How to divide the prey fairly?

The dragon (3) and tiger (4) should get an equal portion since their marginal contributions are identical:

$$v(\{3\} \cup A) = v(\{4\} \cup A)$$
$$\forall A \subseteq \{1, 2\}$$

The pig (1) should not eat at all, since his contribution is null to any coalition: $v(\{1\} \cup A) = v(A)$

$$\forall A \subseteq \{1, 2, 3, 4\}.$$

Weighted voting game (Nassau)

This example show it is meaningless to measure the voting power by the number of votes of every town. Towns 5 and 6 are null players since there is no coalition A such that $\sum_{i \in A} w_i \in \{56, 57\}$. This means that neither 5 nor 6 can ever cast a decisive vote.

The axioms of Shapley value are not superfluous

1) Remove efficiency and define

$$\varphi := 2\varphi^S.$$

Then φ satisfies symmetry, additivity, null player property.

2) Remove NPP and define

$$\varphi_i(v) = \frac{v(N)}{n}.$$

Then φ satisfies SYM, ADD, EFF.

3) Remove SYM and define

$$\varphi_i(v) = v(\{1, \dots, i\}) - v(\{1, \dots, i-1\})$$

Then φ is ADD, EFF, NPP.

4) Remove ADD and define

$$\varphi_i(v) = \begin{cases} 0 & i \text{ null} \\ \frac{v(N)}{n-d} & \text{otherwise} \end{cases}$$

where d is the number of null players in the game. Then φ satisfies EFF, SYM, NPP.

Coefficients in the Shapley value form a probability distribution

$$p_i(A) = \frac{|A|!\,(n-|A|-1)!}{n!} \qquad \forall i \in N, \; \forall A \subseteq N \setminus i$$

Then
$$\sum_{A \subseteq N\setminus i} p_i(A) = \frac{1}{n!} \sum_{A \subseteq N \setminus i} |A|!\,(n-|A|-1)! =$$

$$= \frac{1}{n!} \sum_{k=0}^{n-1} \binom{n-1}{k} k!\,(n-k-1)!$$

$$= \frac{1}{n!} \sum_{k=0}^{n-1} \frac{(n-1)!\, \cancel{k!\,(n-k-1)!}}{\cancel{k!\,(n-k-1)!}} =$$

$$= \frac{1}{n!}(n-1)!\cdot n = \underline{\underline{1}}$$

The equality of 2 formulas for φ_i^S

$$\varphi_i(v) = \frac{1}{n!} \sum_{A \subseteq N \setminus i} \boxed{|A|!\,(n-|A|-1)!} \, \boxed{\big(v(A \cup i) - v(A)\big)}$$

$$= \frac{1}{n!} \sum_{A \subseteq N \setminus i} \; \sum_{\substack{\pi \in \Pi \\ A_i^\pi = A}} \boxed{\left[v(A_i^\pi \cup i) - v(A_i^\pi) \right]}$$

$$= \frac{1}{n!} \sum_{\pi \in \Pi} \underbrace{\sum_{\substack{A \subseteq N \setminus i \\ A_i^\pi = A}}}_{\text{only one summand!}} \left[v(A_i^\pi \cup i) - v(A_i^\pi) \right]$$

$$= \frac{1}{n!} \sum_{\pi \in \Pi} \left[v(A_i^\pi \cup i) - v(A_i^\pi) \right]$$