(1)

A simple game that is not a weighted voting game

$N = \{1,2,3,4\}$

$$\forall A \subseteq N: \quad v(A) = \begin{cases} 0 & A = 1,2,3,4,13,24,\emptyset \\ & \quad \text{loosing} \\ 1 & \text{otherwise} \\ & \quad \text{winning} \end{cases}$$

The geometrical meaning of the claim is that the vertices of the Boolean hypercube $\{0,1\}^4$ coloured by • and • cannot be separated by a hyperplane. We proceed by the contradiction and assume that v is a weighted voting game, where (w_1, w_2, w_3, w_4) is the vector of weights and q is the quota. Then

$$\underbrace{w_1 + w_2 \geq q \quad \text{and} \quad w_3 + w_4 \geq q}$$

$$\Downarrow$$

$$\boxed{\sum_{i \in N} w_i \geq 2q}$$

At the same time,

$$\underbrace{w_1 + w_3 < q \quad \text{and} \quad w_2 + w_4 < q}$$

$$\boxed{\sum_{i \in N} w_i < 2q} \qquad \text{contradiction}$$

② A weighted voting game on 3 players

$$v(A) = \begin{cases} 1 & \sum_{i \in A} w_i \geq q \\ 0 & \text{otherwise} \end{cases}$$

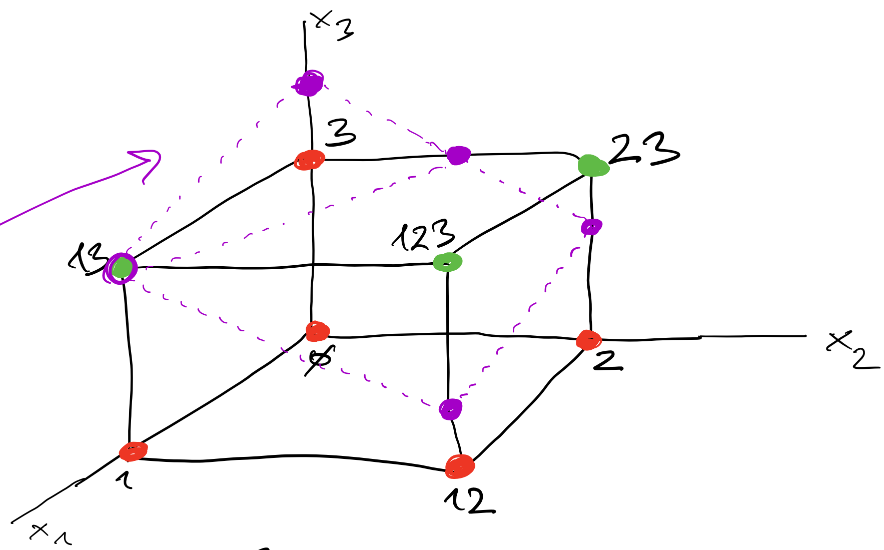
$(w_1, w_2, w_3) = (1, 2, 3)$

$q = 4$

$$= \begin{cases} 1 & A = 123, 13, 23 \\ 0 & A = \emptyset, 1, 2, 3, 12 \end{cases}$$

winning

loosing



The hyperplane : $\sum_{i=1}^{3} w_i x_i = q$

$$x_1 + 2x_2 + 3x_3 = 4$$

Points on the hyperplane : $(0, 0, 4/3)$, $(1, 0, 1)$, $(0, 1/2, 1)$, $(1, 1, 1/3)$, $(0, 1, 2/3)$

③ Every simple game is a vector weighted voting game

Let $v : \mathcal{P}(N) \to \mathbb{R}$ be a simple game.
We want to find K vectors $w^1, \dots, w^K \in \mathbb{R}^n_+$ and quotas $q_1, \dots, q_K > 0$ such that

$$\forall A \subseteq N \qquad v(A) = \begin{cases} 1 & \text{if } \sum_{i \in A} w_i^j \geq q^j \quad \forall j = 1, \dots, K \\ 0 & \text{otherwise.} \end{cases} \quad (*)$$

To this end, let $A^1, \dots, A^K$ be all the nonempty loosing coalitions in v, that is,

$$v(A^j) = 0 \qquad \forall j = 1, \dots, K.$$

For each $j = 1, \dots, K$, define $q^j = 1$, and

$$w_i^j = \begin{cases} 1 & \text{if } i \notin A^j \\ 0 & \text{if } i \in A^j \end{cases} \qquad \forall i \in N.$$

Observe that every pair (w^j, q^j) defines the weighted voting game

$$v^j(A) = \begin{cases} 1 & \sum_{i \in A} w_i^j \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

Equivalently, a coalition A is winning in v^j

$$\iff$$

$$A \not\subseteq A^j.$$

Now, assume that A is winning in v, $v(A) = 1$.

We will show that A is winning in v^j for every $j = 1, \dots, k$. By way of contradiction, let there be some j such that

$$A \subseteq A^j.$$

However, monotonicity of v yields

$$1 = v(A) \leq v(A^j) = 0,$$

a contradiction. Therefore, $A \nsubseteq A^j$ $\forall j = 1, \dots, k$.

In the last step we consider $v(A) = 0$. Then necessarily $A = A^j$ for some j, which implies

$$\sum_{i \in A^j} w_i^j = 0 < 1 = q^j.$$

This finishes the proof since we verified that v satisfies (*).

(4) Reform of the UN Security Council

The idea is that at least 2 permanent members are needed to veto a proposal. This defines a simple game

$$v(A) = \begin{cases} 1 & |A \cap \{1, \dots, 5\}| \geq 4 \text{ and } |A| \geq 9 \\ & \\ 0 & \text{otherwise} \end{cases}$$

Let $i \in \{6, \dots, 15\}$ be a non-permanent member. Then

$$\varphi_i^S(v) = \frac{1}{15!}\left[\underbrace{\binom{9}{3} \cdot 8!\,6!}_{\#\text{ swings for } i \text{ if } |A \cap \{1,\dots,5\}| = 5} + \underbrace{5 \cdot \binom{9}{4} \cdot 8!\,6!}_{\#\text{ swings for } i \text{ if } |A \cap \{1,\dots,5\}| = 4} \right]$$

$$= \frac{8!\,6!\left(\binom{9}{3} + 5\binom{9}{4}\right)}{15!} \doteq \underline{0.016}$$

Then, for any $j \in \{1, \dots, 5\}$,

$$\varphi_j^S(v) = \frac{1}{5}\left(1 - 10\,\varphi_i^S(v)\right) \doteq \underline{0.168}$$

(5) Voting power and quota

We consider a weighted voting game v with weights w and quota q, and a weighted voting game v' with weights w and quota $w(N) + 1 - q$. Then $\forall i \in N$:

$$\boxed{\varphi_i^S(v) = \varphi_i^S(v')} \quad (*)$$

We will prove $(*)$ by establishing a bijection between the permutations making i pivotal in v and the permutations making i pivotal in v'.

Let $i \in N$ be pivotal in v for a permutation π, that is,

$$w(A_i^{\pi}) < q \quad \text{and} \quad w(A_i^{\pi} \cup i) \geq q.$$

Define $\pi'(j) := \pi(n+1-j)$ $\forall j \in N$.

Then

$$\begin{aligned} w(A_i^{\pi'}) &= w(N) - w(A_i^{\pi} \cup i) \\ &\leq w(N) - q < w(N) - q + 1 \end{aligned}$$

and

$$\begin{aligned} w(A_i^{\pi'} \cup i) &= w(N) - w(A_i^{\pi}) \\ &> w(N) - q > w(N) - q + 1, \end{aligned}$$

which means that i is pivotal for π' in game v'. An analogous argument shows the converse.