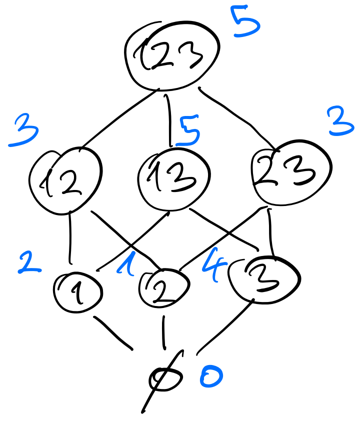
Opening example

Cost game : allocation is a vector $(x_1,x_2,x_3) \in \mathbb{R}^3$ such that $x_1+x_2+x_3 = 5$.



The cost allocations

$$(\underline{2},1,2) \text{ and } (\underline{2},0,3)$$

can be accepted.

However, $\frac{1}{3}(5,5,5)$ will not since player 2 objects.

Similarly, $(x_1,x_2,3+\varepsilon)$ is not stable allocation since coalition $\{2,3\}$ objects:

$c(\{2,3\}) = 3 < x_2+3+\varepsilon$, when $\varepsilon > 0$.

UN Security Council

Our goal is to quantify the voting power $\varphi_i(v)$ of each member i in this game. It is reasonable to assume that

$$\varphi_i(v) = \varphi_j(v)$$

whenever i,j are (non) permanent members.

Simple games and superadditivity (SA)

Let v be a simple game. The following are equivalent:

1) v is SA
2) If $v(A)=1$, then $v(N \setminus A)=0$ $\forall A \subseteq N$.

Proof

1) $\Rightarrow$ 2) Let v be SA and $v(A)=1$. Then

$$\underbrace{v(A)}_{1} + v(N\setminus A) \le v(N) = 1$$

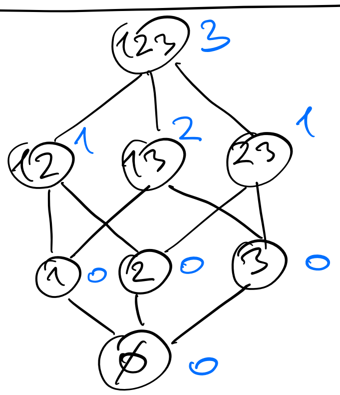
$$\Downarrow$$

$$v(N\setminus A) = 0.$$

2) $\Rightarrow$ 1) Let $A, B \subseteq N$ and $A \cap B = \emptyset$. We want to prove $v(A) + v(B) \le v(A \cup B)$. If $v(A) = v(B) = 0$, then this is trivial. If $v(A) = 1$, then $v(B) \le v(N \setminus A) = 0$, therefore $v(B) = 0$ and $v(A \cup B) = 1$.

Example (core)

v



Core: $x_1, x_2, x_3 \ge 0$

$$\begin{aligned}
x_1 + x_2 &\ge 1 \\
x_2 + x_3 &\ge 1 \\
x_1 + x_3 &\ge 2 \\
x_1 + x_2 + x_3 &= 3
\end{aligned}$$

This is a trapezoid with vertices $(0,1,2)$, $(1,0,2)$, $(2,0,1)$, $(2,1,0)$

The core of a simple majority voting

$C(v) = \emptyset$ since

$$\left.\begin{aligned}
x_1 + x_2 &\ge 1 \\
x_1 + x_3 &\ge 1 \\
x_2 + x_3 &\ge 1
\end{aligned}\right\} \oplus \Rightarrow 2\underbrace{(x_1 + x_2 + x_3)}_{1} \ge 3$$

contradiction!

In conclusion, there is no stable allocation. This is intuitively clear. Suppose for example that Player 1 and 2 form a coalition and distribute their worth uniformly,

$$x = \left(\tfrac{1}{2}, \tfrac{1}{2}, 0\right).$$

Then Player 3 may object and offer Player 1 to form a coalition while accepting the allocation

$$y = \left(\tfrac{1}{2} - \varepsilon, \tfrac{1}{2}, \varepsilon\right)$$

for some small $\varepsilon = 10^{-3}$. Now, any coalition may raise objections, and the process never stops.

LP to decide core nonemptiness

$$\begin{aligned} &\text{Minimize} && x_1 + \cdots + x_n \\ &\text{s.t.} && \sum_{i \in A} x_i \geq v(A) \qquad \forall A \subseteq N \end{aligned}$$

Let the minimizer be $x \in \mathbb{R}^n$ and the optimal value be $v(N)$, that is, $x(N) = v(N)$. Then necessarily $x \in C(v)$.

Conversely, let $C(v) \neq \emptyset$. Then the LP is feasible and any feasible point x satisfies $x(N) \geq v(N)$. Any point $x \in C(v)$ satisfies $x(N) = v(N)$, which is the optimal value.