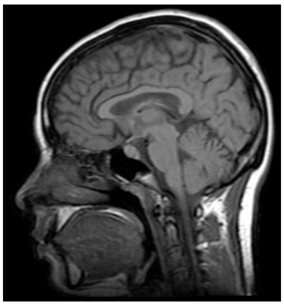
$$\Rightarrow \quad \tilde{h}[k] = \langle \varphi(\cdot),\ \tilde{\varphi}_2(\cdot + k)\rangle$$

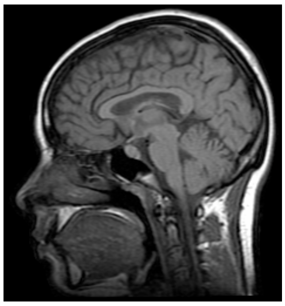
SPLINES: IMAGING APPLICATIONS

- Sampling and interpolation
  - Interpolation, re-sampling, grid conversion
  - Image reconstruction
  - Geometric correction
- Feature extraction
  - Contours, ridges
  - Differential geometry
  - Shape and active contour models
- Image matching
  - Stereo
  - Image registration (multimodal, rigid-body or elastic)
  - Optical flow

Spline approximation: LS resizing

Approximation at arbitrary scales: differential approach using splines



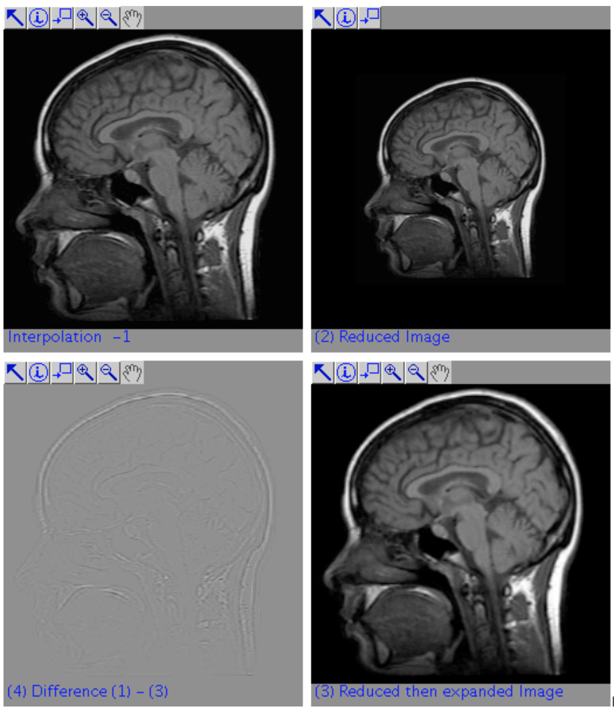


Orthogonal projection onto V_a (cubic spline)

$$a = 1 \longrightarrow 10$$

Application: image resizing

- Resizing algorithm
  - Interpolation
  - Linear splines
  - scaling= 70%



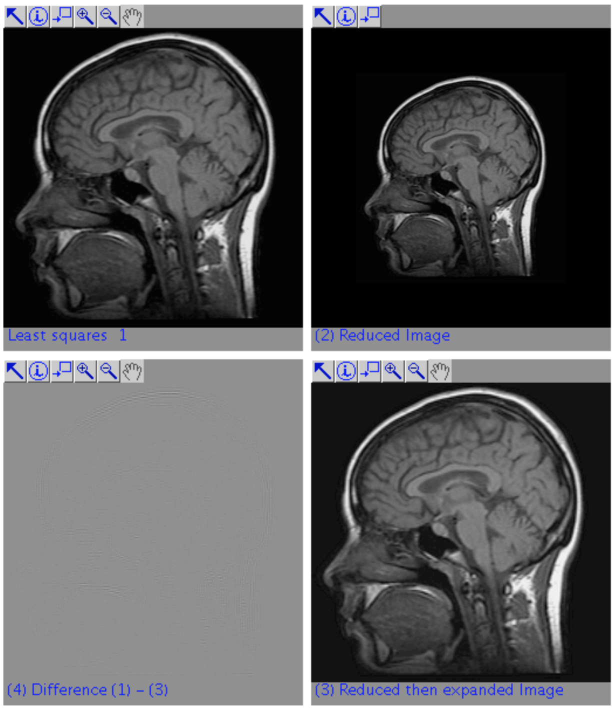
SNR=22.94 dB

Application: image resizing (LS)

- Resizing algorithm
  - Orthogonal projector
  - Linear splines
  - scaling= 70%

SNR=28.359 dB

+ 5.419 dB



(Munoz et al., *IEEE Trans. Imag. Proc*, 2001)

B-spline derivatives

- Derivative operator

$$\mathrm{D}f(x) = \frac{df(x)}{dx} \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad (j\omega) \times \hat{f}(\omega)$$

- Finite-difference operator (centered)

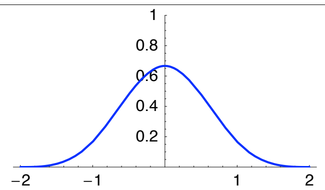
$$\Delta f(x) \triangleq f(x+\tfrac{1}{2}) - f(x-\tfrac{1}{2}) \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad (e^{j\omega/2} - e^{-j\omega/2}) \times \hat{f}(\omega)$$

- Derivative of a B-spline (exact)

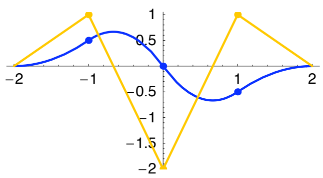
$$\mathrm{D}^m \beta^n(x) = \Delta^m \beta^{n-m}(x)$$

Discrete operator

Reduction of degree



Example: cubic B-spline



Sketch of proof:

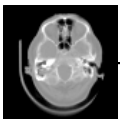
$$\hat{\beta}^n(\omega) = \mathrm{sinc}\left(\frac{\omega}{2\pi}\right)^{n+1} = \left(\frac{e^{j\omega/2} - e^{-j\omega/2}}{j\omega}\right)^{n+1}$$

$$\Rightarrow (j\omega)^m \times \hat{\beta}^n(\omega) = (e^{j\omega/2} - e^{-j\omega/2})^m \times \left(\frac{e^{j\omega/2} - e^{-j\omega/2}}{j\omega}\right)^{n+1-m}$$

Cubic-spline image differentials

Convolution-based implementation

$f(k,l)$



→ 2D filtering (separable) → $c[k,l]$ → Differential mask → $\frac{\partial^{p+q}}{\partial x^p \partial y^q} f(k,l)$

$$\left.\frac{\partial^{p+q}}{\partial x^p \partial y^q}\varphi(x,y)\right|_{x=k,y=l}$$

JAVA code available: http://bigwww.epfl.ch/

- Laplacian

$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} : \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -8 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

- Hessian masks

$$\frac{\partial^2}{\partial x^2} : \frac{1}{6}\begin{bmatrix} 1 & -2 & 1 \\ 4 & -8 & 4 \\ 1 & -2 & 1 \end{bmatrix} \qquad \frac{\partial^2}{\partial x \partial y} : \frac{1}{2\cdot 2}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\frac{\partial^2}{\partial y^2} : \frac{1}{6}\begin{bmatrix} 1 & 4 & 1 \\ -2 & -8 & -2 \\ 1 & 4 & 1 \end{bmatrix}$$

- Gradient masks

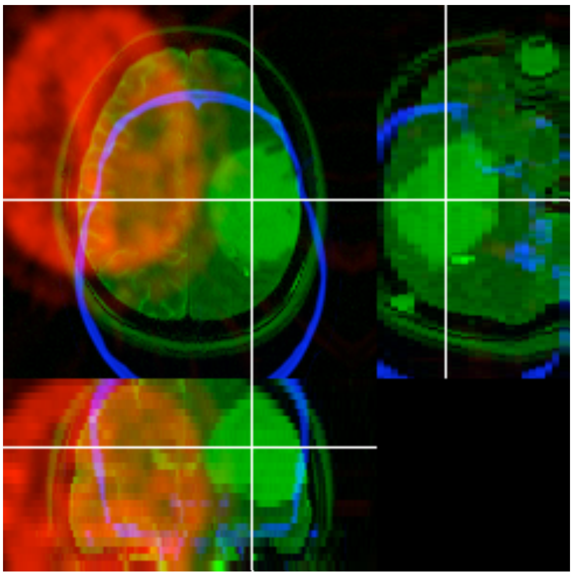
$$\frac{\partial}{\partial x} : \frac{1}{6\cdot 2}\begin{bmatrix} -1 & 0 & 1 \\ -4 & 0 & 4 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\frac{\partial}{\partial y} : \frac{1}{6\cdot 2}\begin{bmatrix} -1 & -4 & -1 \\ 0 & 0 & 0 \\ 1 & 4 & 1 \end{bmatrix}$$

Multi-modal image registration

Specificities of the approach

- Criterion: mutual-information
- Cubic-spline model
  - high quality
  - sub-pixel accuracy
- Multiresolution strategy
- Marquardt-Levenberg-like optimizer
  - Speed
  - Robustness



Thévenaz and Unser, *IEEE Trans. Imag Proc*, 2000

CONCLUSION

- Generalized interpolation
  - Same as standard interpolation, except for a **prefiltering** step
  - Offers more flexibility
  - Best cost/performance tradeoff (splines)
  - Infinite-support interpolator at finite cost
- Special case of polynomial splines
  - Simple to manipulate
  - Smooth and well-behaved
  - Excellent approximation properties
  - Multiresolution properties
- Unifying formulation for continuous/discrete image processing
  - Tools: digital filters, convolution operators
  - Efficient recursive filtering solutions
  - Flexibility: piecewise-constant to bandlimited

Splines: the end of the tunnel

- Survey article on interpolation, *IEEE TMI*, 2000
  Comparison of 31 interpolation algorithms:
  "It [the cubic B-spline interpolator] produces one of the best results in terms of similarity to the original images, and of the top methods, it runs fastest."
- Addendum on spline interpolation, *IEEE TMI*, 2001
  "Therefore, high-degree B-splines are preferable interpolators for numerous applications in medical imaging, particularly if high precision is required."
  (Lehmann et *al*)
- Recent evaluation of interpolation, *Med. Image Anal.*, 2001
  Comparison of 126 interpolation algorithms:
  " The results show that spline interpolation is to be preferred over all other methods, both for its accuracy and its relatively low cost."

(Meijering et *al*)

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  - A. Muñoz Barrutia, T. Blu, M. Unser, "Least-Squares Image Resizing Using Finite Differences," *IEEE Trans. Image Processing*, vol. 10, no. 9, pp. 1365-1378, September 2001.
- Generalized interpolation
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- Preprints and demos: **http://bigwww.epfl.ch/**

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Sampling and interpolation for biomedical imaging: Part II

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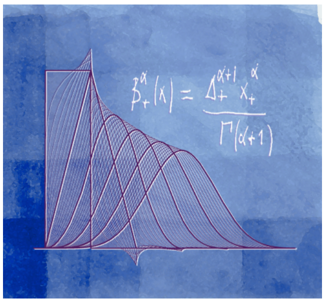


ISBI 2006, Tutorial, Washington DC, April 2006

SAMPLING: 50+ years after Shannon

Analog/physical world — Discrete domain

Continuous signals: $L_2(\mathbb{R})$

Signal subspace

sampling →

← interpolation

Discrete signals: $\ell_2(\mathbb{Z})$

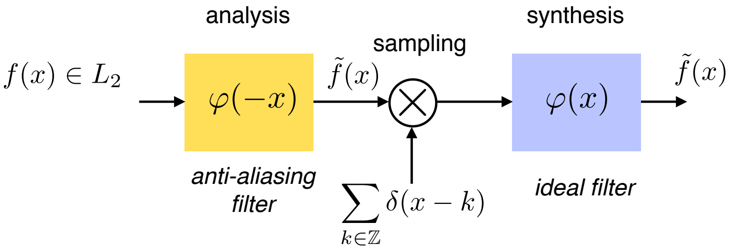
- reconstruction algorithms
- signal processing
- image analysis
- ...

- Introduction: Shannon revisited
- Sampling preliminaries
- Sampling revisited
- Quantitative approximation theory
- Interpolation/approximation in the presence of noise

$\Big\}$ Review paper on sampling (first four items)

Shannon's sampling reinterpreted

- Generating function: $\varphi(x) = \mathrm{sinc}(x)$
- Subspace of bandlimited functions: $V(\varphi) = \mathrm{span}\{\varphi(x-k)\}_{k\in\mathbb{Z}}$



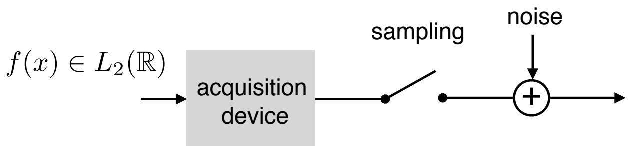
- Analysis: $\tilde{f}(k) = \langle \mathrm{sinc}(x-k), f(x)\rangle$
- Synthesis: $\tilde{f}(x) = \sum_{k\in Z} \tilde{f}(k)\,\mathrm{sinc}(x-k)$

- Orthogonal basis: $\langle \mathrm{sinc}(x-k), \mathrm{sinc}(x-l)\rangle = \delta_{k-l}$ [Hardy, 1941]

⇒ Orthogonal projection operator !

Generalized sampling: roadmap

- Nonideal acquisition system



Measurements:

$g[k] = (h * f)(x)|_{x=k} \quad + n[k]$

Goal: Specify φ and the reconstruction algorithm so that $\tilde{f}(x)$ is a good approximation of $f(x)$

Reconstruction algorithm

Continuous-domain model

$\tilde{f}(x) = \sum_{k\in\mathbb{Z}} c[k]\varphi(x-k)$

Riesz-basis property

signal coefficients

$\{c[k]\}_{k\in\mathbb{Z}}$

Interpolation problem

Discrete signal

$\{f[k]\}_{k\in\mathbb{Z}}$

SAMPLING PRELIMINARIES

- Function and sequence spaces
- Smoothness conditions and sampling
- Shift-invariant subspaces
- Equivalent basis functions

Continuous-domain signals

Mathematical representation: a function of the continuous variable $x \in \mathbb{R}$

- Lebesgue's space of finite-energy functions
  - $L_2(\mathbb{R}) = \left\{ f(x), x \in \mathbb{R} : \int_{x\in\mathbb{R}} |f(x)|^2 \mathrm{d}x < +\infty \right\}$
  - L_2-inner product: $\langle f, g \rangle = \int_{x\in\mathbb{R}} f(x) g^*(x) \mathrm{d}x$
  - L_2-norm: $\|f\|_{L_2} = \left(\int_{x\in\mathbb{R}} |f(x)|^2 \mathrm{d}x \right)^{1/2} = \sqrt{\langle f, f \rangle}$
- Fourier transform
  - Integral definition: $\hat{f}(\omega) = \int_{x\in\mathbb{R}} f(x) e^{-j\omega x} \mathrm{d}x$
  - Parseval relation: $\|f\|_{L_2}^2 = \frac{1}{2\pi} \int_{\omega\in\mathbb{R}} |\hat{f}(\omega)|^2 \mathrm{d}\omega$

Discrete-domain signals

Mathematical representation: a sequence indexed by the discrete variable $k \in \mathbb{Z}$

- Space of finite-energy sequences
  - $\ell_2(\mathbb{Z}) = \left\{ a[k], k \in \mathbb{Z} : \sum_{k \in \mathbb{Z}} |a[k]|^2 < +\infty \right\}$
  - ℓ_2-norm: $\|a\|_{\ell_2} = \left(\sum_{k \in \mathbb{Z}} |a[k]|^2 \right)^{1/2}$

- Discrete-time Fourier transform
  - z-transform: $A(z) = \sum_{k \in \mathbb{Z}} a[k] z^{-k}$
  - Fourier transform: $A(e^{j\omega}) = \sum_{k \in \mathbb{Z}} a[k] e^{-j\omega k}$

Smoothness conditions and sampling

- Sobolev's space of order $s \in \mathbb{R}^+$

$$W_2^s(\mathbb{R}) = \left\{ f(x), x \in \mathbb{R} : \int_{\omega \in \mathbb{R}} (1 + |\omega|^{2s}) |\hat{f}(\omega)|^2 \mathrm{d}\omega < +\infty \right\}$$

f and all its derivatives up to (fractional) order s are in L_2

- Mathematical requirements for ideal sampling
  - The input signal $f(x)$ should be continuous
  - The samples $f[k] = f(x)|_{x=k}$ should be in ℓ_2

Theorem
Let $f(x) \in W_2^s$ with $s > \frac{1}{2}$. Then, the samples of f at the integers, $f[k] = f(x)|_{x=k}$, are in ℓ_2 and

$$F(e^{j\omega}) = \sum_{k \in \mathbb{Z}} f[k] e^{-j\omega k} = \sum_{n \in \mathbb{Z}} \hat{f}(\omega + 2\pi n) \qquad \text{a.e.}$$

Generalized (*almost everywhere*) version of Poisson's formula [Blu-U., 1999]

Shift-invariant spaces

Integer-shift-invariant subspace associated with a generating function φ (e.g., B-spline):

$$V(\varphi) = \left\{ f(x) = \sum_{k\in\mathbb{Z}} c[k]\varphi(x-k) : c \in \ell_2(\mathbb{Z}) \right\}$$

Generating function: $\varphi(x) \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad \hat{\varphi}(\omega) = \int_{x\in\mathbb{R}} \varphi(x) e^{-j\omega x} \mathrm{d}x$

- Autocorrelation (or Gram) sequence

$$a_\varphi[k] \triangleq \langle \varphi(\cdot), \varphi(\cdot - k)\rangle \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad A_\varphi(e^{j\omega}) = \sum_{n\in\mathbb{Z}} |\hat{\varphi}(\omega + 2\pi n)|^2$$

- Riesz-basis condition

Positive-definite Gram sequence: $0 < A^2 \leq \sum_{n\in\mathbb{Z}} A_\varphi(e^{j\omega}) \leq B^2 < +\infty$

$$\Updownarrow$$

$$A \cdot \|c\|_{\ell_2} \leq \underbrace{\left\| \textstyle\sum_{k\in\mathbb{Z}} c[k]\varphi(x-k) \right\|_{L_2}}_{\|f\|_{L_2}} \leq B \cdot \|c\|_{\ell_2}$$

Orthonormal basis $\Leftrightarrow$ $a_\varphi[k] = \delta_k$ $\Leftrightarrow$ $A_\varphi(e^{j\omega}) = 1$ $\Leftrightarrow$ $\|c\|_{\ell_2} = \|f\|_{L_2}$ (Parseval)

Example of sampling spaces

- Piecewise-constant functions

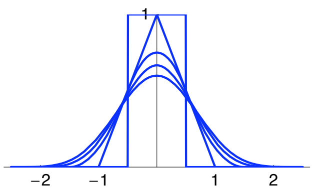
$\varphi(x) = \mathrm{rect}(x) = \beta^0(x)$ $\qquad a_\varphi[k] = \delta_k \quad \Leftrightarrow$ the basis is orthonormal

- bandlimited functions

$\varphi(x) = \mathrm{sinc}(x)$ $\qquad \sum_{n\in\mathbb{Z}} |\hat{\varphi}(\omega + 2\pi n)|^2 = 1 \quad \Leftrightarrow$ the basis is orthonormal

- Polynomial splines of degree n

$$\varphi(x) = \beta^n(x) = (\underbrace{\beta^0 * \beta^0 \cdots * \beta^0}_{(n+1)\text{ times}})(x)$$



Autocorrelation sequence: $a_{\beta^n}[k] = (\beta^n * \beta^n)(x)|_{x=k} = \beta^{2n+1}(k)$

Proposition. The B-spline of degree n, $\beta^n(x)$, generates a Riesz basis with lower and upper Riesz bounds $A = \inf_\omega\{A_{\beta^n}(e^{j\omega})\} \geq \left(\frac{2}{\pi}\right)^{n+1}$ and $B = \sup_\omega\{A_{\beta^n}(e^{j\omega})\} = 1$.

Equivalent and dual basis functions

- Equivalent basis functions: $\varphi_{\mathrm{eq}}(x) = \sum_{k\in\mathbb{Z}} p[k]\varphi(x-k)$

Proposition. Let φ be a valid (Riesz) generator of $V(\varphi) = \mathrm{span}\{\varphi(x-k)\}_{k\in\mathbb{Z}}$. Then, φ_{eq} also generates a Riesz basis of $V(\varphi)$ iff.

$$0 < C_1 \le |P(e^{j\omega})|^2 \le C_2 < +\infty \qquad \text{(almost everywhere)}$$

- Dual basis function

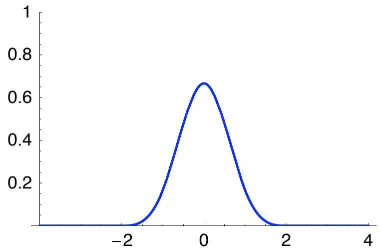
Unique function $\mathring{\varphi} \in V(\varphi)$ such that $\langle \varphi(x), \mathring{\varphi}(x-k)\rangle = \delta_k$ (biorthogonality)

Together, φ and $\mathring{\varphi}$ operate as if they were an orthogonal basis; i.e., the orthogonal projector of any function $f \in L_2$ onto $V(\varphi)$ is given by

$$\mathrm{P}_{V(\varphi)} f(x) = \sum_{k\in\mathbb{Z}} \underbrace{\langle f, \mathring{\varphi}(\cdot - k)\rangle}_{c[k]} \varphi(x-k)$$

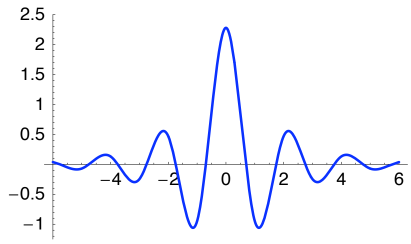
Example: four equivalent cubic-spline bases

- Cubic B-spline: $\varphi(x) = \beta^3(x)$



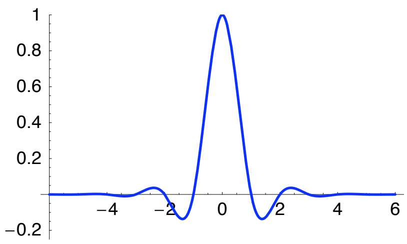
Compact support

- Dual spline: $\mathring{\varphi}(x)$



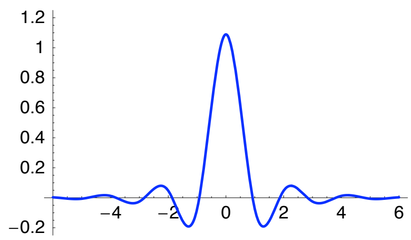
Biorthogonality: $\langle \varphi(x), \mathring{\varphi}(x-k)\rangle = \delta_k$

- Interpolating spline: $\varphi_{\mathrm{int}}(x)$



Interpolation: $\langle \varphi_{\mathrm{int}}(x), \delta(x-k)\rangle = \delta_k$

- Orthogonal spline: $\varphi_{\mathrm{ortho}}(x)$

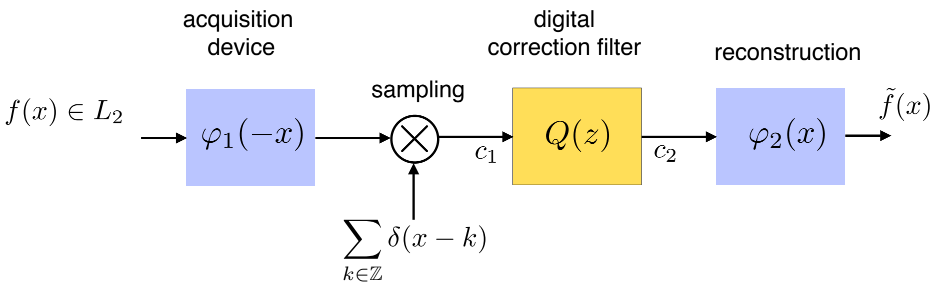


Orthogonality: $\langle \varphi_{\mathrm{ortho}}(x), \varphi_{\mathrm{ortho}}(x-k)\rangle = \delta_k$

SAMPLING REVISITED

- Generalized sampling system
- Generalized sampling theorem
- Consistent sampling: properties
- Performance analysis
- Applications

Generalized sampling system



- $\varphi_1(-x)$: prefilter (acquisition system)
- $\varphi_2(x)$: generating function (reconstruction subspace)

- Constraints
  - Consistent measurements: $\langle \tilde{f}, \varphi_1(\cdot - k)\rangle = c_1[k] = \langle f, \varphi_1(\cdot - k)\rangle, \ \forall k \in \mathbb{Z}$
  - Linearity and integer-shift invariance

⇒ Digital filtering solution: $\tilde{f}(x) = \sum_{n\in\mathbb{Z}} \underbrace{(q * c_1)[k]}_{c_2[k]} \varphi_2(x-k)$

Generalized sampling theorem

Cross-correlation sequence: $a_{12}[k] = \langle \varphi_1(\cdot - k), \varphi_2(\cdot) \rangle \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad A_{12}(e^{j\omega})$

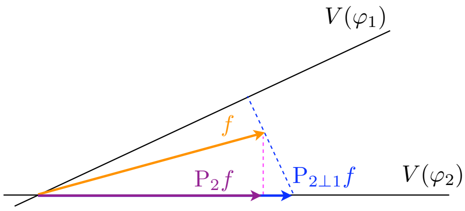
Consistent sampling theorem

Let $A_{12}(e^{j\omega}) \geq m > 0$. Then, there exists a unique solution $\tilde{f} \in V(\varphi_2)$ that is consistent with f in the sense that $c_1[k] = \langle f, \varphi_1(\cdot - k) \rangle = \langle \tilde{f}, \varphi_1(\cdot - k) \rangle$

$$\tilde{f}(x) = \mathrm{P}_{2\perp 1} f(x) = \sum_{n \in \mathbb{Z}} (q * c_1)[k] \varphi_2(x - k) \quad \text{with} \quad Q(z) = \frac{1}{\sum_{k \in \mathbb{Z}} a_{12}[k] z^{-k}}$$

- Geometric interpretation

$\tilde{f} = \mathrm{P}_{2\perp 1} f$ is the projection of f onto $V(\varphi_2)$ perpendicular to $V(\varphi_1)$.



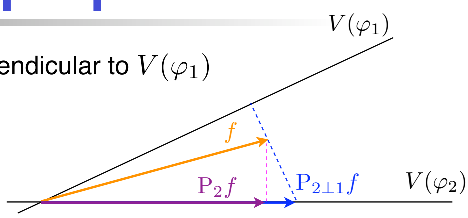
Orthogonality of error:

$$\langle f - \tilde{f}, \varphi_1(\cdot - k) \rangle = \underbrace{\langle f, \varphi_1(\cdot - k) \rangle}_{c_1[k]} - \underbrace{\langle \tilde{f}, \varphi_1(\cdot - k) \rangle}_{c_1[k]} = 0$$

(consistency)

Consistent sampling: properties

$\tilde{f} = \mathrm{P}_{2\perp 1} f$: oblique projection onto $V(\varphi_2)$ perpendicular to $V(\varphi_1)$



- Generalization of Shannon's theorem

  Every signal $f \in V(\varphi_2)$ can be reconstructed exactly

  - Flexibility and realism

    - φ_1 and φ_2 can be selected freely
    - They need not be biorthogonal (unlike wavelet pairs)

- Special case: least-squares approximation

  $\varphi_1 \in V(\varphi_2) \quad \Rightarrow \quad V(\varphi_1) = V(\varphi_2) \quad \Rightarrow \quad \mathrm{P}_{2\perp 1} = \mathrm{P}_2$ (orthogonal projection)

  Minimun-error approximation: $\tilde{f}(x) = \mathrm{P}_2 f(x) = \sum_{k \in \mathbb{Z}} \underbrace{\langle f, \mathring{\varphi}_2(\cdot - k) \rangle}_{(c_1 * q)[k]} \varphi_2(x - k)$

Application 1: interpolation revisited

- Interpolation constraint

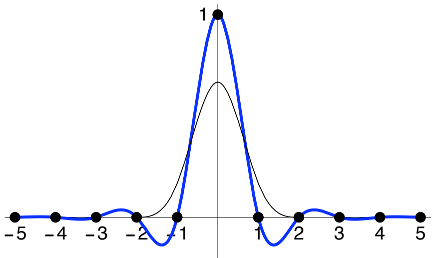
$$c_1[k] = f(x)|_{x=k} = \langle \delta(\cdot - k), f \rangle$$

- Interpolator = consistent ideal sampling system
  - Ideal sampler: $\varphi_1(x) = \delta(x)$
  - Reconstruction function: $\varphi_2(x) = \varphi(x)$
  - Cross-correlation: $a_{12}[k] = \langle \delta(\cdot - k), \varphi(\cdot) \rangle = \varphi(k)$

- Reconstruction/interpolation formula

$$Q_{\text{int}}(z) = \frac{1}{\sum_{k\in\mathbb{Z}} \varphi(k) z^{-k}}$$

$$\begin{aligned} f(x) &= \sum_{k\in\mathbb{Z}} \overbrace{(f * q_{\text{int}})[k]}^{c[k]} \; \varphi(x-k) \\ &= \sum_{k\in\mathbb{Z}} f[k] \; \varphi_{\text{int}}(x-k) \end{aligned}$$



Example: cubic-spline interpolant

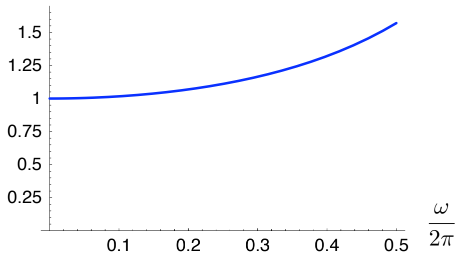
$$\varphi_{\text{int}}(x) = \sum_{k\in\mathbb{Z}} q_{\text{int}}[k] \; \varphi(x-k)$$

Application 2: consistent image display

- Problem specification
  - Ideal acquisition device: $\varphi_1(x,y) = \mathrm{sinc}(x) \cdot \mathrm{sinc}(y)$
  - LCD display: $\varphi_2(x,y) = \mathrm{rect}(x) \cdot \mathrm{rect}(y)$

- Separable image-enhancement filter

$$A_{12}(e^{j\omega}) = \sum_{n\in\mathbb{Z}} \hat{\varphi}_1^*(\omega + 2\pi n)\hat{\varphi}_2(\omega + 2\pi n) \quad \Rightarrow \quad Q(e^{j\omega}) = \frac{1}{\mathrm{sinc}\left(\frac{\omega}{2\pi}\right)}$$



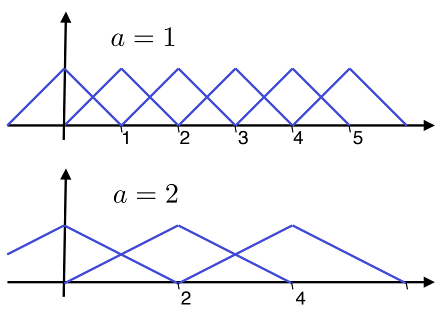
QUANTITATIVE APPROXIMATION THEORY

- Order of approximation
- Fourier-domain prediction of the L_2-error
- Strang-Fix conditions
- Spline case
- Asymptotic form of the error
- Optimized basis functions (MOMS)
- Comparison of interpolators

Order of approximation

- General "shift-invariant" space at scale a

$$V_a(\varphi) = \left\{ s_a(x) = \sum_{k\in\mathbb{Z}} c[k]\varphi\left(\frac{x}{a} - k\right) : c \in \ell_2 \right\}$$



- Projection operator

$$\forall f \in L_2, \qquad \mathrm{P}_a f = \arg \min_{s_a \in V_a} \|f - s_a\|_{L_2}$$

- Order of approximation

Definition

A scaling/generating function φ has order of approximation L iff.

$$\forall f \in W_2^L, \qquad \|f - \mathrm{P}_a f\|_{L_2} \le C \cdot a^L \cdot \|f^{(L)}\|_{L_2}$$

Fourier-domain prediction of the L_2-error

Theorem [Blu-U., 1999]

Let $\mathrm{P}_a f$ denote the orthogonal projection of f onto $V_a(\varphi)$ (at scale a). Then,

$$\forall f \in W_2^s, \qquad \|f - \mathrm{P}_a f\|_{L_2} = \left(\int_{-\infty}^{+\infty} |\hat{f}(\omega)|^2 E_\varphi(a\omega) \frac{d\omega}{2\pi} \right)^{1/2} + o(a^s)$$

where

$$E_\varphi(\omega) = 1 - \frac{|\hat{\varphi}(\omega)|^2}{\sum_{k\in\mathbb{Z}} |\hat{\varphi}(\omega + 2\pi k)|^2}$$

Fourier-transform notation: $\hat{f}(\omega) = \int_{-\infty}^{+\infty} f(x) e^{-j\omega x} \mathrm{d}x$

Strang-Fix conditions of order L

Let $\varphi(x)$ satisfy the Riesz-basis condition. Then, the following Strang-Fix conditions of order L are equivalent:

(1) $\hat{\varphi}(0) = 1$, and $\hat{\varphi}^{(n)}(2\pi k) = 0$ for $\begin{cases} k \neq 0 \\ n = 0 \ldots L-1 \end{cases}$

(2) $\varphi(x)$ reproduces the polynomials of degree $L-1$; i.e., there exist weights $p_n[k]$ such that

$x^n = \sum_{k\in\mathbb{Z}} p_n[k] \varphi(x-k)$, for $n = 0 \ldots L-1$

(3) $E_\varphi(\omega) = \frac{C_L^2}{(2L)!} \cdot \omega^{2L} + O(\omega^{2L+2})$

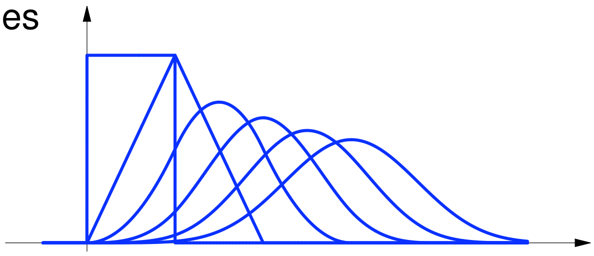
(4) $\forall f \in W_2^L, \qquad \|f - \mathrm{P}_a f\|_{L_2} = O(a^L)$

Polynomial splines

- Basis functions: causal B-splines

$$\beta_+^n(x) = \left(\beta_+^{n-1} * \beta_+^0\right)(x)$$

$$\beta_+^0(x) = \begin{cases} 1, & \text{for } 0 \le x < 1 \\ 0, & \text{otherwise.} \end{cases}$$



- Fourier-domain formula

$$\hat{\beta}_+^n(\omega) = \left(\frac{1-e^{-j\omega}}{j\omega}\right)^{n+1}$$

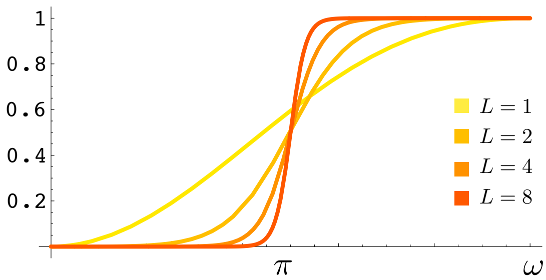
- Order of approximation

$$\hat{\beta}_+^n(2\pi k + \Delta\omega) = O(|\Delta\omega|^{n+1}) \text{ for } k \neq 0$$

$$\Longrightarrow \qquad \beta_+^n \text{ has order of approximation } L = n+1$$

Spline approximation

- Fourier approximation kernel



$$E_{\beta^n}(\omega) = \frac{\sum_{k\neq 0} |\hat{\beta}^n(\omega+2\pi k)|^2}{\sum_{k\in\mathbb{Z}} |\hat{\beta}^n(\omega+2\pi k)|^2}$$

Order: $L = n+1$

- Link with Riemann's zeta function

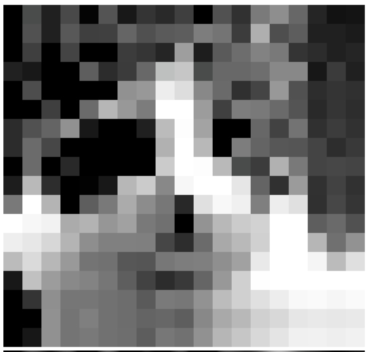
$$\boxed{\zeta(z) = \sum_{n=1}^{+\infty} n^{-z}}$$

$$\begin{aligned} E_{\beta^n}(\omega) &= |2\sin(\omega/2)|^{2n+2} \frac{\sum_{k\neq 0} \frac{1}{|\omega+2\pi k)|^{2n+2}}}{\sum_{k\in\mathbb{Z}} |\hat{\beta}^n(\omega+2\pi k)|^2} \\ &= \frac{2\zeta(2n+2))}{(2\pi)^{2n+2}} \cdot \omega^{2n+2} + O(|\omega|^{2n+4}) \end{aligned}$$

Spline reconstruction of a PET-scan

Piecewise constant

$L = 1$



Cubic spline

$L = 4$

Asymptotic form of the error

Theorem [U.-Daubechies, 1997]

Let φ be an Lth order function. Then, asymptotically, as $a \to 0$,

$$\forall f \in W_2^L, \qquad \|f - \mathrm{P}_a f\|_{L_2} = C_L \cdot a^L \cdot \|f^{(L)}\|_{L_2}$$

where

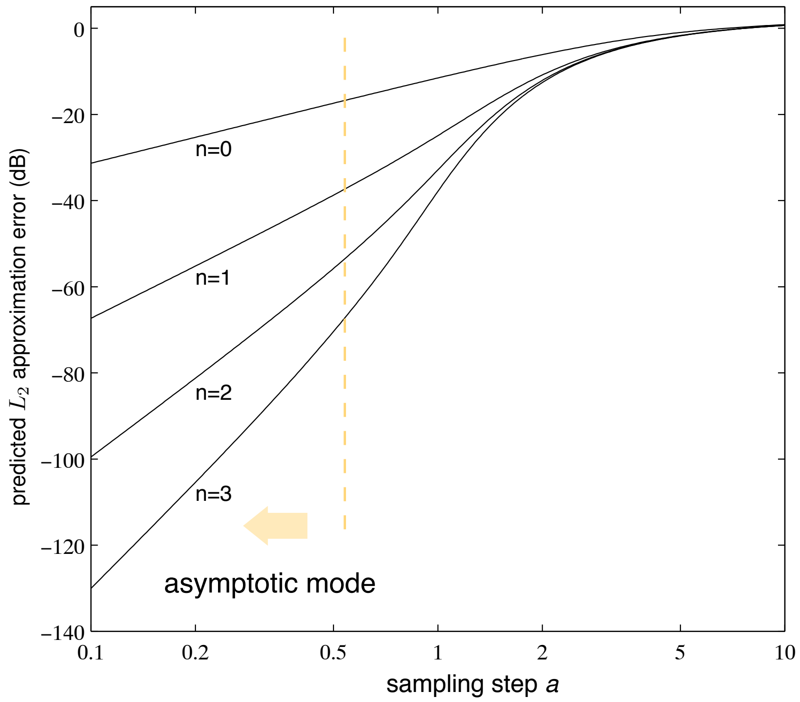
$$C_L = \frac{1}{L!}\sqrt{2\sum_{n=1}^{+\infty}\left|\hat{\varphi}^{(L)}(2\pi n)\right|^2} \qquad (= \sqrt{\frac{E_\varphi^{(2L)}(0)}{(2L)!}})$$

- Special case: splines of order $L = n + 1$

$$C_{L,\text{splines}} = \frac{\sqrt{2\zeta(2L)}}{(2\pi)^L} = \sqrt{\frac{B_{2L}}{(2L)!}}$$

(Bernoulli number of order $2L$)

Characteristic decay of the error for splines



Least squares approximation of the function $f(x) = e^{-x^2/2}$

Optimized basis functions (MOMS)

- Motivation
  - Cost of prefiltering is negligible (in 2D and 3D)
  - Computational cost depends on kernel size W
  - Order of approximation is a strong determinant of quality

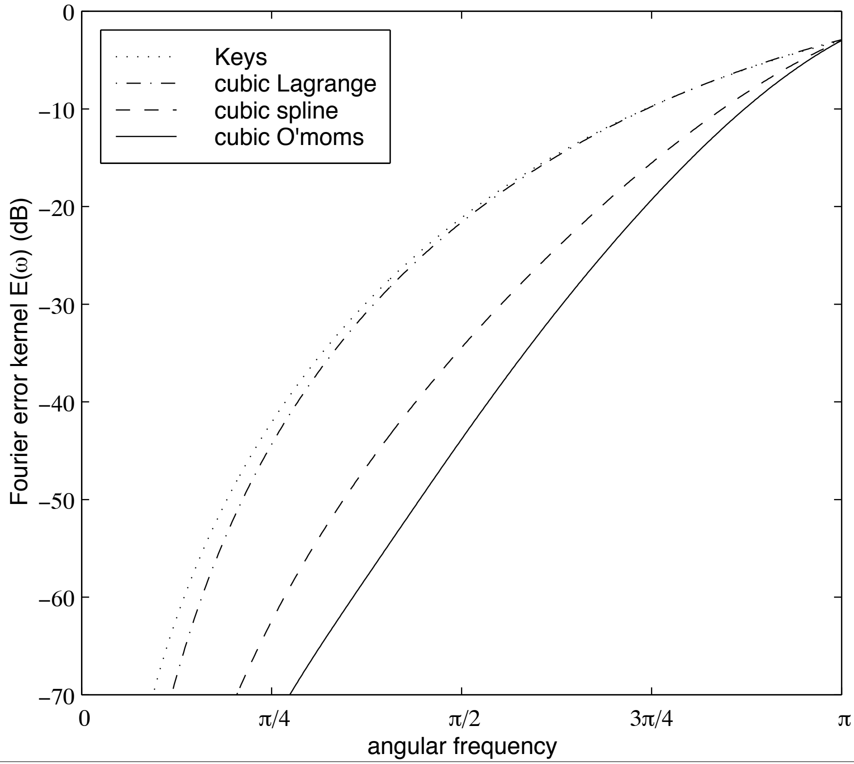
QUESTION: What are the basis functions with maximum order of approximation and minimum support ?

ANSWER: Shortest functions of order L (MOMS) $\varphi_{\text{moms}}(x) = \sum_{k=0}^{L-1} a_k \mathrm{D}^k \beta^{L-1}(x)$

- Most interesting MOMS
  - B-splines: smoothest ($\beta^{L-1} \in \dot{C}^{L-1}$) and only refinable MOMS
  - Shaum's piecewise-polynomial interpolants (no prefilter)
  - OMOMS: smallest approximation constant C_L

$$\varphi_{\text{opt}}^3(x) = \beta^3(x) + \frac{1}{42}\frac{\mathrm{d}^2\beta^3(x)}{\mathrm{d}x^2}$$

Comparisons of cubic interpolators of size *W*=4



2-29

INTERPOLATION IN THE PRESENCE OF NOISE

- Interpolation and regularization
- Smoothing splines
- General concept of an L-spline
- Optimal Wiener-like estimators

2-30

Spline-fitting with noisy data

- Context
  - Input data $\{f[k]\}_{k\in\mathbb{Z}}$ corrupted by noise
  - Model: continuously defined function $s(x)$
  - Data term: $\xi_{\text{data}} = \sum_{k\in\mathbb{Z}} |f[k] - s(k)|^2$ (discrete domain)
  - Spline energy: $\xi_{\text{spline}} = \|\mathrm{D}^m s\|_{L_2}^2$ (continuous domain)

- Possible formulations

$$\boxed{\min_{s(x)\in L_2} \xi_{\text{spline}} \quad \text{subject to } \xi_{\text{data}} = C_1}$$

A priori knowledge of noise variance

OR

$$\boxed{\min_{s(x)\in L_2} \xi_{\text{data}} \quad \text{subject to } \xi_{\text{spline}} \leq C_2}$$

A priori knowledge of signal class

$\Updownarrow$

$$\min_{s(x)\in L_2} \{\xi_{\text{data}} + \lambda\, \xi_{\text{spline}}\}$$

(Tikhonov-like regularization)

Lagrange multiplier

Regularized fit: smoothing splines

- B-spline representation: $s(x) = \sum_{k\in\mathbb{Z}} c[k]\beta^n(x-k)$

- Smoothing splines

Discrete, noisy input:

$f[k] = s(k) + n[k]$ $\longrightarrow$ Smoothing algorithm $\longrightarrow$ $c[k]$

Theorem: The solution (among all functions) of the smoothing spline problem

$$\min_{s(x)} \left\{ \sum_{k\in\mathbb{Z}} |f[k] - s(k)|^2 + \lambda \int_{-\infty}^{+\infty} |\mathrm{D}^m s(x)|^2 dx \right\}$$

is a cardinal spline of degree $2m-1$. Its coefficients $c[k] = h_\lambda * f[k]$ can be obtained by suitable recursive digital filtering of the input samples $f[k]$.

- Special case: the draftman's spline

The minimum-curvature interpolant is obtained by setting $m = 2$ and $\lambda \to 0$.

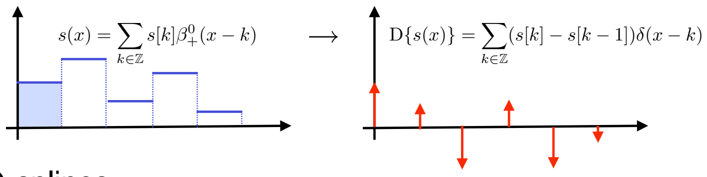
It is a cubic spline !

General concept of an L-spline

$\mathrm{L}\{\cdot\}$: differential operator (shift-invariant) $\delta(x)$: Dirac distribution

Definition
The function $s(x)$ is a ***cardinal* L-*spline*** (with knots at the integers) iff.

$$\mathrm{L}\{s(x)\} = \sum_{k\in\mathbb{Z}} a[k]\delta(x-k)$$



- Special cases
  - Piecewise-constant = D-splines
  - Polynomial splines = D^{n+1}-splines

Justification:
$\mathrm{D}^{n+1}\{\beta_+^n(x)\} = \Delta_+^{n+1}\{\delta(x)\} = \sum_{k\in\mathbb{Z}} d[k]\delta(x-k) \quad \xleftrightarrow{\mathcal{F}} \quad D(e^{j\omega}) = (1-e^{-j\omega})^{n+1}$

Existence of B-spline-like bases

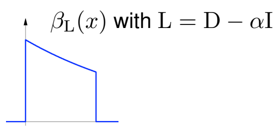
$\mathrm{L}\{\cdot\}$: generalized differential operator of order $s > \frac{1}{2}$

- Riesz-basis representation

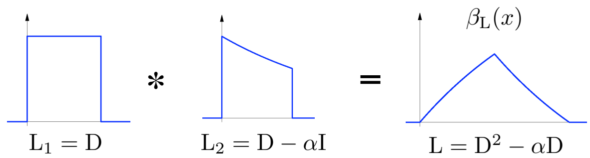
Cardinal L-splines generally admit a B-spline-like representation

$$s(x) = \sum_{k\in\mathbb{Z}} c[k]\beta_{\mathrm{L}}(x-k)$$

Example: first-order exponential B-spline



- Composition properties

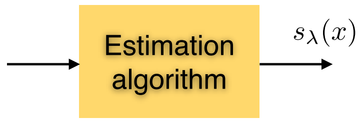


  - Higher-order B-splines:
    $\beta_{\mathrm{L}_1}(x)$ and $\beta_{\mathrm{L}_2}(x)$ are B-spline generators for the cardinal L_1- and L_2-splines.
    Then, $\beta_{\mathrm{L}_1}(x) * \beta_{\mathrm{L}_2}(x)$ is a generator for the $(\mathrm{L}_1\mathrm{L}_2)$-splines.
  - Positive-definite operators: If $\beta_{\mathrm{L}}(x)$ generates a Riesz basis for the L-splines, then $\varphi(x) = \beta_{\mathrm{L}}(x) * \beta_{\mathrm{L}}(-x)$ generates a Riesz basis for the $(\mathrm{L}^*\mathrm{L})$-splines and the interpolation problem in $V(\varphi)$ is well posed.

Generalized smoothing splines

- Generalized spline energy: $\xi_{\text{spline}} = \|\mathrm{L}s\|_{L_2}^2$
- Generalized smoothing-spline fit

Discrete, noisy input:
$f[k] = s(k) + n[k]$



Theorem: The solution (among all functions) of the generalized smoothing problem

$$\min_{s(x)} \left\{ \sum_{k\in\mathbb{Z}} |f[k] - s(k)|^2 + \lambda \int_{-\infty}^{+\infty} |\mathrm{L}s(x)|^2 dx \right\}$$

is a cardinal $\mathrm{L}^*\mathrm{L}$-spline.

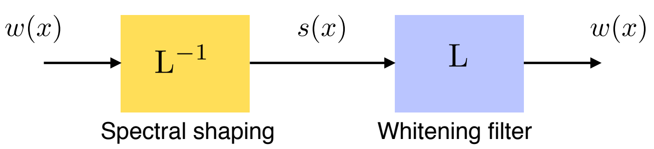
The solution has a B-spline representation $s_\lambda(x) = \sum_{k\in\mathbb{Z}} c[k]\varphi(x-k)$, the coefficients of which are obtained by suitable filtering of the input data (generalized smoothing-spline algorithm).

Stochastic signal models

- Wide-sense stationary process
  - Realization of the stochastic process: $s(x)$
  - Zero-mean: $E\{s(x)\} = 0$
  - Autocorrelation function: $E\{s(y) \cdot s(y-x)\} = c_s(x) \in L_2$
  - Spectral density function: $C_s(\omega) = \int_{x\in\mathbb{R}} c_s(x) e^{-j\omega x} \mathrm{d}x \in L_2$
- Stochastic differential equation

$\mathrm{L}\{s(x)\} = w(x)$ (driven by white Gaussian noise)

$c_w(x) = \sigma_0^2 \delta(x)$



$C_w(\omega) = \sigma_0^2$

$$C_s(\omega) = \frac{\sigma_0^2}{|\hat{L}(\omega)|^2}$$

MMSE estimation in the presence of noise

- Statistical hypotheses
  - Discrete measurements (signal + noise): $f[k] = s(k) + n[k]$
  - Signal autocorrelation: $c_s(x)$ such that $\mathrm{L}^*\mathrm{L}\{c_s(x)\} = \sigma_0^2 \cdot \delta(x)$
  - Discrete white noise with variance $\sigma^2 \Rightarrow c_n[k] = \sigma^2 \cdot \delta[k]$

- MMSE continuous-domain signal estimation

Theorem
Under the above assumptions, the linear Minimum-Mean Square Error Estimator of $s(x)$ at position $x = x_0$, given the measurements $\{f[k]\}_{k\in\mathbb{Z}}$, is $s_\lambda(x_0)$ with $\lambda = \frac{\sigma^2}{\sigma_0^2}$, where $s_\lambda(x)$ is the $\mathrm{L}^*\mathrm{L}$-smoothing-spline fit of $\{f[k]\}_{k\in\mathbb{Z}}$ given by the generalized smoothing-spline algorithm.

Remark: optimal overall estimators if one adds the assumption of Gaussianity

CONCLUSION

- Generalized sampling
  - Unifying Hilbert-space formulation: Riesz basis, etc.
  - Approximation point of view:
    projection operators (oblique vs. orthogonal)
  - Increased flexibility; closer to real-world systems
  - Generality: nonideal sampling, interpolation, etc...
- Quest for the “optimal” representation space
  - Not bandlimited ! (prohibitive cost, ringing, etc.)
  - Quantitative approximation theory: L_2-estimates, asymptotics
  - Optimized functions: MOMS
  - Signal-adapted design ?
- Interpolation/approximation in the presence of noise
  - Regularization theory: smoothing splines
  - Stochastic formulation: new, hybrid form of Wiener filter

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