**Surname and name**: ____________________

| Task | 1 | 2 | 3 | 4 | 5 | *Total* |
|---|---|---|---|---|---|---|
| Maximum | 10 | 10 | 10 | 5 | 5 | 40 |
| *Points* | | | | | | |

1. Consider a two-player zero-sum game where the payoff of the row player is given by the matrix

$$\begin{bmatrix} 2 & 1 & 0 \\ 2 & 0 & 3 \\ -1 & 3 & -3 \end{bmatrix}.$$

   (a) (2 pts) Write the formula for the expected payoff of the column player when the row player uses the uniform distribution on all the strategies and the column player plays a generic mixed strategy $q$.

   (b) (2 pts) Does this game have a pure Nash equilibrium? Why?

   (c) (3 pts) Verify that the mixed strategy for the row player $\left(0, \frac{2}{3}, \frac{1}{3}\right)$ and the mixed strategy for the column player $\left(\frac{1}{5}, \frac{3}{5}, \frac{1}{5}\right)$ form a Nash equilibrium.

   (d) (3 pts) Write the linear programs for computing the maximin and minimax strategies.

**Solution:**

(a) The payoff matrix of the column player is

$$\begin{bmatrix} -2 & -1 & 0 \\ -2 & 0 & -3 \\ 1 & -3 & 3 \end{bmatrix}.$$

Therefore, expressing the vector of mixed strategies of the column player as $q = (q_1, q_2, q_3)$, we get the expected payoff

$$\tfrac{1}{3}(-3q_1 - 4q_2) = -q_1 - \tfrac{4}{3}q_2.$$

(b) No. It is easy to check that the matrix has no saddle point. (c) This is a straightforward verification using the definition of NE (minmax/maxmin strategies) for two-player zero-sum games. The value of the game is 1. (d) $\max v_1$ s.t. $2p_1 + 2p_2 - p_3 \geq v_1$, $p_1 + 3p_3 \geq v_1$, $3p_2 - 3p_3 \geq v_1$, $p_1 + p_2 + p_3 = 1$, $p_1, p_2, p_3 \geq 0$, $v_1 \in \mathbb{R}$, and $\min v_2$ s.t. $2q_1 + q_2 \leq v_2$, $2q_1 + 3q_3 \leq v_2$, $-q_1 + 3q_2 - 3q_3 \leq v_2$, $q_1 + q_2 + q_3 = 1$, $q_1, q_2, q_3 \geq 0$, $v_2 \in \mathbb{R}$.

2. (a) (2 pts) Is it possible to transform a simultaneous-move game—where all players choose their actions at the same time and the game may last for multiple rounds, such as Repeated Rock-Paper-Scissors or Matching Pennies—into an equivalent extensive-form game represented as a tree where players make decisions sequentially? If yes, describe the method for constructing such a transformation. If not, provide a clear explanation of why this transformation is not feasible.

(b) (5 pts) Imagine a two-player card game where both players start with $N$ cards each. In each turn a player secretly plays one of their cards and removes it from their hand without showing it to the other player. The game starts with Player 1, and after every turn, the player who acts alternates. Players don't draw any new cards after the game begins. The game ends when neither player has cards left to play. What is the total number of possible histories, terminal histories and infosets for each player?

(c) (3 pts) In this two-player game, each player can choose one of the actions: 1, 2, or 3. Before the game begins, Nature randomly selects a number between 2 and 5, with each number equally likely. The players aim to choose actions such that the sum of their actions matches the number chosen by Nature. The game proceeds as follows: First, Player 1 observes the number chosen by Nature and selects an action. Next, Player 2 chooses their action, but Player 2 does not know the number chosen by Nature. They only have access to Player 1's chosen action. Write down all possible sequences of play in this game and list the pure strategies for Player 2.

**Solution:**

1. Yes, it may be transformed. Each round in the simultaneous move game will be split into $n$ rounds in extensive form, where $n$ is the number of players. In each extensive-form round, only a single player plays without revealing any information to the opponents. Because of that, the order of players when splitting the game does not matter.

2. Histories: The game starts with a single history and $N$ possible actions by player 1. After playing any action, the other player has in each history out of $N$ histories $N$ actions. In total, after both players play a single action, there will be $N^2$ histories, but players will have only $N-1$ actions. This repeats until neither player has any cards. The total amount is then $|H| = 1 + N + N^2 + N^2(N-1) + N^2(N-1)^2 + ... + N^2(N-1)^2 \cdots 2^2 \cdot 1^2 = 1 + \sum_{i=0}^{N-1} \prod_{j=0}^{i} (N-j)^2 + \prod_{j=0}^{i} (N-j) \cdot \prod_{j=0}^{i-1} (N-j)$. Terminal histories: The last term of the $|H|$, which is $|Z| = N^2(N-1)^2 \cdots 2^2 \cdot 1^2 = \prod_{j=0}^{N} (N-j)^2 = (N!)^2$. Infosets: Since neither player shows any information to the opponent, the amount of infosets is equal to the amount of action sequences for both players $|I| = 1 + N + N(N-1) + \cdots + N! = 1 + \sum_{i=0}^{N} \prod_{j=0}^{i} (N-j)$

3. $\Sigma_2 = \{\emptyset\} \cup \{A_i A_j, \text{ where } i, j \in 1, 2, 3\}$
   $\mathcal{S}_2 = \{A_1 A_i A_2 A_j A_3 A_k, \text{ where } i, j, k \in 1, 2, 3\}$

3. The Parliament of a country consists of 100 seats. Each Member of Parliament (MP) belongs to one of two categories: *civil* or *military*. Following the recent general elections, four political parties secured representation in Parliament:

- *Red Party*: 11 MPs (5 military),
- *Green Party*: 17 MPs (0 military),
- *Blue Party*: 42 MPs (15 military),
- *White Party*: 30 MPs (20 military).

A proposed law is approved if it receives a simple majority of all MPs (at least 51 votes) and a simple majority of military MPs (at least 21 votes).

(a) (2 pts) Represent this scenario as a vector weighted voting game.

(b) (3 pts) Can the game be represented as a weighted voting game?

(c) (2 pts) Compute the Shapley-Shubik index for each party.

(d) (2 pts) Compute the Banzhaf index for each party.

(e) (1 pts) Is there a veto party in the game?

**Solution:**

(a) $\mathbf{w}^1 = (11, 17, 42, 30)$, $q^1 = 51$ and $\mathbf{w}^2 = (5, 0, 15, 20)$, $q^2 = 21$. (b) Yes. For example, $\mathbf{w} = (1, 1, 2, 3)$ and $q = 5$. (c) Shapley-Shubik: $\frac{1}{24} \cdot (2, 2, 6, 14)$. (d) Banzhaf: $\frac{1}{8} \cdot (1, 1, 3, 5)$. (e) Yes. The White Party is a vetoer since the grand coalition would be losing without it.

4. (5 pts) Find the unique mixed Bayesian Nash equilibrium of this game and calculate the associated payoffs to each player type.

Consider the following Bayesian simultaneous-move game with players $a_1$ and $a_2$. Player $a_2$ has two types $a_2^l$ and $a_2^r$, and two actions $L$ and $R$. Player $a_1$ has only one type and actions $U$ and $D$. Both types of player $a_2$ are equally likely.

| | $a_2^l$ (50%) $L$ | $a_2^l$ (50%) $R$ | $a_2^r$ (50%) $L$ | $a_2^r$ (50%) $R$ |
|---|---|---|---|---|
| $a_1$ $U$ | $3, 3$ | $1, 4$ | $1, 3$ | $0, 4$ |
| $a_1$ $D$ | $1, 4$ | $3, 3$ | $1, 2$ | $1, 3$ |

**Solution:**

1. $a_1$: $\left[\frac{1}{2}, \frac{1}{2}\right]$; $a_2$: $\left[\frac{3}{4}, \frac{1}{4}\right]$, $[0, 1]$
2. $a_1$: $\frac{5}{4}$; $a_2$: $\frac{7}{2}$, $\frac{7}{2}$

5. (5 pts) Find a symmetric equilibrium in a first-price auction with 5 bidders whose independent private valuations are drawn from the power distribution $F(x) = x^2$, where $x \in [0, 1]$.

*Hints in case you need to re-derive some formulas: The cumulative distribution function of the k'th lowest value in a sample of m has the formula* $F_{km}(x) = \sum_{j=k}^{m} \binom{m}{j} F^j(x)(1 - F(x)^{m-j}$*; and the formula for integration by parts is* $\int_a^b u(x)v'(x) = [u(x)v(x)]_a^b - \int_a^b u'(x)v(x)\, dx$.

**Solution:**

The optimal strategy is $E[Y|Y < v]$ or just $v - \int_0^v \frac{F_Y(x)}{F_Y(v)}\, dx$. The CDF of the highest value among opponents is the probability that all of them are below some $x$, i.e. $F^4(x) = x^8$. Plugging it in, we get $v - \frac{1}{v^8} \int_0^v x^8\, dx$ which is $v - \frac{1}{v^8} [\frac{x^9}{9}]_0^v$ or $\frac{8v}{9}$.