Active Contours — Snakes

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2007–2013

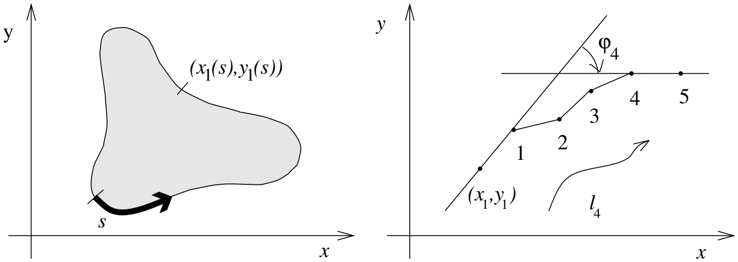
Snake principles

- Initial curve (manual)
- Curve evolves using image data
- ... until finds desired boundary
- Criterion = image term + smoothness (internal) term + shape term

(show animate heart.gif, ventricles_movie.gif)

Traditional snakes

Curve is parameterized as $\mathbf{v}(s) = [x(s), y(s)]$ with $s \in [0, 1]$



(a) (b)

Minimize energy

$$E^*_{\mathrm{snake}} = \int_0^1 E_{\mathrm{snake}}\big(\mathbf{v}(s)\big)\, \mathrm{d}s$$
$$= \int_0^1 \Big(E_{\mathrm{int}}\big(\mathbf{v}(s)\big) + E_{\mathrm{image}}\big(\mathbf{v}(s)\big) + E_{\mathrm{con}}\big(\mathbf{v}(s)\big)\Big)\, \mathrm{d}s\,,$$

Internal energy term

$$E_{\mathrm{int}} = \frac{\alpha}{2}\left|\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}s}\right|^2 + \frac{\beta}{2}\left|\frac{\mathrm{d}^2\mathbf{v}}{\mathrm{d}s^2}\right|^2 ,$$

α, β specify *elasticity* and *stiffness*. Can depend on s.

Image energy term

$$E_{\mathrm{image}} = w_{\mathrm{line}}\, E_{\mathrm{line}} + w_{\mathrm{edge}}\, E_{\mathrm{edge}}$$

- Line functional attracts to white/black parts

$$E_{\mathrm{line}} = \pm f(x, y)$$

- Edge functional attracts to strong edges

$$E_{\mathrm{edge}} = -\left|\nabla f(x, y)\right|^2$$

  smooth/denoise before/after taking the gradient

- Other application-dependent image energy terms.

Shape term — prefer likely shapes.

Euler-Lagrange equations

$$E^*_{\text{snake}} = \int_0^1 E_{\text{snake}}\big(\mathbf{v}(s), \mathbf{v}'(s)\big)\mathrm{d}s = \int_0^1 E_{\text{snake}}\big(\mathbf{v}, \mathbf{v}_s\big)\mathrm{d}s$$

For optimal $\mathbf{v}(s)$ it must hold

$$\frac{\mathrm{d}}{\mathrm{d}s} E_{\mathbf{v}_s} - E_{\mathbf{v}} = 0$$

Substituting for E_{int} in $E_{\text{snake}} = E_{\text{int}} + E_{\text{image}}$:

$$-\frac{\mathrm{d}}{\mathrm{d}s}\left(\alpha(s)\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}s}\right) + \frac{\mathrm{d}^2}{\mathrm{d}s^2}\left(\beta(s)\frac{\mathrm{d}^2\mathbf{v}}{\mathrm{d}s^2}\right) + \nabla E_{\text{image}}\Big(\mathbf{v}(s)\Big) = 0$$

Supposing constant α, β

$$-\alpha\frac{\mathrm{d}^2\mathbf{v}}{\mathrm{d}s^2} + \beta\frac{\mathrm{d}^4\mathbf{v}}{\mathrm{d}s^4} + \underbrace{\nabla E_{\text{image}}\Big(\mathbf{v}(s)\Big)}_{\mathbf{f}_E} = 0$$

where $\mathbf{f}_E$ is an external force

Solving EL equations

Euler-Lagrange equation for $\mathbf{v}(s)$

$$-\alpha\frac{\mathrm{d}^2\mathbf{v}}{\mathrm{d}s^2} + \beta\frac{\mathrm{d}^4\mathbf{v}}{\mathrm{d}s^4} + \kappa\mathbf{f}_E = 0$$

Gradient descent — time evolution converges to a solution

$$\frac{\partial\mathbf{v}}{\partial t} = \alpha\,\frac{\partial^2\mathbf{v}}{\partial s^2} - \beta\,\frac{\partial^4\mathbf{v}}{\partial s^4} + \kappa\,\mathbf{f}_\mathsf{E}$$

Balloon force

What to do, when no image information is available? Grow/shrink.

$$\frac{\partial \mathbf{v}}{\partial t} = \alpha \, \frac{\partial^2 \mathbf{v}}{\partial s^2} - \beta \, \frac{\partial^4 \mathbf{v}}{\partial s^4} + \kappa \, \mathbf{f}_\mathrm{E} + \lambda \, \mathbf{f}_\mathrm{B}$$

Balloon force $\mathbf{f}_\mathrm{B}$ perpendicular to the snake curve.

Discretization and implementation

- Unit time steps $\Delta t = 1$
- Snake is represented by two vectors containing the x and y coordinates of a sequence of points on the snake curve.
- Distance between subsequent points is maintained close to 1 pixel.
- Resample if needed.
- Snake is supposed to be closed and non-intersecting.

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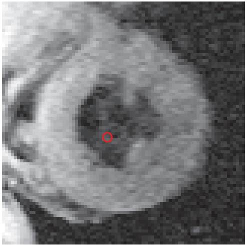
- Snake is supposed to be closed and non-intersecting.
- Derivatives approximated by discrete convolution

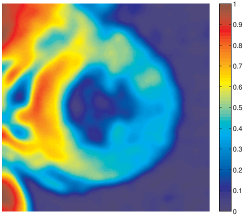
$$\alpha \frac{\partial^2 \mathbf{v}}{\partial s^2} - \beta \frac{\partial^4 \mathbf{v}}{\partial s^4} \bigg|_{s=s_i} \approx h * \begin{bmatrix} x(s_i) \\ y(s_i) \end{bmatrix} .$$

- Stop when area no longer changes.

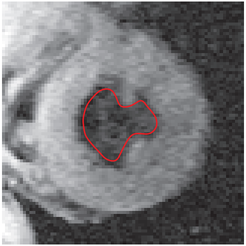
Snake example 1



MRI image



Energy



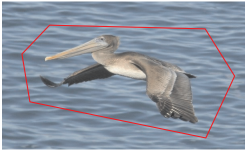
result

Energy = smoothed image

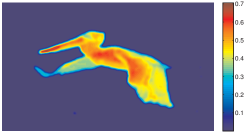
$\alpha = 0.1$, $\beta = 0.01$, $\kappa = 0.2$, $\lambda = 0.05$.

Growing balloon force.

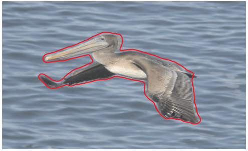
Snake example 2



input image



energy



result

Energy = image converted to grayscale, thresholded, smoothed.
$\alpha = 0.1$, $\beta = 0.1$, $\kappa = 0.3$, $\lambda = -0.05$
Shrinking balloon force.

Gradient vector flow (GVF) snakes

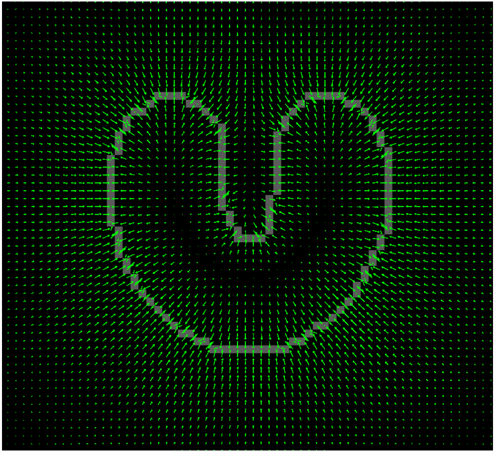
(Xu and Prince)

- Image gives information close to edges
- No information in flat region
- An *edge map* f is high where we want the snake to be attracted, i.e. $\mathbf{f}_E = \nabla f$
- GVF provides a smooth interpolation $\mathbf{g} = (u, v)$ everywhere from f
- Alternative to balloon force, less parameter tuning.

GVF field

Minimize

$$\iint \mu \left(u_x^2 + u_y^2 + v_x^2 + v_y^2\right) + \|\nabla f\|^2 \, \|\mathbf{g} - \nabla f\|^2 \, \mathrm{d}x \, \mathrm{d}y$$



GVF minimization

$$\iint \mu \left(u_x^2 + u_y^2 + v_x^2 + v_y^2\right) + \|\nabla f\|^2 \, \|\mathbf{g} - \nabla f\|^2 \, \mathrm{d}x \, \mathrm{d}y$$

At minimum, Euler-Lagrange equations must hold

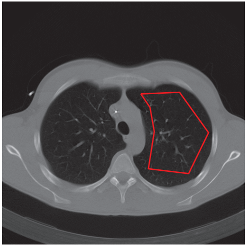
$$\begin{aligned}
\mu \, \Delta u - \left(u - f_x\right) \left(f_x^2 + f_y^2\right) &= 0 \, , \\
\mu \, \Delta v - \left(v - f_y\right) \left(f_x^2 + f_y^2\right) &= 0 \, ,
\end{aligned}$$

Solved by gradient descent / time evolution:

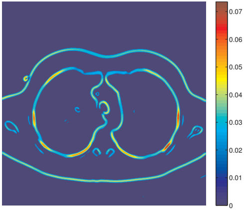
$$\begin{aligned}
u_t(x,y,t) &= \mu \, \Delta u(x,y,t) - \left(u(x,y,t) - f_x(x,y)\right) \left(f_x(x,y)^2 + f_y(x,y)^2\right) \\
v_t(x,y,t) &= \mu \, \Delta v(x,y,t) - \left(v(x,y,t) - f_y(x,y)\right) \left(f_x(x,y)^2 + f_y(x,y)^2\right)
\end{aligned}$$

- Equations are discretized and solved by numeric integration with a fixed time step on a uniform grid.
- Multiresolution needed for speed and robustness.

GVF example



Lung slice



Energy

result

Energy $=$ thresholded smoothed edge map $E = \|\nabla G_\sigma * f\|$
No balloon force needed.