1st-price auction with $n=2$ bidders and uniformly distributed private values

$V_1, V_2 \sim$ Uniform over $[0,1]$
and independent (IPV assumption)

1) We will prove that bidding half the private value is an equilibrium strategy s^* for every player i, where $s^*(v) = \frac{1}{2} v$. Specifically, we show that the strategy profile (s^*, s^*) is a BNE.

Let player 1 bids $b_1 \geq 0$ while player 2 uses pure strategy s^*. Then

$$
\begin{aligned}
U_1(b_1, s^*(V_2)) &= U_1\left(b_1, \tfrac{1}{2}V_2\right) = \mathbf{P}\left[\tfrac{1}{2}V_2 \leq b_1\right](v_1 - b_1) \\
&= \mathbf{P}\left[V_2 \leq 2b_1\right](v_1 - b_1) \\
&= \min(2b_1, 1)(v_1 - b_1). \\
&= \begin{cases} 2b_1(v_1 - b_1), & b_1 \in [0, \frac{1}{2}], \\ v_1 - b_1, & b_1 \in [\frac{1}{2}, \infty). \end{cases}
\end{aligned}
$$

The maximum of $U_1(b_1, s^*)$ is $b_1 = \frac{1}{2} v_1$, so this is the best response strategy equal to s^*, which shows that (s^*, s^*) is a BNE.

2) We can directly compute the *expected revenue* of the seller, which is equal to

$$\mathbf{E}\left[\max\left(\frac{V_1}{2}, \frac{V_2}{2}\right)\right] = \frac{1}{2}\mathbf{E}\left[\underbrace{\max(V_1, V_2)}_{Z :=}\right].$$

The distribution function of random variable Z is

$$F_Z(x) = \mathbf{P}[Z \leq x] = \mathbf{P}[\max(V_1, V_2) \leq x]$$

$$= \mathbf{P}[V_1 \leq x, V_2 \leq x]$$

$$= \mathbf{P}[V_1 \leq x] \cdot \mathbf{P}[V_2 \leq x]$$

$$= x^2 \quad \forall x \in [0, 1]$$

This means that the pdf of Z is $f(x) \begin{cases} = 2x & \forall x \in [0, 1), \\ = 0 & \text{otherwise}. \end{cases}$

The expected revenue is then

$$\frac{1}{2}\mathbf{E}(Z) = \frac{1}{2}\int_0^1 x \cdot (2x)\, dx = \int_0^1 x^2 dx$$

$$= 1/3$$

2-nd price auction with $n=2$ bidders and uniformly distributed private values

The assumptions are same as above.
We know that the equilibrium strategy is to bid truthfully,

$$s^*(v) = v \quad \forall v \in [0,1].$$

The seller's expected revenue is

$$\mathbb{E}\Big[\underbrace{\min(V_1,V_2)}_{V_1+V_2-\max(V_1,V_2)}\Big].$$

We know that $\mathbb{E}[V_1] = \mathbb{E}[V_2] = 1/2$ and $\mathbb{E}[\max(V_1,V_2)] = 2/3$, so

$$\mathbb{E}[\min(V_1,V_2)] = 1/3,$$

which is the same as in the above example.