Surname and name: ______________________________

Task	1	2	3	4	5	*Total*
Maximum	10	10	10	6	4	40
Points						

1. Consider a two-player game where the payoffs of the players are given by

	a	b	c
d	$3,3$	$5,1$	$2,4$
e	$1,5$	$4,4$	$3,2$
f	$4,2$	$2,3$	$1,1$

   (a) (2 pts) Does every player have a pure strategy which weakly or strongly dominates every other strategy?

   (b) (2 pts) Does the game have a Nash equilibrium in pure strategies?

   (c) (3 pts) Consider the mixed strategies $p(d) = p(f) = 0.5$ and $q(a) = q(c) = 0.5$. Is (p, q) a mixed Nash equilibrium?

   (d) (3 pts) Assume that every player uses a uniform distribution as a mixed strategy. Is such a pair a mixed Nash equilibrium?

Solution:

(a) No. It is checked easily by enumerating all strategies of each player. (b) No. Easy check by the enumeration of all 9 strategy profiles. (c) Yes. We get $U_1(p,q) = 5/2 = U_1(d,q) = U_1(f,q)$ while $U_1(e,q) = 2$, and $U_2(p,q) = 5/2 = U_2(p,a) = U_2(p,c)$ while $U_2(p,b) = 2$. (d) No. The counterexample is easy to find. Let now r_1 and r_2 be the two uniform distributions. Then $U_1(r_1, r_2) = 25/9$ but $U_1(d,q) = 10/3$.

2. (a) (2 pts) Is there an extensive-form game without chance moves that can be transformed into a normal-form game in such a way that some terminal rewards from the extensive-form game are missing? Either provide an example of such a game or explain why this cannot occur.

   (b) (5 pts) Consider a two-player zero-sum game with perfect recall and no chance moves. Suppose the longest trajectory in the game has length D, and the infoset with the highest number of available actions contains A actions. Describe how the game would look if it had the minimum and maximum possible number of infosets. Additionally, provide a formula for the number of histories, terminal histories, and infosets in such a game.

   (c) (3 pts) Consider a modified version of the Matching Pennies game, which remains zero-sum. First, Player 1 chooses the type of game: either they want to match the opponent's coin or not. Player 2 observes this choice and then selects either heads or tails. After that, Player 1 makes their move, choosing either heads or tails without observing Player 2's choice. Finally, based on the chosen game type and the players' decisions, a reward of +1 is given to either Player 1 or Player 2. What are the variables in the linear program used to find the Nash equilibrium for both Player 1 and Player 2?

Solution:

1. Yes, any game with absent-mindness.
2. The least amount of infosets is having all infosets (except the last one at depth D) with 1 action and the last one with A actions. The amount of histories is then $D+A$, where terminals are A. Each non-terminal history is it's own infoset, so there are D of them. The Maximal amount of infosets is in perfect information game, where each infoset has the A actions. The amount of histories is $|H| = 1 + A + A^2 \dots A^D = \sum_{i=0}^{D} A^i$, where the last term are terminal histories. Again each non-hsitory corresponds to a single infoset, this means there are $|I| = \sum_{i=0}^{D-1} A^i$ infosets.
3. Program for player 1: Variables: Realization plans for player 1 $r_1(\emptyset), r_1(M), r_1(N)$, $r_1(MH), r_1(MT), r_1(NH), r_1(NT)$. Expected value of opponent's infosets $v_2(M), v_2(N)$. Program for player 2: Variables: Realization plans for player 2 $r_2(\emptyset), r_2(MH), r_2(MT)$, $r_2(NH), r_2(NT)$. Expected value of opponent's infoset $v_1(\emptyset), v_1(M), v_1(N)$.

3. The decision (yes/no) about a future investment of a company depends on a weighted majority voting among 4 shareholders. Their shares are captured by the vector $\mathbf{w} = (40, 30, 20, 10)$. This means that the investment proposal is approved by any coalition of shareholders with the total share ≥ 51.
   (a) (3 pts) Model the described scenario as a simple voting game.
   (b) (3 pts) Compute the Banzhaf index.
   (c) (2 pts) Is this game supermodular?
   (d) (2 pts) Does the game have nonempty core?

Solution:
(a) $v(A) = 1$ if $\sum_{i \in A} w_i \geq 51$, and $v(A) = 0$ if $\sum_{i \in A} w_i \leq 50$, for any coalition $A \subseteq N = \{1, 2, 3, 4\}$. (b) Banzhaf index is $(\frac{5}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8})$. (c) No. The counterexample is $1 = v(123) + v(1) \leq v(12) + v(13) = 2$. (d) Since the game has no veto players, the core is empty.

4. (6 pts) Find the unique mixed Bayesian Nash equilibrium of the following game with players a_1 and a_2. Player a_2 has two types a_2^l and a_2^r, and three actions L, M, and R. Player a_1 has only one type and two actions U and D. The probability of each type of a_2 is indicated above

the corresponding table.

a_2^l (75%)

a_1		L	M	R
	U	$0,1$	$4,0$	$4,0$
	D	$4,2$	$0,1$	$0,8$

a_2^r (25%)

	L	M	R
U	$8,0$	$0,4$	$4,3$
D	$0,3$	$8,2$	$3,1$

Solution:

1. a_1 — $\left[\frac{6}{7}, \frac{1}{7}\right]$; a_2 — $\left[\frac{1}{6}, 0, \frac{5}{6}\right]$, $[0, 1, 0]$

5. (4 pts) What is the result of the following VCG auction with three bidders and two items A and B? The table below indicates the bidder's valuations for each bundle of items. Decide who wins which items and what they will pay for them.

	v_i			
	A	B	AB	Payment
b_1	6	6	9	
b_2	7	3	10	
b_3	9	3	13	

Solution:

Bidder 1 wins item B and pays 4 for it, Bidder 3 wins item A and pays 7 for it.