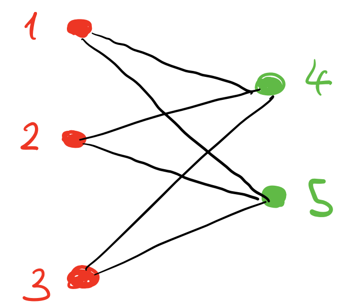
Example of a zero-sum polymatrix game

$N = \{\underbrace{1,2,3}_{\text{evaders}},\underbrace{4,5}_{\text{inspectors}}\}$



- evaders try to escape through exit points
- every inspector may block only the exit point in his authority while any evader may pick any exit

Let S_i $(i=4,5)$ be the disjoint sets of exit points in the authority of inspector i. Then every evader $i \in \{1,2,3\}$ has a strategy set

$$S_i = S_4 \cup S_5 .$$

Further, define payoffs for any evader $i \in \{1,2,3\}$ and any $j \in \{4,5\}$:

Evader: $$u_{ij}(s_i,s_j) = \begin{cases} 0 & s_i \notin S_j \\ 1-\delta(s_i,s_j) & s_i \in S_j \end{cases}$$

Inspector: $$u_{ji}(s_j,s_i) = \delta(s_j,s_i)$$

Then $u_i(s_1,\dots,s_5) = 1-\delta(s_i,s_{p(i)})$, where $p(i)$ is the unique index 4 or 5 s.t. $s_i \in S_{p(i)}$, and

$$u_j(s_1, \ldots, s_5) = \left| \{ i = 1,2,3 \mid s_i = s_j \} \right|$$

$= \#$ evaders caught by j.

The resulting game is polymatrix and constant-sum, since

$$\sum_{i=1}^{5} u_i(s_1, \ldots, s_5) = \# \text{ evaders caught} + \# \text{ evaders escaped} = 3$$

LP for a zero-sum polymatrix game

Let (p^*, w^*) be an optimal solution. Then $\max_{s_i} U_i(s_i, p^*_{-i}) \leq w_i \quad \forall i \in N$,

and

$$\sum_{i \in N} w_i \geq \sum_{i \in N} \max_{s_i} U_i(s_i, p^*_{-i})$$

$$= \max_{p \in \Delta} \sum_{i \in N} U_i(p_i, p^*_{-i})$$

$$\geq \sum_{i \in N} U_i(p^*) = 0.$$

Therefore, in order for a pair (p^*, w^*) to be optimal, necessarily

$$w_i^* = \max_{s_i} U_i(s_i, p^*_{-i}) = U_i(p^*)$$

for every $i \in N$. The last equality means that p^* is a NE.

Existence of pure NE for potential games

Let $P : S \to \mathbb{R}$ be the potential. We will show that any

$$s^* \in \arg\max_{s \in S} P(s)$$

is a NE. Indeed, we get

$$P(s^*) \geq P(s_i, s^*_{-i}) \quad \forall s_i$$

$$\Updownarrow$$

$$\underbrace{P(s^*) - P(s_i, s^*_{-i})}_{u_i(s^*) - u_i(s_i, s^*_{-i})} \geq 0$$

$$\Updownarrow$$

s^* is a NE

Examples of potential games

1) Bach or Stravinski

2,1	0,0
0,0	1,2

2	1
0	2

potential

2) Prisoners' dilemma

	g	s
g	-1,-1	-4,0
s	0,-4	-3,-3

0	1
1	2

potential

the number of players choosing strategy "s"

3) Congestion games

FP - failure of convergence

	d	e	f
a	0,0	1,0	0,1
b	0,1	0,0	1,0
c	1,0	0,1	0,0

The unique NE is (p_1^*, p_2^*), where each p_i^* is a uniform probability distribution.

However, the empirical distributions fail to converge when FP starts at (a, e).

BR dynamics – failure of convergence

$$\begin{array}{c|cc} & c & d \\ \hline a & 1 & -1 \\ b & -1 & 1 \end{array}$$

Starting from (a,c), the dynamics cycle

$$\begin{tikzcd} (a,c) \arrow[r] & (a,d) \arrow[d] \\ (b,c) \arrow[u] & (b,d) \arrow[l] \end{tikzcd}$$

The empirical frequencies correspond asymptotically to the unique NE of the Matching Pennies.

Note that this is no longer true in the game

1000	−1
−1	1

in which BR dynamics exhibit the same cycling behavior, but the NE is not uniform !

Congestion games

There is a road network and n drivers, where driver $i \in N$ wants to go from one node to another. Each driver chooses a path and incurs a cost due to the congestion on the path.

Notation:

$\pi = (\pi_1, \dots, \pi_n)$ ... paths selected by the players

R ... the set of roads

$n_r(\pi)$ ... # drivers using $r \in R$ (r is on some π_i)

$c_r(k)$ ... the cost of using $r \in R$ when there are k drivers

Cost function of player i is

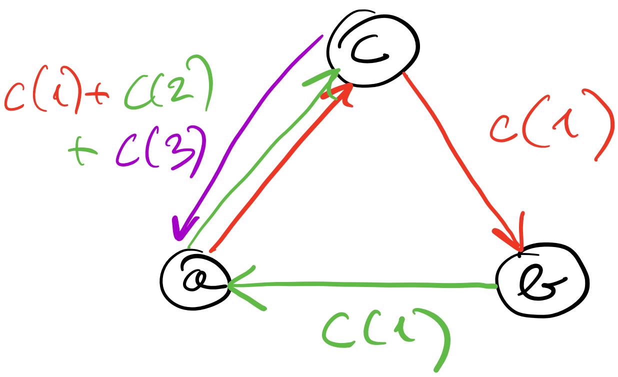
$$f_i(\pi) = \sum_{r \in \pi_i} c_r(n_r(\pi)).$$

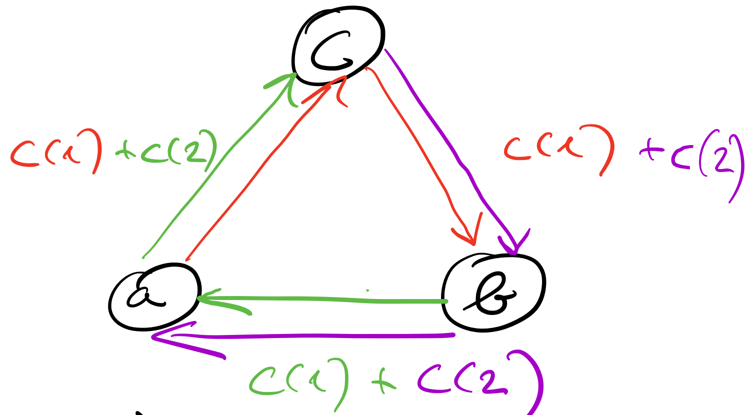
This is a potential game.

It can be checked by induction on N (adding one player at a time) that the potential is

$$P(\pi) = \sum_{r \in R} \sum_{\ell=1}^{n_r(\pi)} c_r(\ell).$$

Example :





$P(\pi) = 3c(1) + c(2) + c(3)$ $\quad$ $P(\pi') = 3c(1) + 3c(2)$

Note that

$$P(\pi') - P(\pi) = 2c(2) - c(3)$$
$$= f(\pi') - f(\pi)$$

where f is the cost function of player with the violet color.