ALTERNATIVES TO NASH EQUILIBRIUM

Tomáš Kroupa

AI Center
Department of Computer Science
Faculty of Electrical Engineering
Czech Technical University in Prague

MOTIVATION

1. Players' decisions may be *correlated*, unlike in Nash equilibrium, where players act independently
2. One player may *publicly disclose* their chosen strategy to the other players, unlike in a Nash Equilibrium setting

CORRELATED EQUILIBRIUM

CORRELATION OF ACTIONS

Bach or Stravinski

	B	S
B	2, 1	0, 0
S	0, 0	1, 2

- Pure NE: (B, B) and (S, S)
- Mixed NE: $p_1^*(B) = 2/3, p_2^*(S) = 2/3$ with utility $0.\overline{6}$ for each player

How to choose between (B, B) and (S, S)?

1. The mediator tosses a fair coin: $\quad$ *heads* $\Rightarrow (B, B)$, *tails* $\Rightarrow (S, S)$
2. Each player receives advice about what action to play
3. If both players accept the recommendation, the utility is 1.5

Extensive-form game with imperfect information $\Gamma(p)$

1. The mediator uses a probability distribution p over

$$\mathbf{S} = S_1 \times \cdots \times S_n$$

   to sample an action profile $\mathbf{s} \in \mathbf{S}$
2. The mediator tells player i only s_i
3. Every player i picks some $s_i' \in S_i$

NASH EQUILIBRIUM IN THE METAGAME

- A strategy of player i in $\Gamma(p)$ is a mapping

$$\sigma_i\colon S_i \to S_i$$

- Player i follows the suggestion of the mediator using strategy

$$\sigma_i^*(s_i) \coloneqq s_i$$

- Strategy profile $(\sigma_1^*, \ldots, \sigma_n^*)$ is a NE in $\Gamma(p)$ if no player has an incentive to deviate from their advice, assuming all others follow theirs

CORRELATED EQUILIBRIUM

Definition

A correlated equilibrium in a strategic game is a probability distribution p such that $(\sigma_1^*, \ldots, \sigma_n^*)$ is a NE in $\Gamma(p)$,

$$\sum_{\mathbf{s}_{-i} \in \mathbf{S}_{-i}} p(\mathbf{s}_{-i} \mid s_i) u_i(s_i', \mathbf{s}_{-i}) \quad \leq \quad \sum_{\mathbf{s}_{-i} \in \mathbf{S}_{-i}} p(\mathbf{s}_{-i} \mid s_i) u_i(s_i, \mathbf{s}_{-i}),$$

for every player i and every $s_i, s_i' \in S_i$ such that $p(s_i) > 0$.

CORRELATED EQUILIBRIUM, EQUIVALENTLY

Proposition

The following are equivalent for a probability distribution p over $\mathbf{S}$.

1. *p is a CE.*
2. *For each player i and all $s_i, s_i' \in S_i$,*

$$\sum_{\mathbf{s}_{-i} \in \mathbf{S}_{-i}} p(s_i, \mathbf{s}_{-i}) u_i(s_i', \mathbf{s}_{-i}) \quad \leq \quad \sum_{\mathbf{s}_{-i} \in \mathbf{S}_{-i}} p(s_i, \mathbf{s}_{-i}) u_i(s_i, \mathbf{s}_{-i}).$$

EXAMPLE

Bach or Stravinski

	B	S
B	$2, 1$	$0, 0$
S	$0, 0$	$1, 2$

$$p(B,S) \leq 2p(B,B)$$
$$2p(S,B) \leq p(S,S)$$
$$2p(S,B) \leq p(B,B)$$
$$p(B,S) \leq 2p(S,S)$$

Some solutions:

1. $p(B,B) = \alpha, \quad p(S,S) = 1-\alpha,$ for any $\alpha \in [0,1]$
2. $p(B,B) = 1$ NE
3. $p(S,S) = 1$ NE
4. $p(B,B) = p(S,S) = 2/9, \quad p(B,S) = 4/9, \quad p(S,B) = 1/9$ NE

EXISTENCE OF CE

Proposition

Any NE $(p_1^*, \dots, p_n^*)$ of a strategic game induces a CE p^* such that

$$p^*(\mathbf{s}) = \prod_{i \in N} p_i^*(s_i) \qquad \forall \mathbf{s} \in \mathbf{S}.$$

COMPUTATION OF CE

Maximize the social welfare

$$\sum_{i \in N} \sum_{\mathbf{s} \in \mathbf{S}} p(\mathbf{s}) u_i(\mathbf{s})$$

subject to the constraint that p is a CE.

Bach or Stravinski

	B	S
B	$2, 1$	$0, 0$
S	$0, 0$	$1, 2$

$$\text{Maximize} \quad 3p(B,B) + 3p(S,S)$$

$$p(B,S) \leq 2p(B,B)$$

$$2p(S,B) \leq p(S,S)$$

$$2p(S,B) \leq p(B,B)$$

$$p(B,S) \leq 2p(S,S)$$

Optimal solution: $p(B,B) = \alpha, \quad p(S,S) = 1 - \alpha, \quad$ for any $\alpha \in [0,1]$

COARSE CORRELATED EQUILIBRIUM

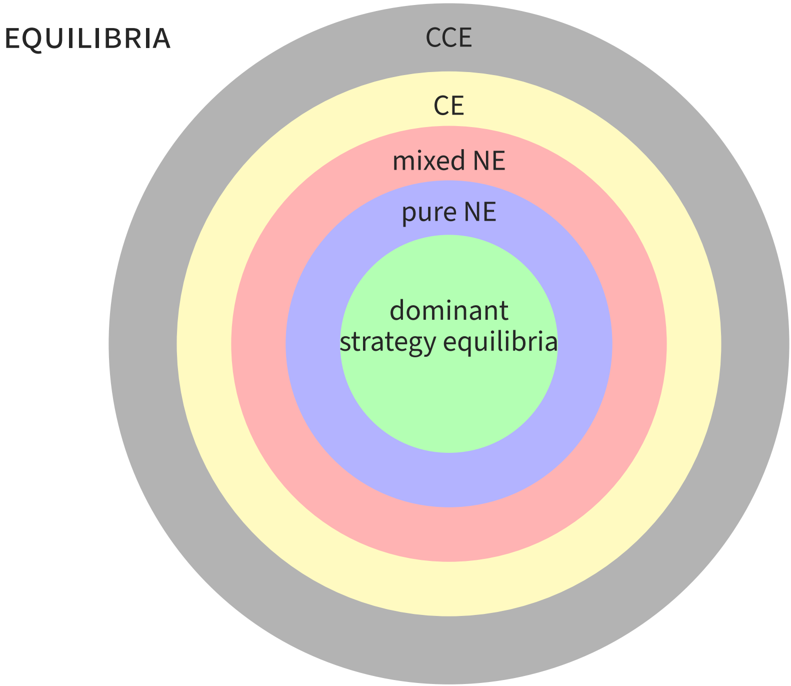
Is the player better off always following p, ignoring the mediator's advice?

Definition

A coarse CE in a strategic game is a probability distribution p such that

$$\sum_{\mathbf{s} \in \mathbf{S}} p(\mathbf{s}) u_i(s_i', \mathbf{s}_{-i}) \leq \sum_{\mathbf{s} \in \mathbf{S}} p(\mathbf{s}) u_i(\mathbf{s})$$

for every player i and every $s_i' \in S_i$.


EQUILIBRIA
CCE
CE
mixed NE
pure NE
dominant
strategy equilibria

CCE = NE FOR SOME GAMES

Matching pennies

	h	t
h	1	-1
t	-1	1

Every CCE induces a NE.

Prisoner's dilemma

	q	s
q	$-1, -1$	$-4, 0$
s	$0, -4$	$-3, -3$

Every CCE induces a NE.

Proposition

Every CCE p in a two-player zero-sum game induces a NE (p_1, p_2), where

$$p_1(s_1) := \sum_{t_2 \in S_2} p(s_1, t_2) \quad \textit{and} \quad p_2(s_2) := \sum_{t_1 \in S_1} p(t_1, s_2) \quad \forall s_1 \in S_1, s_2 \in S_2.$$

STACKELBERG EQUILIBRIUM

PUBLIC COMMITMENT TO AN ACTION

Example (Conitzer, 2006)

	c	d
a	$2, 1$	$4, 0$
b	$1, 0$	$3, 1$

- Strategy profile (a, c) is the only NE

The row player (leader) publicly commits to

- action b, the column player (follower) plays d and utility is $(3, 1)$
- mixed strategy $p_1(a) = p_1(b) = 1/2$, the follower is indifferent between $c \Rightarrow U_1(p_1, c) = 1.5$ and $d \Rightarrow U_1(p_1, d) = 3.5$

TWO-PLAYER STACKELBERG GAME

Player 1 (leader) and player 2 (follower) interact as follows:

1. The leader publicly commits to a mixed strategy $p_1 \in \Delta_1$
2. The follower then selects a pure strategy $s_2 \in \mathrm{BR}_2(p_1)$

Bilevel optimization

The leader maximizes $U_1(p_1, s_2)$ depending on $s_2 \in \mathrm{BR}_2(p_1)$, which is typically non-unique. We need a tie-breaking rule.

TIE-BREAKING RULES

1. If $|\mathrm{BR}_2(p_1)| = 1$ for every $p_1 \in \Delta_1$, the leader simply solves

$$\max_{p_1 \in \Delta_1} U_1(p_1, \mathrm{BR}_2(p_1))$$

2. Otherwise we assume that the follower breaks ties
   - to the disadvantage of the leader
   - in favor of the leader

WEAK AND STRONG STACKELBERG EQUILIBRIUM

The follower picks $s_2 \in \mathrm{BR}_2(p_1)$

1. to the disadvantage of the leader:

$$\max_{p_1 \in \Delta_1} \min_{s_2 \in \mathrm{BR}_2(p_1)} U_1(p_1, s_2)$$

2. in favor of the leader:

$$\max_{p_1 \in \Delta_1} \max_{s_2 \in \mathrm{BR}_2(p_1)} U_1(p_1, s_2)$$

Definition

1. Weak SE (p_1^*, s_2^*) is a solution to the 1st problem.
2. Strong SE (p_1^*, s_2^*) is a solution to the 2nd problem.

WEAK SE MAY NOT EXIST

Example

$$\begin{array}{c|c|c|} & c & d \\ \hline a & 2,1 & 4,0 \\ b & 1,0 & 3,1 \end{array}$$

$$p_1 \coloneqq p_1(a)$$

$$\mathrm{BR}_2(p_1) = \begin{cases} d & 0 \le p_1 < 1/2 \\ \{c, d\} & p_1 = 1/2 \\ c & 1/2 < p_1 \le 1 \end{cases}$$

1. Weak SE doesn't exist since there is no maximizer of function

$$p_1 \in [0,1] \quad \mapsto \min_{s_2 \in \mathrm{BR}_2(p_1)} U_1(p_1, s_2) = \begin{cases} p_1 + 3 & 0 \le p_1 < 1/2 \\ p_1 + 1 & 1/2 \le p \le 1 \end{cases}$$

2. Strong SE for the leader is $p_1^* = 1/2$

HOW TO COMPUTE STRONG SE?

$$\max_{p_1 \in \Delta_1} \max_{s_2 \in \mathrm{BR}_2(p_1)} U_1(p_1, s_2) = \max_{s_2 \in S_2} \max_{\substack{p_1 \in \Delta_1 \\ s_2 \in \mathrm{BR}_2(p_1)}} U_1(p_1, s_2)$$

Algorithm based on LP

- For each $s_2 \in S_2$ solve the LP:

$$\begin{aligned} \max \quad & U_1(p_1, s_2) \\ \text{subject to} \quad & U_2(p_1, s_2) \geq U_2(p_1, t_2) \qquad \forall t_2 \in S_2 \\ & p_1 \in \Delta_1 \end{aligned}$$

- Strong SE p_1^* is the optimal solution for an LP with the maximal value

SE IN TWO-PLAYER ZERO-SUM GAMES

Proposition
In any two-player zero-sum game, weak SE and strong SE coincide, and both are equal to the set of NE.

- In a two-player zero-sum game, whether a player publicly discloses their strategy or not is inconsequential
- This stands in stark contrast with general-sum games