DCGI

DEPARTMENT OF COMPUTER GRAPHICS AND INTERACTION

ARRANGEMENTS (uspořádání)

PETR FELKEL

FEL CTU PRAGUE

felkel@fel.cvut.cz

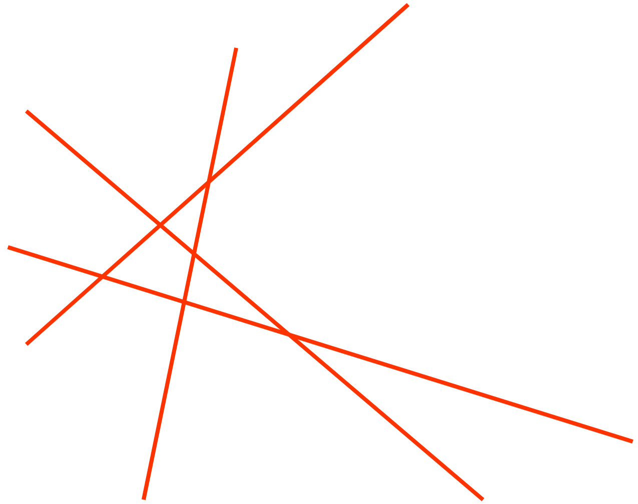
https://cw.felk.cvut.cz/doku.php/courses/a4m39vg/start

Based on [Berg], [Mount]

Version from 3.12.2020

Talk overview

- Arrangements of lines
  - Incremental construction
  - Topological plane sweep



- Duality – next lesson

Arrangements

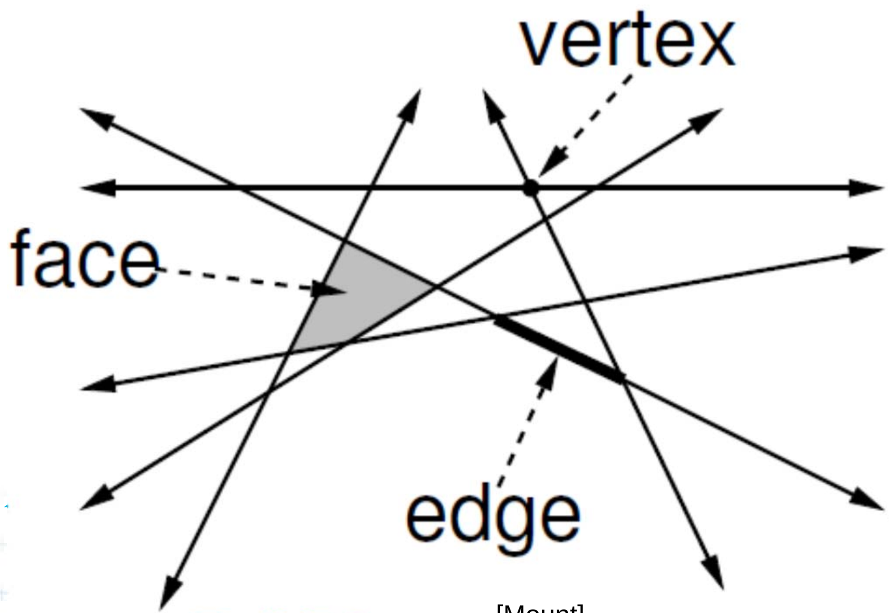
- The next most important structure in CG after CH, VD, and DT
- Possible in any dimension
  arrangement of (d-1)-dimensional hyperplanes
- We concentrate on arrangement of lines in plane
- Typical application: problems of point sets in dual plane (collinear points, point on circles, …)

Some more applications (see CGAL)

- Finding the minimum-area triangle defined by a set of points,
- computation of the sorted angular sequences of points,
- finding the ham-sandwich cut,
- planning the motion of a polygon translating among polygons in the plane,
- computing the offset polygon,
- constructing the farthest-point Voronoi diagram,
- coordinating the motion of two discs moving among obstacles in the plane,
- performing Boolean operations on curved polygons.

Line arrangement

- A finite set L of lines subdivides the plane into a cell complex, called arrangement $A(L)$
- In plane, arrangement defines a planar graph
  - – Vertices – intersections of (2 or more) lines
  - – Edges – intersection free segments (or rays or lines)
  - – Faces – convex regions containing no line (possibly unbounded)



[Mount]

Line arrangement

- Simple arrangement assumption

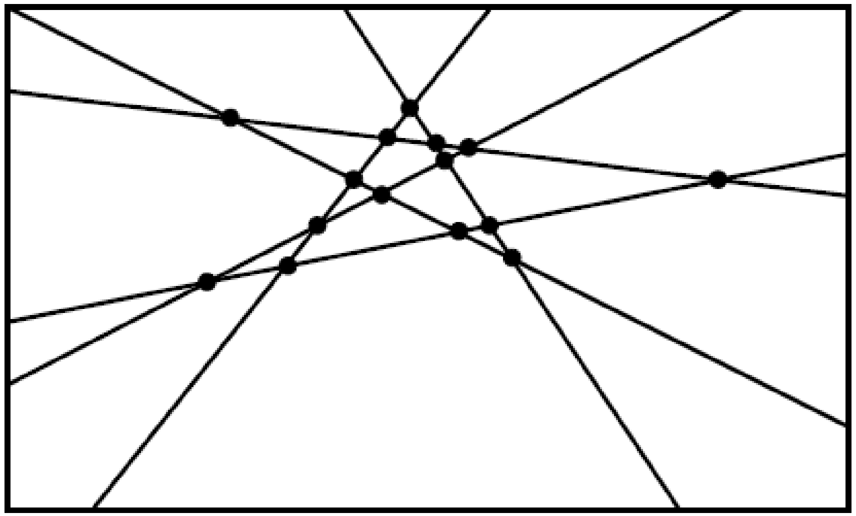
  = no three lines intersect in a single point

  - Can be solved by careful implementation or symbolic perturbation

DCGI

Line arrangement

- Formal problem: graph must have bounded edges
  - Topological fix: add vertex in infinity
  - Geometrical fix: BBOX, often enough as abstract with corners $\{-\infty, -\infty\}, \{\infty, \infty\}$



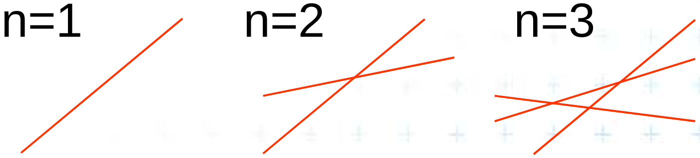
bounding box [Mount]

Combinatorial complexity of line arrangement

- $O(n^2)$
- Given n lines in general position, max numbers are
  - Vertices $\binom{n}{2} = \frac{n(n-1)}{2}$ $\rightarrow$ each line intersect $n-1$ others
  - Edges n^2 $\rightarrow$ $n-1$ intersections create n edges on each of n lines
  - Faces $\frac{n(n+1)}{2} + 1 = \binom{n}{2} + n + 1$

$$f_0 = 1 \qquad \text{(celá rovina)}$$
$$f_n = f_{n-1} + n$$

n=0 n=1 n=2 n=3



$f_0 = 1$ $f_1 = 2$ $f_2 = 4$ $f_3 = 7$

$$f_n = f_0 + \sum_{i=1}^{n} i = \frac{n(n+1)}{2} + 1$$

Construction of line arrangement

$$n^2 \log n^2 = 2n^2 \log n = O(n^2 \log n)$$

(0. Plane sweep method)

- $O(n^2 \log n)$ time and $O(n)$ storage plus $O(n^2)$ storage for the arrangement (n^2 vertices, edges, faces. $\log n^2$ - heap & BVS access)

A. Incremental method

- $O(n^2)$ time and $O(n^2)$ storage
- Optimal method

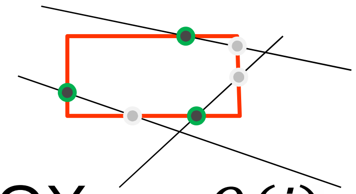
B. Topological plane sweep

- $O(n^2)$ time and $O(n)$ storage only
- Does not store the result arrangement
- Useful for applications that may throw out the arrangement after processing

DCGI

A. Incremental construction of arrangement

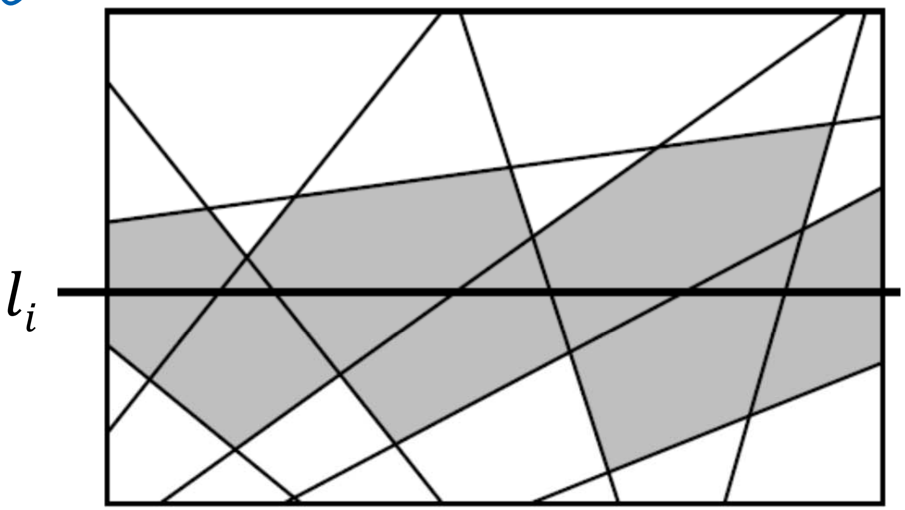
- $O(n^2)$ time, $O(n^2)$ space
  ~size of arrangement => it is an optimal algorithm
- Not randomized – depends on n only, not on order
- Add line l_i one by one $(i = 1 \dots n)$



  – Find the leftmost intersection with the BBOX among $2(i-1)+4$ edges already on the BBOX …$O(i)$
  – Trace the line through the arrangement $A(L_{i-1})$ and split the intersected faces …$O(i)$ – why? See later
  – Update the subdivision (cell split) …$O(1)$
- Altogether $O(ni) = O(n^2)$

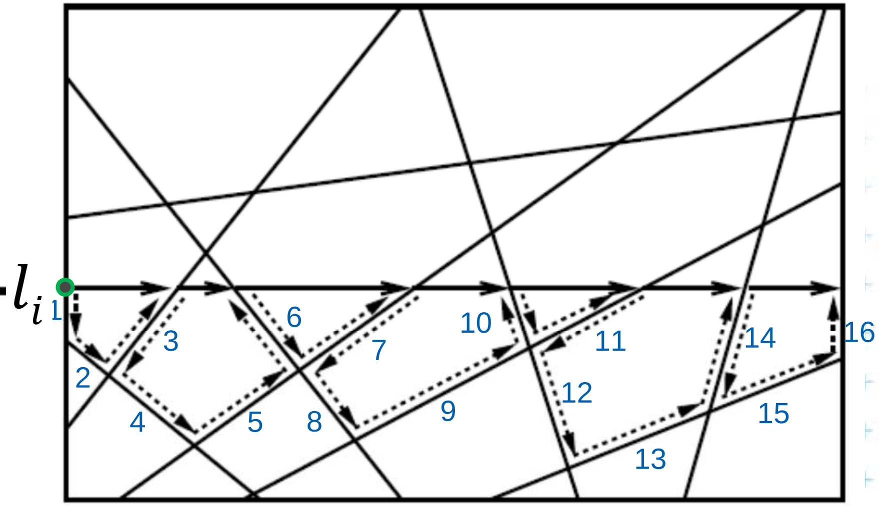
A. Tracing the line through the arrangement

- Walk around edges of current face (face walking)
- Determine if the line l_i intersects current edge e
- When intersection found, jump to the face on the other side of edge e



The zone of l_i

n=8 lines, 7 faces in the zone, 16 edges tested of max 48



Walking the lower part of the zone

[Berg]

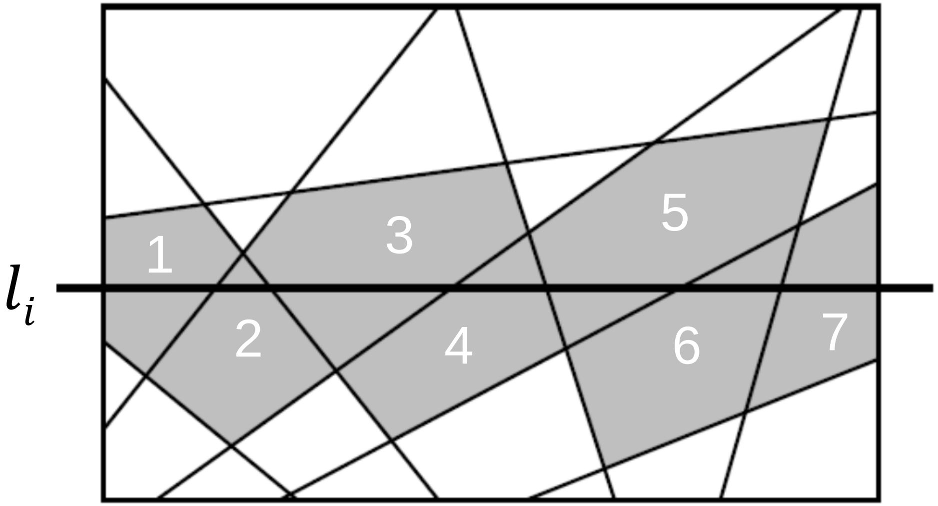
A. Incremental construction of arrangement

Arrangement(*L*)
Input: Set of lines *L* in general position (no 3 intersect in 1 common point)
Output: Line arrangement *A*(*L*) (resp. part of the arrangement stored in BBOX *B*(*L*) containing all the vertices of *A*(*L*))

1. Compute the BBOX *B*(*L*) containing all the vertices of *A*(*L*) …$O(n^2)$
2. Construct DCEL for the subdivision induced by BBOX *B*(*L*) …$O(1)$
3. **for** *i* = *1* **to** *n* ***do*** ***// insert line*** $\boldsymbol{l_i}$
4. find edge *e*, where line l_i intersects the BBOX of 2(*i*-*1*)+4 edges …$O(i)$
5. *f* = bounded face incident to the edge *e*
6. **while** *f* is in *B*(*L*) (bounded face f = f is in the BBOX) …$O(i)$
7. split *f* and set *f* to be the next intersected face across the intersected edge
8. update the DCEL (split the cell) …$O(1)$

See later…

The Zone of edge l_i



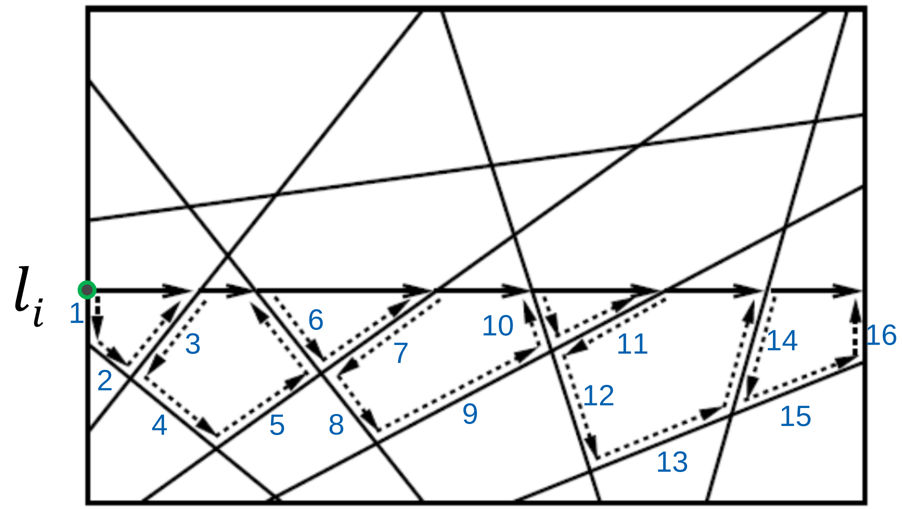
The zone of l_i for $i = 9$

Zone $Z_A(l_i)$ = set of i faces of $A(L)$ intersected by l_i

l_i crosses max $i - 1$ lines $\Rightarrow$ i faces

l_9: $i - 1 = 8$ lines, 7 of max 9 faces in the zone

Edges in the cells of the zone



Total number of edges in all zone faces

Naïve upper bound

edge l_i passes max i faces … $O(i)$

each face bounded by at most i lines

} $O(i^2)$????

Tight upper bound $6i = O(i)$

n=8 lines, 16 edges tested of max 48

Tracing the line through the arrangement

- Number of traversed edges determines the insertion complexity
- Naïve estimation would be $O(i^2)$ traversed edges (i faces, i lines per face, i^2 edges)
- According to the Zone theorem, it is $O(i)$ edges only!

Zone theorem

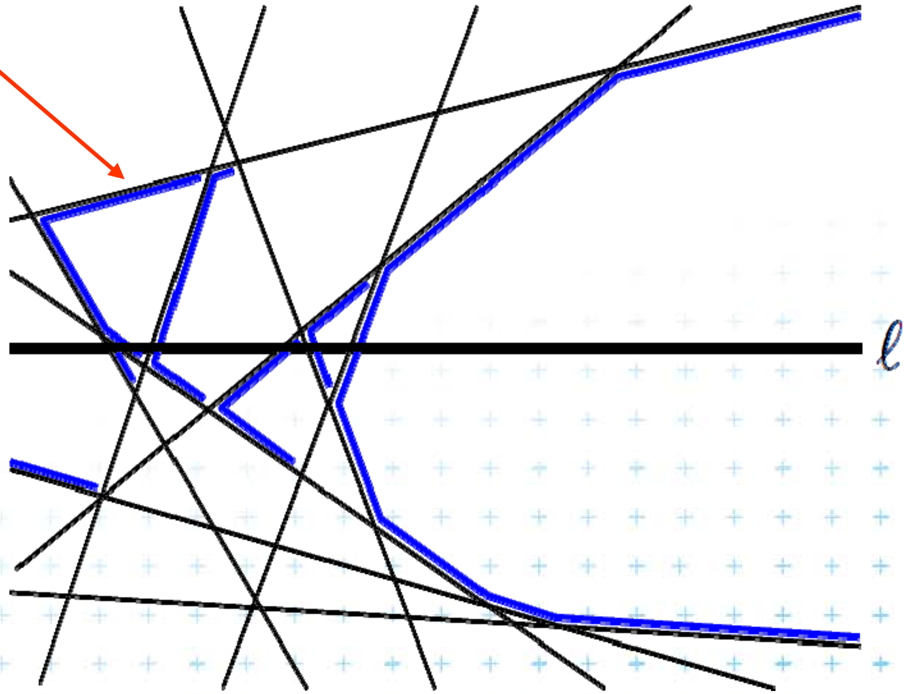
= given an arrangement $A(L)$ of n lines in the plane and given any line l in the plane, the total number of edges in all the cells of the zone $Z_A(L)$ is at most $6n$.

DCGI

Key idea of a proof

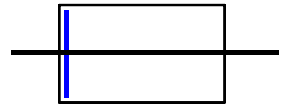
[Mount 2014, page 75]

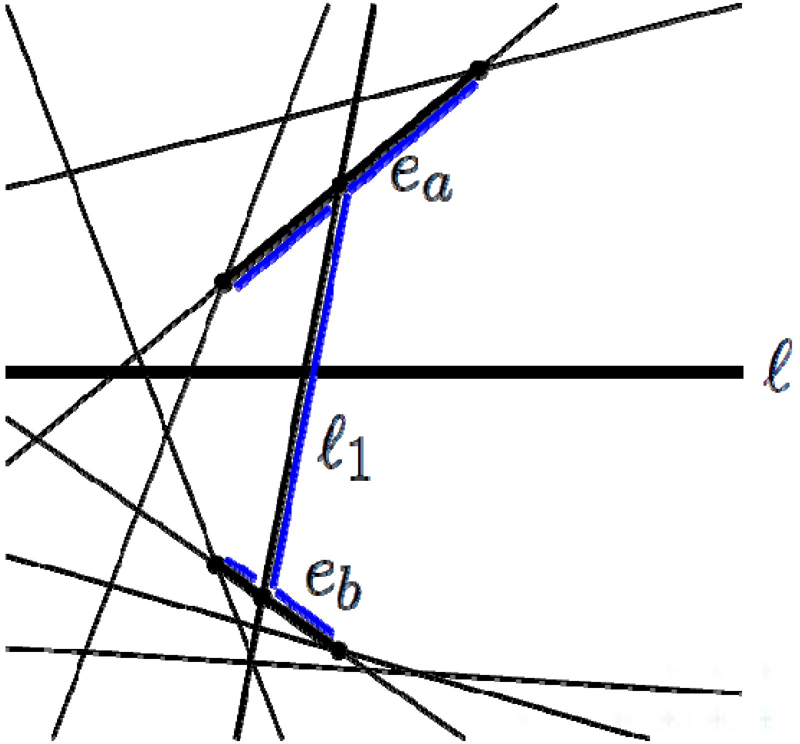
- Find a way to add up edges so that each line will induce a constant number of edges
- Split $6n$ edges of the zone into
  - $3n$ left bounding edges
  - $3n$ right bounding edges
  - $6n$ bounding edges total



The proof (left bounding edges) [Mount 2014, page 75]

$n = 1$, one left bounding edge, $1 \leq 3 = 3n$





True for $n - 1$ lines

$\Rightarrow$ holds for n lines

l_1= rightmost line intersecting l

Without l_1

$3(n-1)$ left bounding edges

Insert l_1

$+1$ left bounding edge l_1

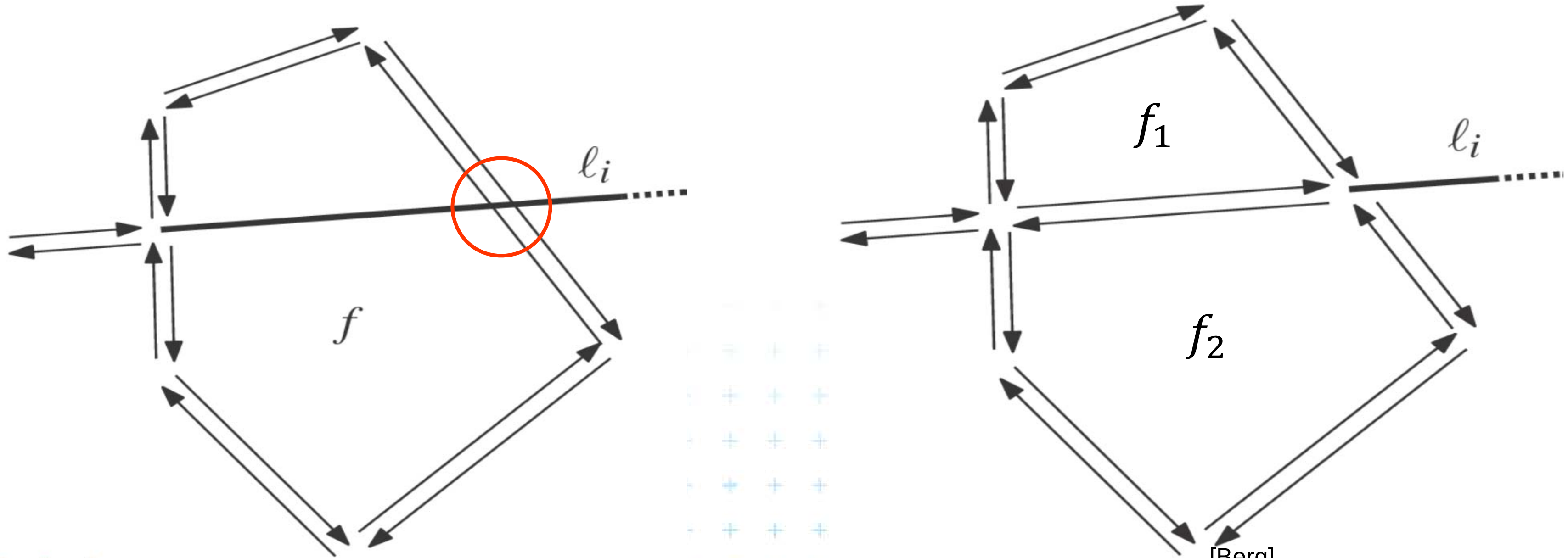
$+2$ split e_a and e_b

$3(n-1) + 3 = 3n \Rightarrow$ hold

or less if right bounding edges

Cell split in *O*(1)

- 1 new vertex
- 2 new face records, 1 face record (f) destroyed
- 3x2 new half-edges, 2 half-edges destroyed
- update pointers   … O(1)



[Berg]

Complexity of incremental algorithm

- n insertions
- $O(i) = O(n)$ time for one line insertion
  instead of $O(i^2)$
  (Zone theorem)

=> Complexity: $\underbrace{O(n^2)}_{\text{bbox}} + \underbrace{n\,O(i)}_{\text{edges walked}} = O(n^2)$

B. Topological plane sweep algorithm

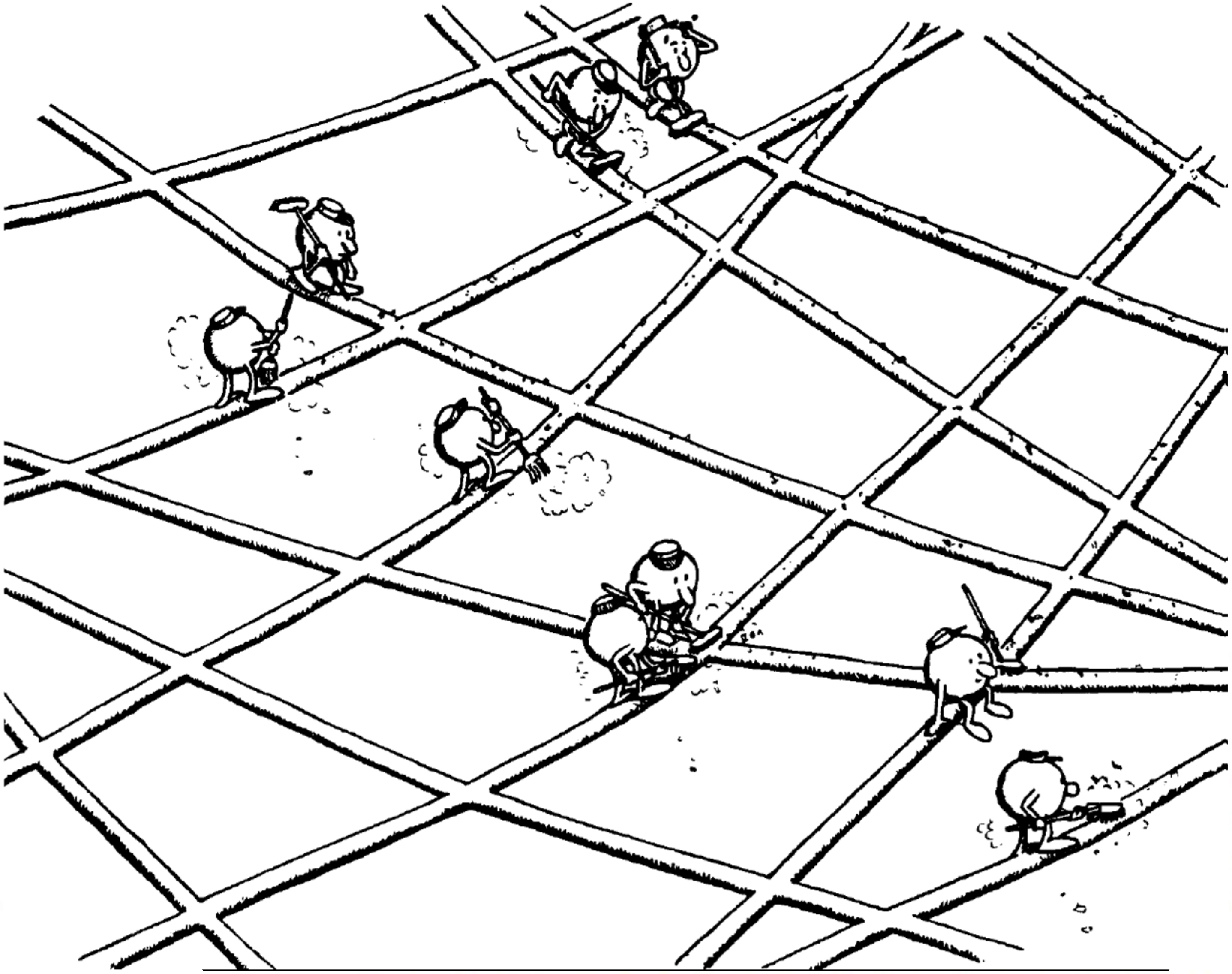
- Complete arrangement needs $O(n^2)$ storage
- Often we need just to process each arrangement element just once – and we can throw it out then
- Classical Sweep line algorithm (for arrangement of lines)
  - needs $O(n)$ storage
  - needs $\log n$ for heap manipulation in $O(n^2)$ event points

  => $O(n^2 \log n)$ algorithm
- Topological sweep line - TSL
  - no $O(\log n)$ factor in time complexity in $O(n^2)$ event points
  - array of n neighbors and a stack of ready vertices $O(1)$

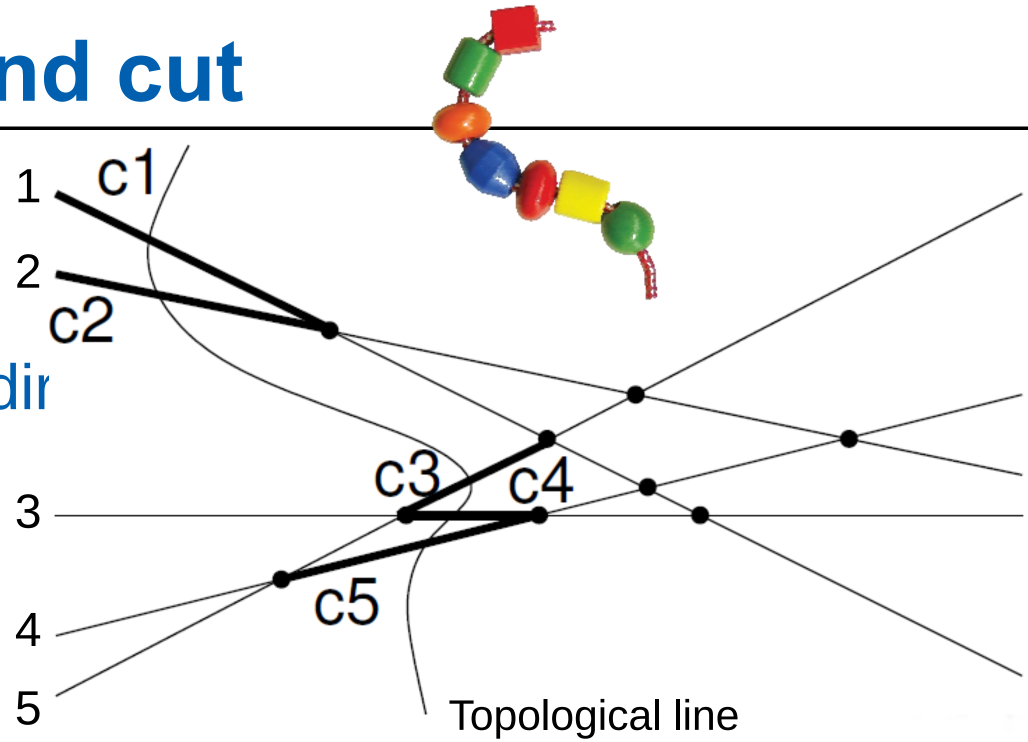
  => $O(n^2)$ algorithm

DCGI

Illustration from Edelsbrunner & Guibas



Topological line and cut



Topological line (curve)
(an intuitive notion)

- Monotonic curve in y-dir
- intersects each line exactly once (as a sweep line)

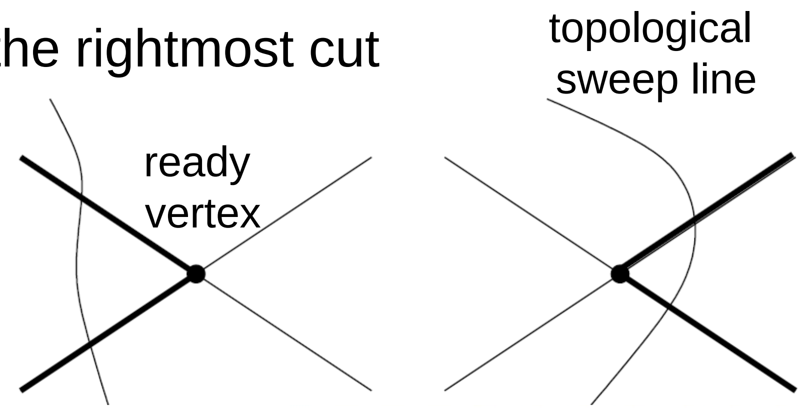
Cut in an arrangement A

- is an ordered sequence of edges $c_1, c_2, \dots, c_n$ in A (one taken from each line), such that for $1 \le i \le n-1$, c_i and c_{i+1} are incident to the same face of A and c_i is above and c_{i+1} below the face
- Edges in the cut are not necessarily connected (as c_2 and c_3)

www.nejbaby.cz

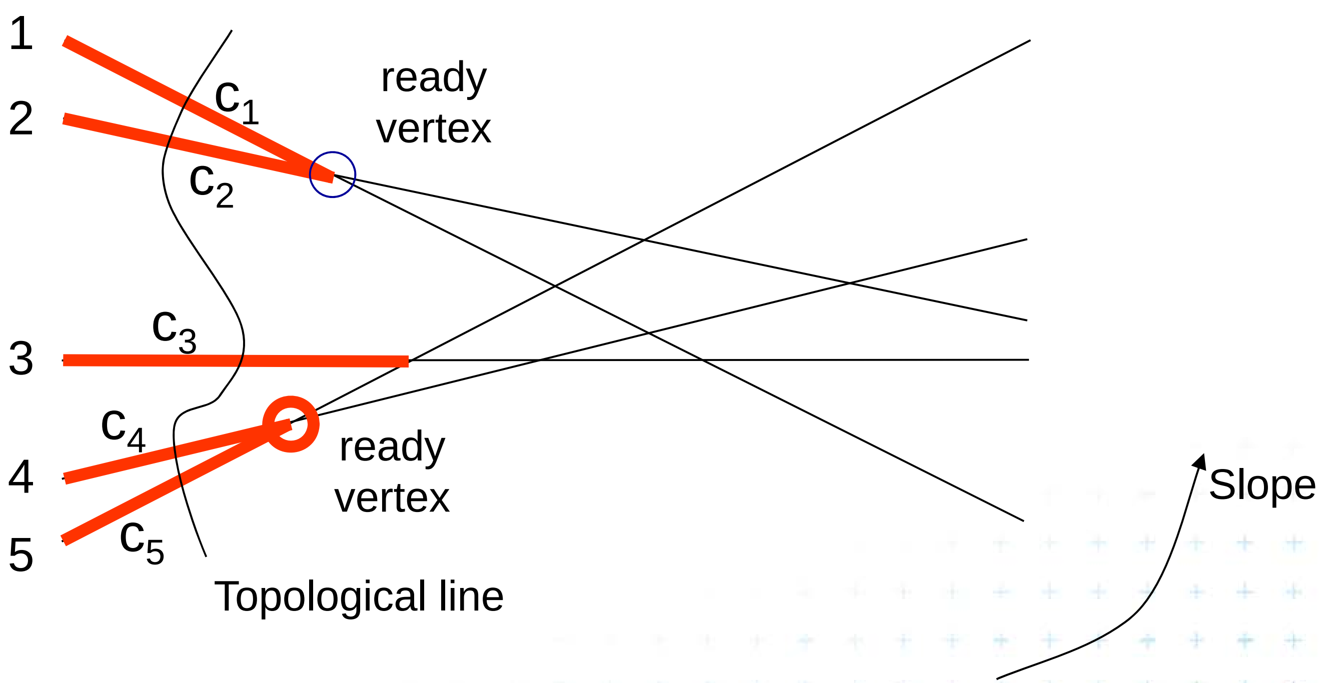
Topological plane sweep algorithm

- Starts at the leftmost cut
  - Consist of left-unbounded edges of A (ending at $-\infty$)
  - Computed in $O(n \log n)$ time – order of slopes
- The sweep line is
  - pushed from the leftmost cut to the rightmost cut
  - Advances in elementary steps



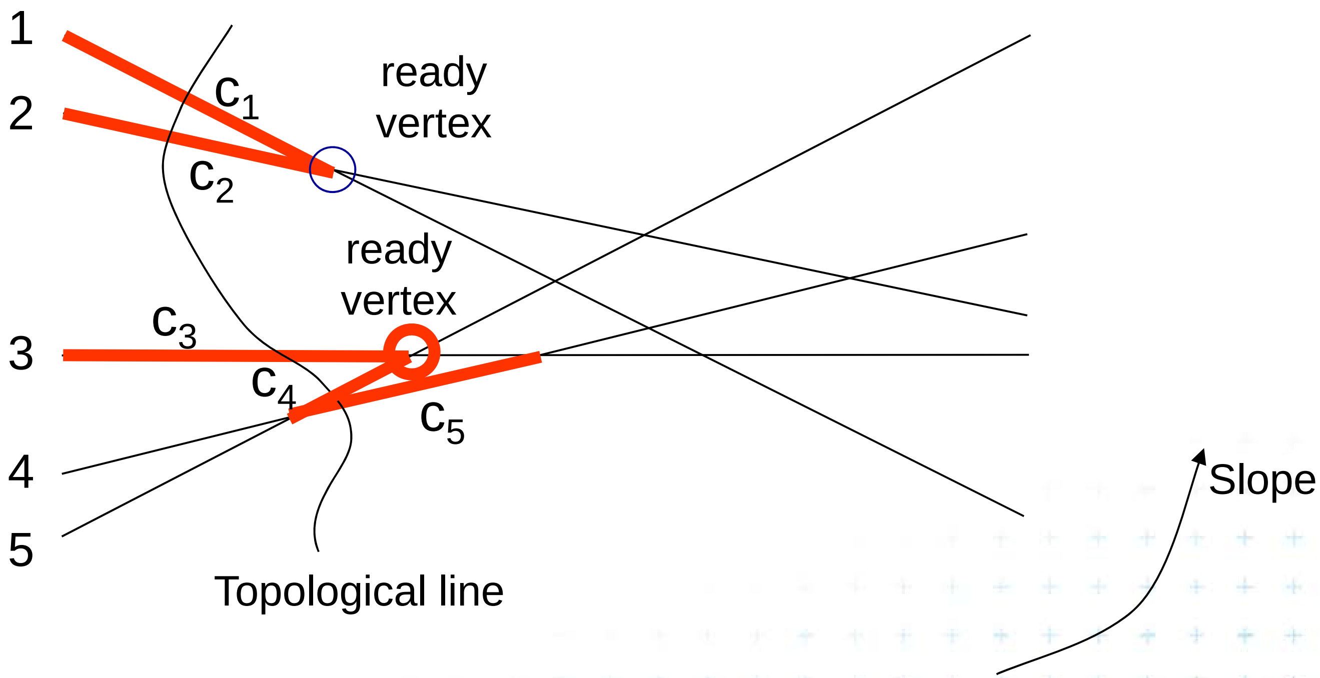
- Elementary step
  - = Processing of any *ready vertex*
    (intersection of consecutive edges at their right-point)
  - Swaps the order of lines along the sweep line
  - Is always possible (e.g., the point with smallest x)
  - Searching of smallest x would need $O(\log n)$ time …

Step 0 – the leftmost cut

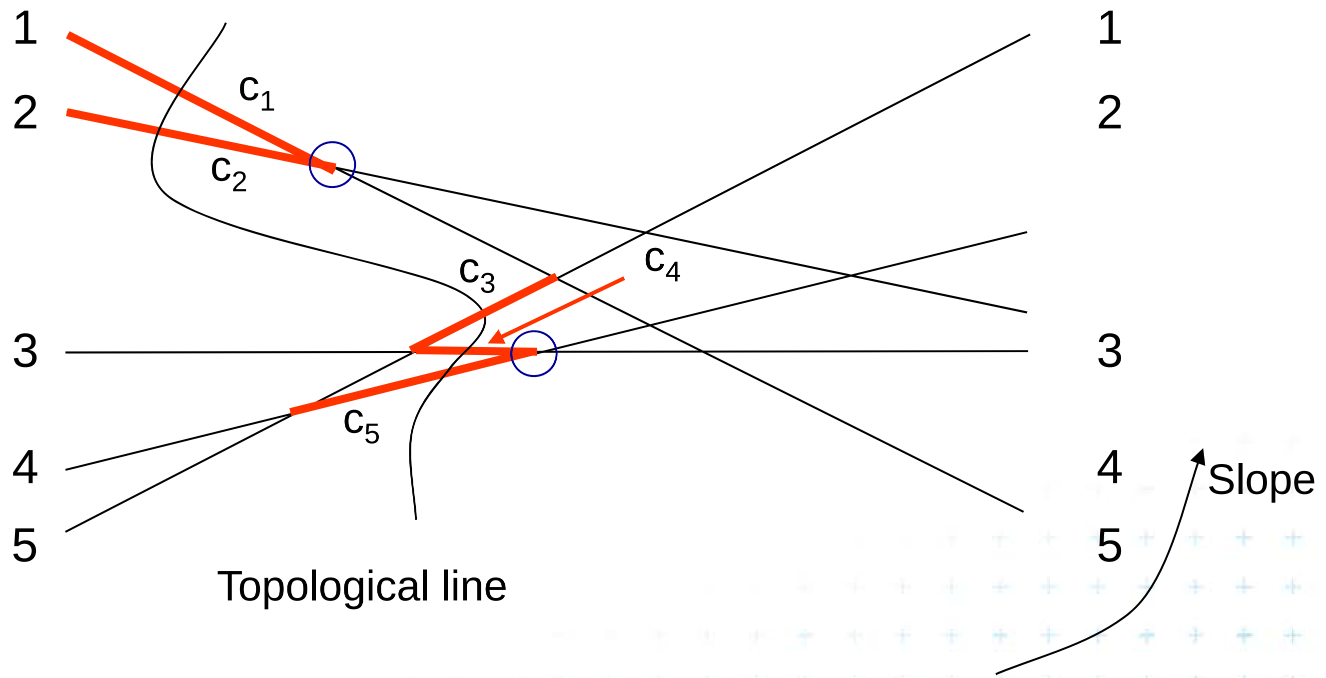


c_i = ordered sequence of edges along the topological sweep line

Step 1 – after processing of c4 x c5

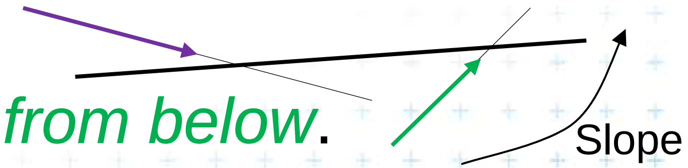


Step 2 – after processing of c3 x c4



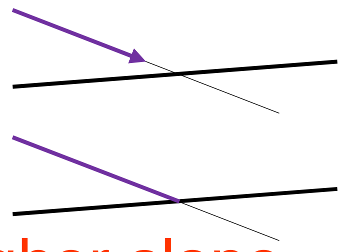
How to determine the next right point?

- Elementary step (intersection at edges right-point)
  - Is always possible (e.g., the point with smallest x)
  - But searching the smallest x would need $O(\log n)$ time
  - We need $O(1)$ time
- Right endpoint of the edge in the cut results from
  - UHT a line of *smaller slope* intersecting it *from above* (traced from L to R) or
  - LHT line of *larger slope* intersecting it *from below*.



- Use Upper and Lower Horizon Trees (UHT, LHT)
  - Common segments of UHT and LHT belong to the cut
  - Intersect the trees, find pairs of consecutive edges
  - use the <u>right points as legal steps (push to stack)</u>

Upper and lower horizon tree



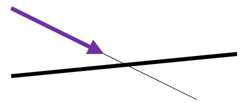
- Upper horizon tree (UHT)
  - Insert lines in order of decreasing slope (cw)
  - When two edges meet, keep the edge with higher slope and trim the inserted edge (with lower slope)
  - To get one tree and not the forest of trees (if not connected) add a vertical line in +∞ (slope +90°)
  - Left endpoints of the edges in the cut do not belong to the tree

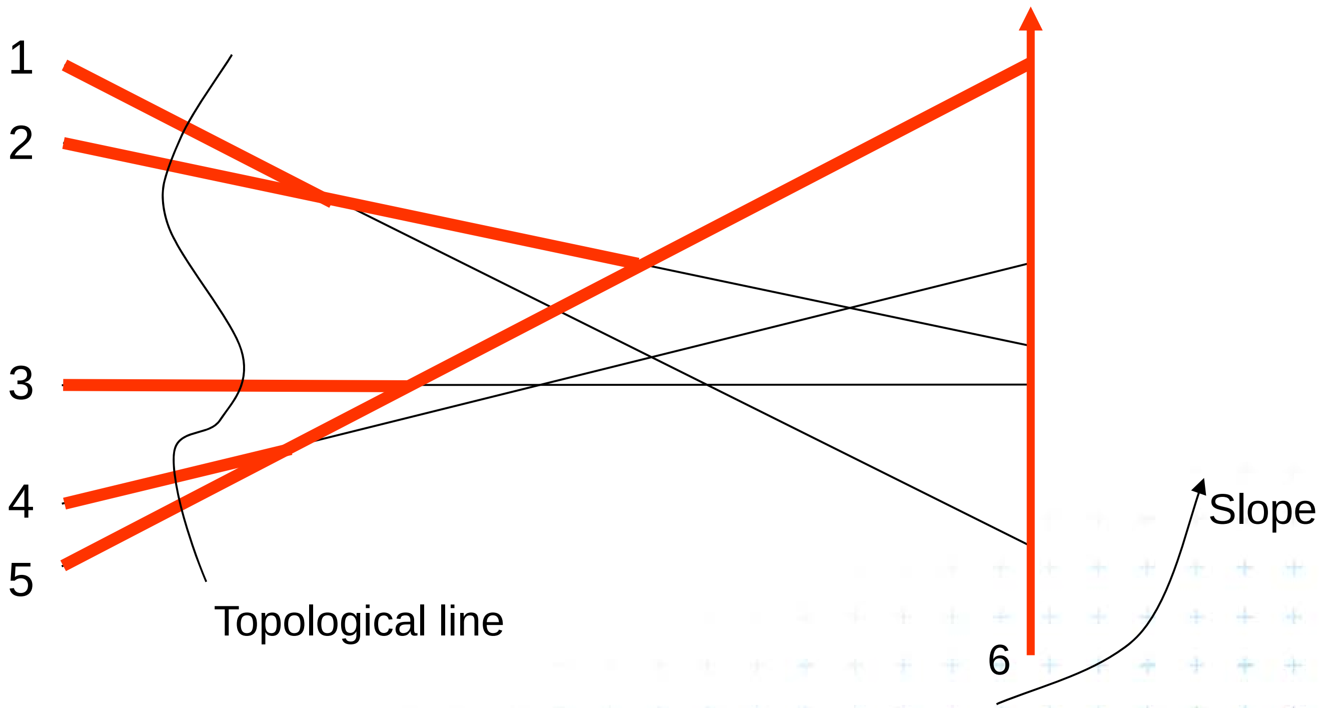
- Lower horizon tree (LHT) construction is symmetrical
- UHT and LHT serve for right endpts determination

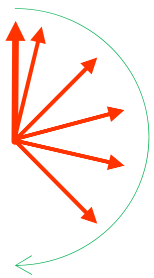
DCGI

Upper horizon tree (UHT) – initial tree

Insert lines in order of decreasing slope ("cw")



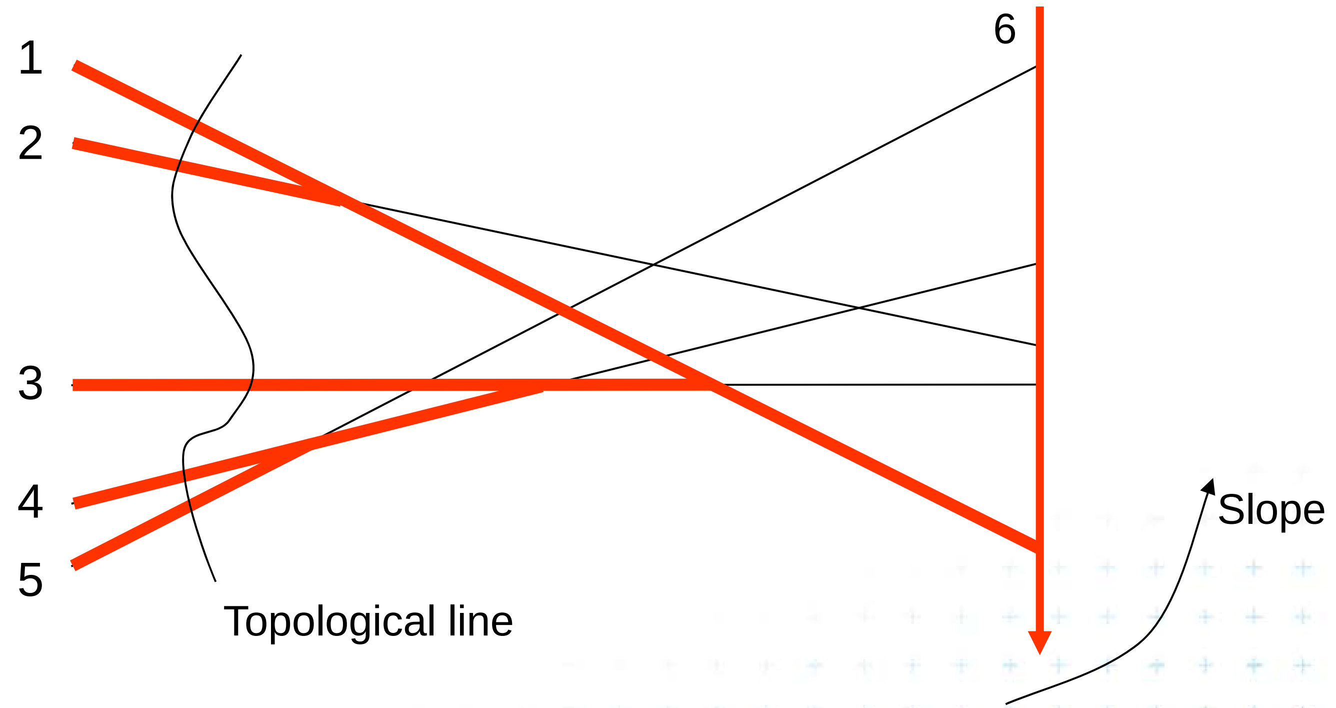


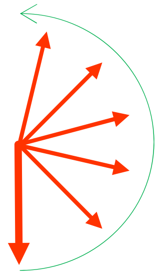


Insertion order: 6, 5, 4, 3, 2, 1

Lower horizon tree (LHT) – initial tree

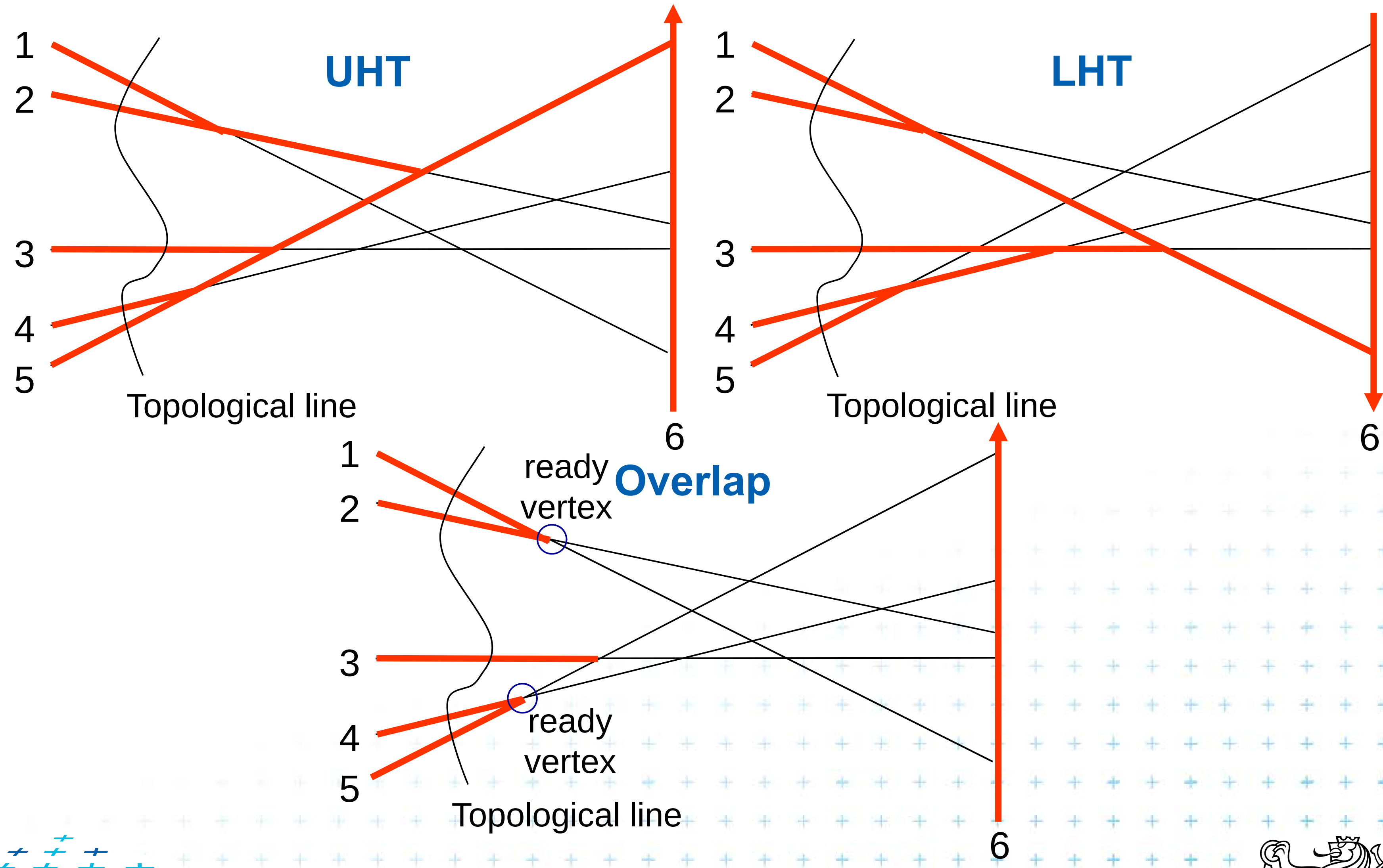
Insert lines in order of increasing slope ("ccw")





Insertion order: 6, 1, 2, 3, 4, 5

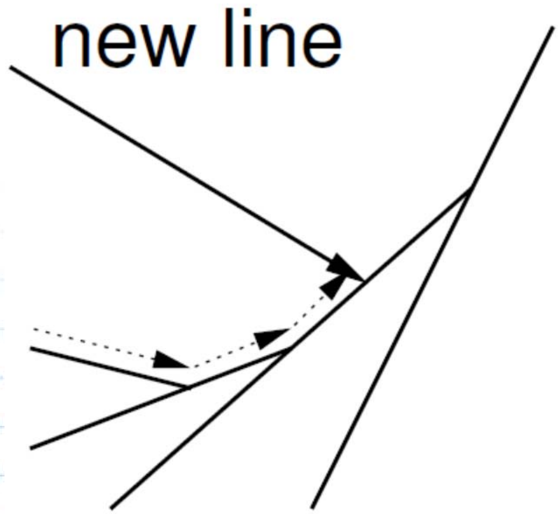
Overlap UHT and LHT – detect ready vertices



Upper horizon tree (UHT) – init. construction

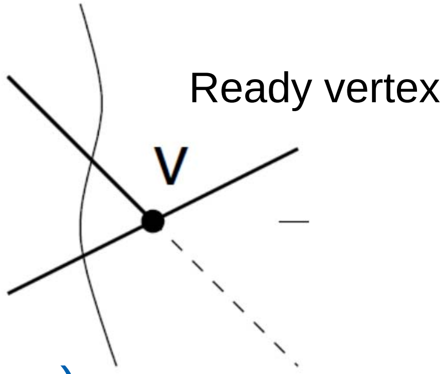
- Insert lines in order of decreasing slope (cw)
- Each new line starts above all the current lines
- The uppermost face = convex polygonal chain
- Walk left to right along the chain
  to determine the intersection
- Never walk twice over a segment
  - Such segment is no longer part of the upper chain
  - $O(n)$ segments in UHT

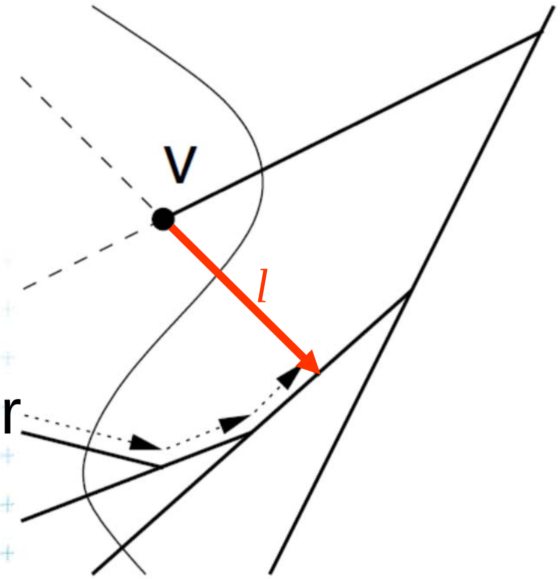
  => $O(n)$ initial construction
  (after $n \log n$ sorting of the lines ~slope)



Upper horizon tree (UHT) – update

- After the elementary step
- Two edges swap position along the sweep line
- Lower edge *l* (lower slope, comes from above)
  - Reenter to UHT
  - Terminate at nearest edge of UHT
  - Start in edge below in the current cut
  - Traverse the face in CCW order
  - Intersection must exist, as *l* has lower slope than the other edge from *v* and both belong to the same face





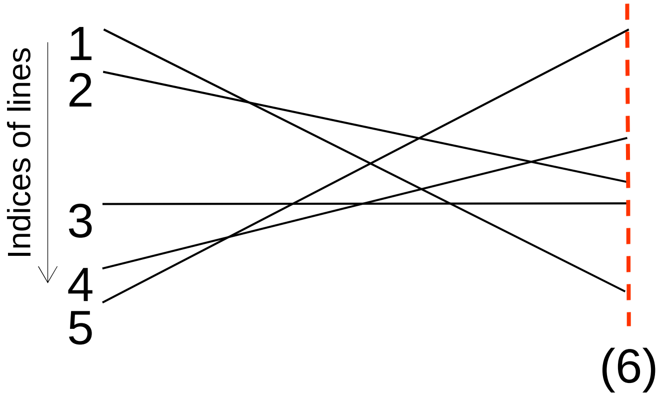
Data structures for topological sweep alg.

Topological sweep line algorithm uses 5 arrays:

1) Line equation coefficients – *E* [1:*n*]
2) Upper horizon tree – UHT [1:*n*]
3) Lower horizon tree – LHT [1:*n*]
4) Order of lines cut by the sweep line – C [1:*n*]
5) Edges along the sweep line – N [1:*n*]
6) Stack for ready vertices (events) – S

(*n* number of lines)

1) Line equation coefficients *E* [1:*n*]



- Array of line equation coefs. *E*
  - Contains coefficients a_i and b_i of line equations $y = a_i x + b_i$
  - *E* is indexed by the line index
  - Lines are ordered according to their slope (angle from -90° to 90°)

Slope

Array of line equations E

$y = a_i x + b_i$

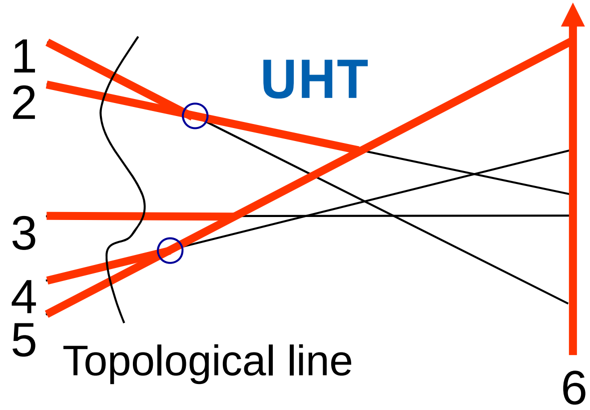
1	a_1	b_1
2	a_2	b_2
3	a_3	b_3
4	a_4	b_4
5	a_5	b_5

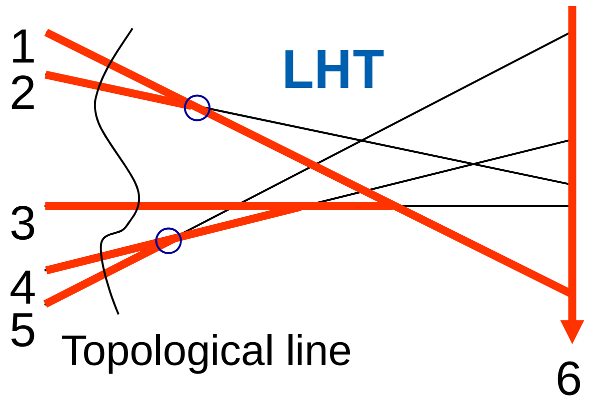
DCGI

2) and 3) – Horizon trees UHT and LHT

Their intersection is used for searching of legal steps (right points)

- the shorter edge wins





UHT array

Delimiting lines indices

1	$-\infty$	2
2	$-\infty$	5
3	$-\infty$	5
4	$-\infty$	5
5	$-\infty$	6
6	$-\infty$	$+\infty$

LHT array

Delimiting lines indices

1	$-\infty$	6
2	$-\infty$	1
3	$-\infty$	1
4	$-\infty$	3
5	$-\infty$	4
6	$+\infty$	$-\infty$

- Store pairs of line indices in E that delimit segment l_i to the left and to the right
- Segments are half open
- Unlimited line has "indices"
  $(-\infty, +\infty]$ $(+\infty, -\infty]$
- One additional vertical line
  – prevents the tree from splitting into forest of trees
  – is inserted first and never trimmed
  – is $(-\infty, +\infty]$ for UHT
  – is $(+\infty, -\infty]$ for LHT

4) Order of lines cut by sweep line – C [1:*n*]

- The topological sweep line cuts each line once
- Order of the cuts (along the topological sweep line) is stored in array C as a sequence of line indices
- Array C "points" to the array E of line equations
- For the initial leftmost cut, the order is the same as in E
- Index *ci* addresses *i-th* line from top along the sweep line

CUT Lines C

Indexes of supporting lines

c1	1
c2	2
c3	3
c4	4
c5	5

DCGI

5) Edges along the sweep line – N [1:*n*]

- Edges intersected by the topological sweep line are stored here (edges along the sweep line)
- Instead of endpoints themselves, we store the indices of lines whose intersections delimit the edge
- Order of these edges is the same as in C (both use the index *ci*)
- Index *ci* stores the index of *i-th* edge from top along the sweep line

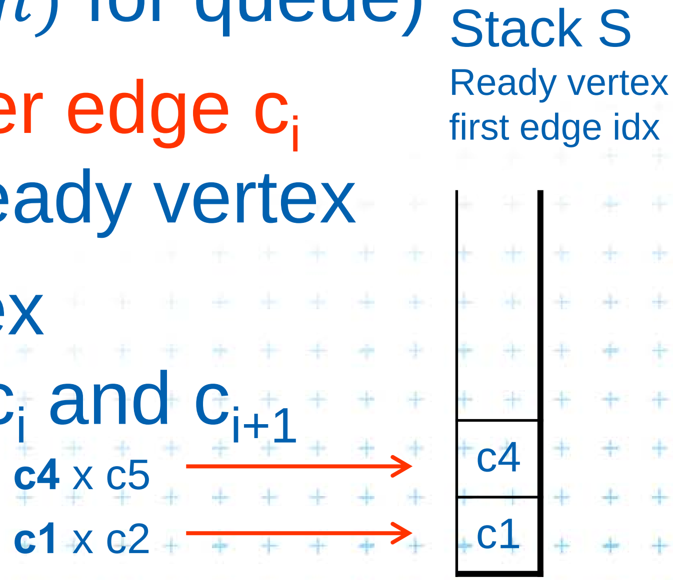
CUT edges N
Pairs of line indices delimiting the edge

c1	$-\infty$	**2**
c2	$-\infty$	**1**
c3	$-\infty$	**5**
c4	$-\infty$	**5**
c5	$-\infty$	**4**

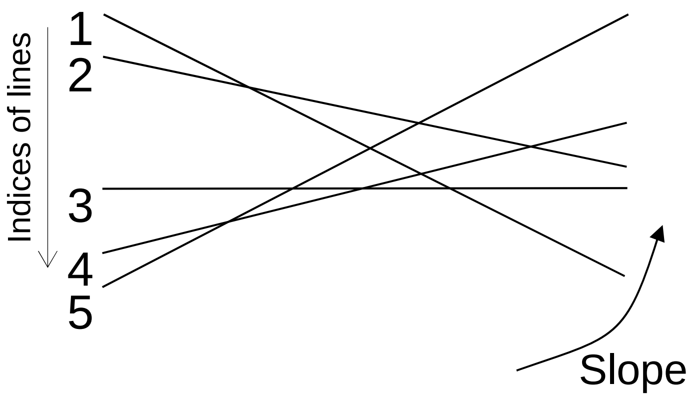
The first edge along the sweep line:
- lies on line C[c1]
- Comes from infinity
- is delimited by edge E[2]

6) Stack S

- The exact order of events is not important
  (event = intersection in ready vertex)
- Alg. can process any “ready vertex”
- Event queue is therefore replaced by a stack
  (faster: $O(1)$ instead of $O(\log n)$ for queue)
- The stack stores just the upper edge c_i
  from the pair intersecting in ready vertex
- Intersection in the ready vertex
  is computed between stored c_i and c_{i+1}



Topological sweep line demo



Array of line equations E

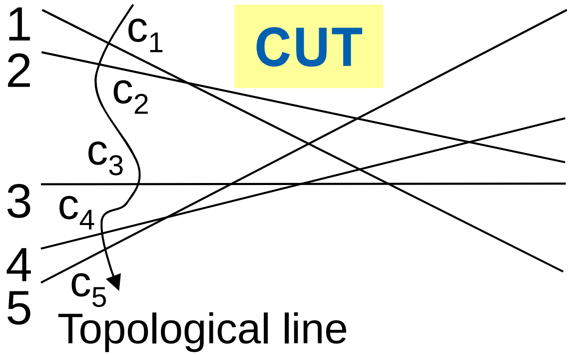
$y = a_i x + b$

Indices of lines		
1	a_1	b_1
2	a_2	b_2
3	a_3	b_3
4	a_4	b_4
5	a_5	b_5

Input

- set of lines *L* in the plane
- ordered in increasing slope (∠ -90° to 90°), simple, not vertical
- line parameters in array *E*

1) Initial leftmost cut - C



- Store the indices of lines in E into the Cut lines array C in increasing slope order

Array of line equations E

$y = a_i x + b$

Indices of lines

1	a_1	b_1
2	a_2	b_2
3	a_3	b_3
4	a_4	b_4
5	a_5	b_5

CUT Lines C

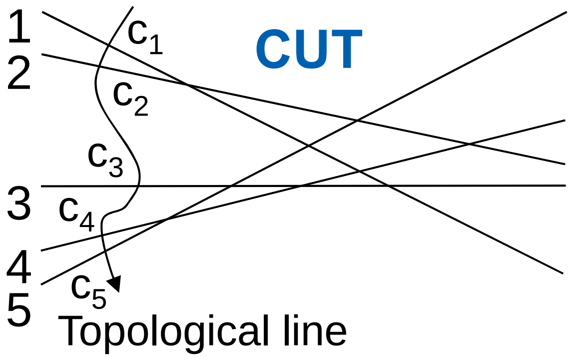
Indexes of sup-porting lines

Line indices along the cut

c1	1
c2	2
c3	3
c4	4
c5	5

DCGI

1) Initial leftmost cut - N



- Prepare array *N* for endpoints of the cut edges (resp. for line indices delimiting these edges)
- Init it by line “ends” $-\infty, +\infty$

Array of line equations E

$y = a_i x + b$

1	a_1	b_1
2	a_2	b_2
3	a_3	b_3
4	a_4	b_4
5	a_5	b_5

indices of lines

CUT edges N

Pairs of line indices delimiting the edge

c1	$-\infty$	∞
c2	$-\infty$	∞
c3	$-\infty$	∞
c4	$-\infty$	∞
c5	$-\infty$	∞

Index of delimiter edge in $-\infty$

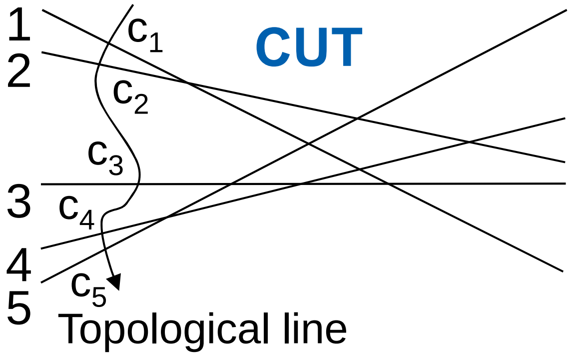
CUT Lines C

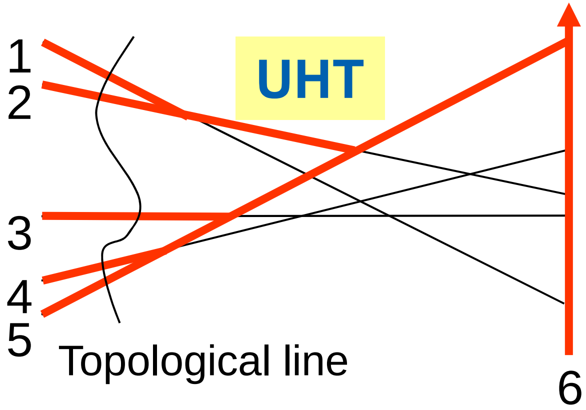
Indexes of sup-porting lines

c1	1
c2	2
c3	3
c4	4
c5	5

DCGI

2a) Compute Upper Horizon Tree - UHT





Array of line equations E

$y = a_i x + b$

1	a_1	b_1
2	a_2	b_2
3	a_3	b_3
4	a_4	b_4
5	a_5	b_5

UHT array

Delimiting lines indices

1	$-\infty$	2
2	$-\infty$	5
3	$-\infty$	5
4	$-\infty$	5
5	$-\infty$	6
6	$-\infty$	$+\infty$

Order of insertion into UHT

Additional "help edge"
Unlimited, bottom-up
Inserted first, never changed

CUT edges N

Pairs of line indices delimiting the edge

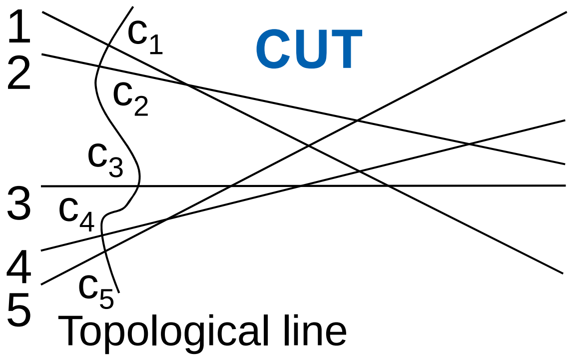
c1	$-\infty$	∞
c2	$-\infty$	∞
c3	$-\infty$	∞
c4	$-\infty$	∞
c5	$-\infty$	∞

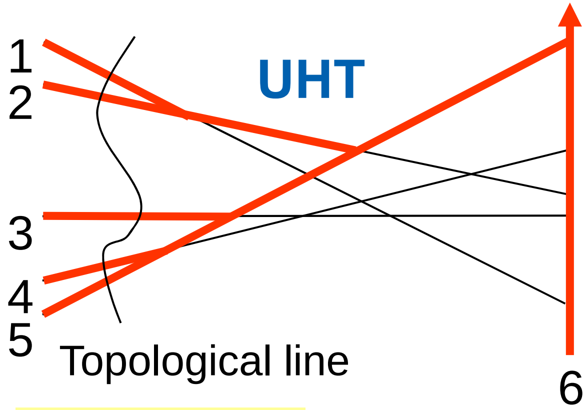
CUT Lines C

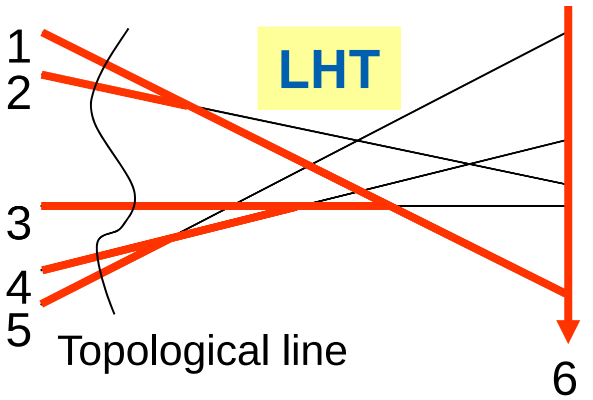
Indexes of supporting lines

c1	1
c2	2
c3	3
c4	4
c5	5

2b) Compute Lower Horizon Tree - LHT







Array of line equations E
$y = a_i x + b$

1	a_1	b_1
2	a_2	b_2
3	a_3	b_3
4	a_4	b_4
5	a_5	b_5

UHT array
Delimiting lines indices

1	$-\infty$	2
2	$-\infty$	5
3	$-\infty$	5
4	$-\infty$	5
5	$-\infty$	6
6	$-\infty$	$+\infty$

LHT array
Delimiting lines indices

1	$-\infty$	6
2	$-\infty$	1
3	$-\infty$	1
4	$-\infty$	3
5	$-\infty$	4
6	$+\infty$	$-\infty$

Order of insertion into LHT

Inserted first, never changed top to bottom

CUT edges N
Pairs of line indices delimiting the edge

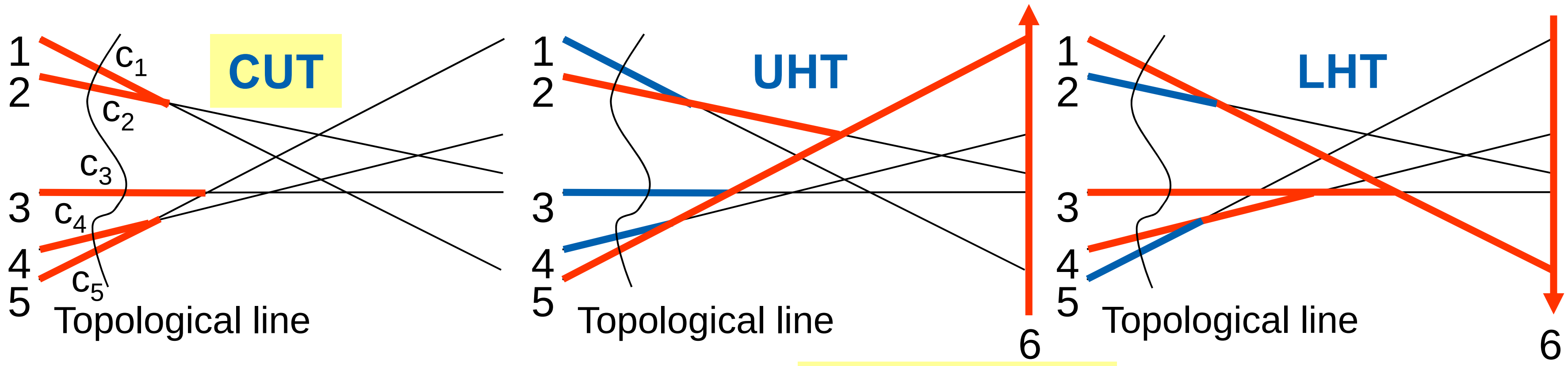
c1	$-\infty$	∞
c2	$-\infty$	∞
c3	$-\infty$	∞
c4	$-\infty$	∞
c5	$-\infty$	∞

CUT Lines C
Indexes of supporting lines

c1	1
c2	2
c3	3
c4	4
c5	5

Stack S
Ready vertex first edge idx

3a) Determine right delimiters of edges - N



Array of line equations E

$y = a_i x + b$

1	a_1	b_1
2	a_2	b_2
3	a_3	b_3
4	a_4	b_4
5	a_5	b_5

UHT array

Delimiting lines indices

1	$-\infty$	2
2	$-\infty$	5
3	$-\infty$	5
4	$-\infty$	5
5	$-\infty$	6
6	$-\infty$	$+\infty$

LHT array

Delimiting lines indices

1	$-\infty$	6
2	$-\infty$	1
3	$-\infty$	1
4	$-\infty$	3
5	$-\infty$	4
6	$+\infty$	$-\infty$

CUT edges N

Pairs of line indices delimiting the edge

c1	$-\infty$	**2**
c2	$-\infty$	**1**
c3	$-\infty$	**5**
c4	$-\infty$	**5**
c5	$-\infty$	**4**

CUT Lines C

Indexes of supporting lines

c1	1
c2	2
c3	3
c4	4
c5	5

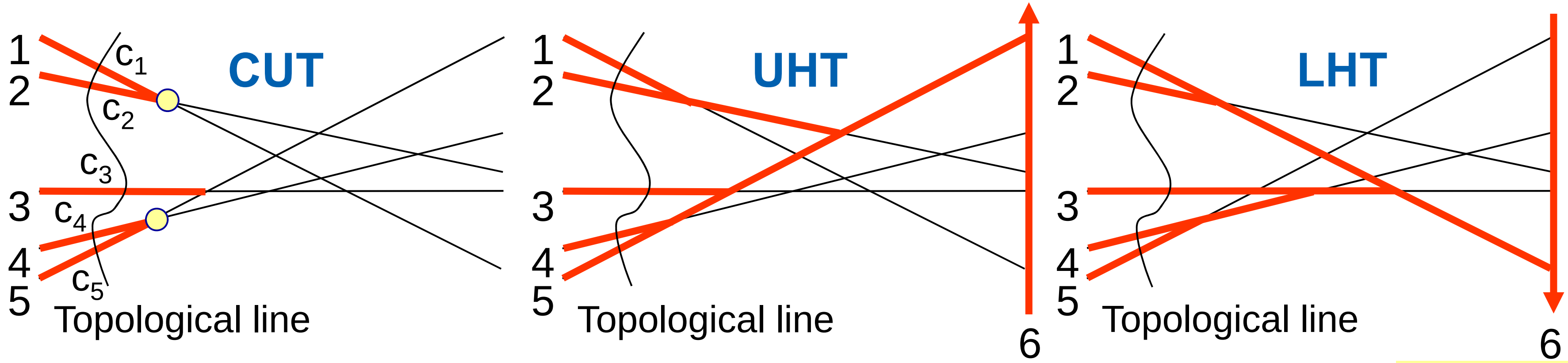
Stack S

Ready vertex first edge idx

Intersect the trees – take the shorter edge

DCGI

3b) Ready vertices = inters. of neighbors – S



Array of line equations E
$y = a_i x + b$

1	a_1	b_1
2	a_2	b_2
3	a_3	b_3
4	a_4	b_4
5	a_5	b_5

UHT array
Delimiting lines indices

1	$-\infty$	2
2	$-\infty$	5
3	$-\infty$	5
4	$-\infty$	5
5	$-\infty$	6
6	$-\infty$	$+\infty$

LHT array
Delimiting lines indices

1	$-\infty$	6
2	$-\infty$	1
3	$-\infty$	1
4	$-\infty$	3
5	$-\infty$	4
6	$+\infty$	$-\infty$

CUT edges N
Pairs of line indices delimiting the edge

c1	$-\infty$	**2**
c2	$-\infty$	**1**
c3	$-\infty$	**5**
c4	$-\infty$	**5**
c5	$-\infty$	**4**

CUT Lines C
Indexes of supporting lines

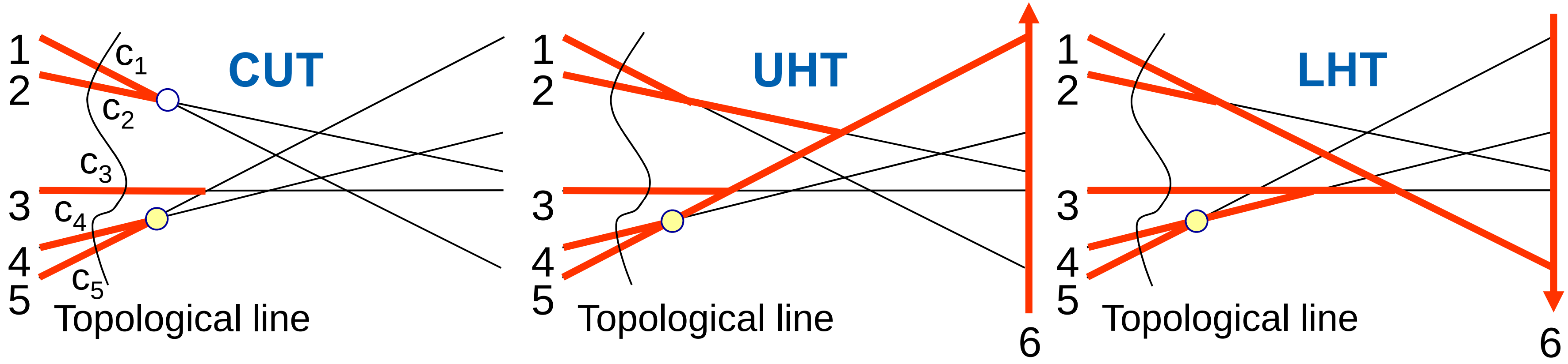
c1	1
c2	2
c3	3
c4	4
c5	5

Stack S
Ready vertex first edge idx

c4
c1

Compute intersections of neighbors – push into stack

4a) Pop ready vertex from S – process c4



Array of line equations E $y = a_i x + b$		
1	a_1	b_1
2	a_2	b_2
3	a_3	b_3
4	a_4	b_4
5	a_5	b_5

UHT array Delimiting lines indices		
1	$-\infty$	2
2	$-\infty$	5
3	$-\infty$	5
4	$-\infty$	5
5	$-\infty$	6
6	$-\infty$	$+\infty$

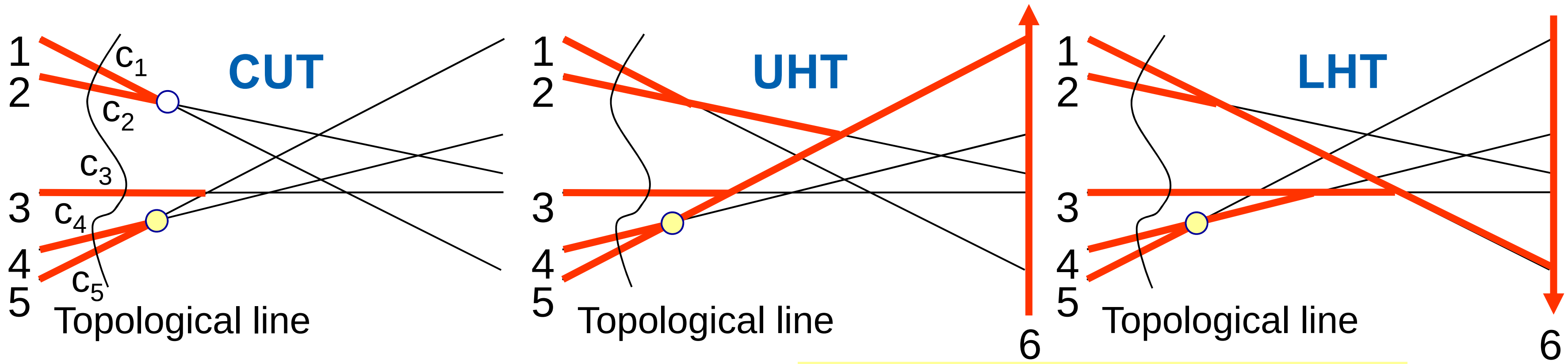
LHT array Delimiting lines indices		
1	$-\infty$	6
2	$-\infty$	1
3	$-\infty$	1
4	$-\infty$	3
5	$-\infty$	4
6	$+\infty$	$-\infty$

CUT edges N Pairs of line indices delimiting the edge		
c1	$-\infty$	2
c2	$-\infty$	1
c3	$-\infty$	5
c4	$-\infty$	5
c5	$-\infty$	4

CUT Lines C Indexes of sup-porting lines	
c1	1
c2	2
c3	3
c4	4
c5	5

Stack S Ready vertex first edge idx
c4
c1

4b) Swap lines c4 and c5 – swap 4 and 5



Array of line equations E
$y = a_i x + b$

1	a_1	b_1
2	a_2	b_2
3	a_3	b_3
4	a_4	b_4
5	a_5	b_5

UHT array
Delimiting lines indices

1	$-\infty$	2
2	$-\infty$	5
3	$-\infty$	5
4	$-\infty$	5
5	$-\infty$	6
6	$-\infty$	$+\infty$

LHT array
Delimiting lines indices

1	$-\infty$	6
2	$-\infty$	1
3	$-\infty$	1
4	$-\infty$	3
5	$-\infty$	4
6	$+\infty$	$-\infty$

CUT edges N
Pairs of line indices delimiting the edge

c1	$-\infty$	2
c2	$-\infty$	1
c3	$-\infty$	5
c4	$-\infty$	4
c5	$-\infty$	5

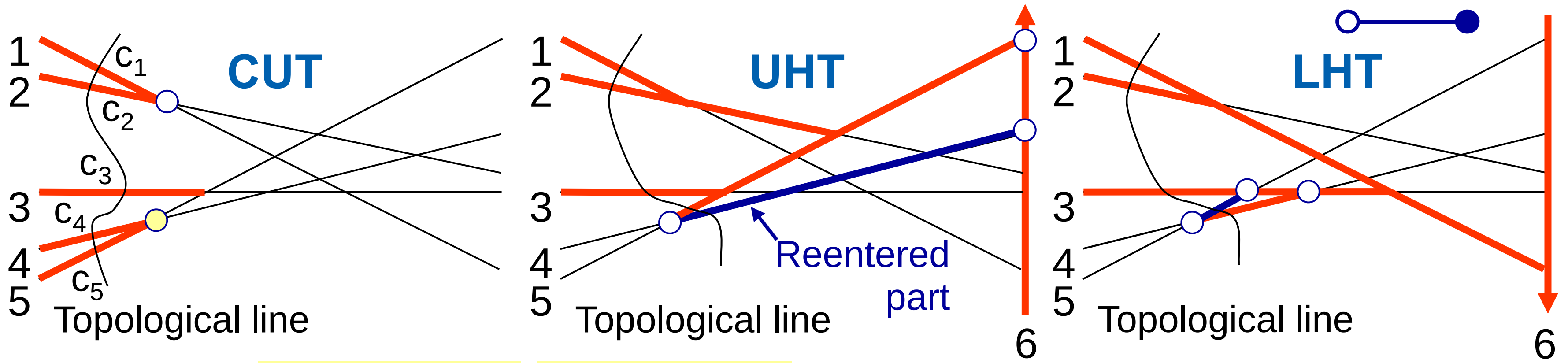
CUT Lines C
Indexes of supporting lines

c1	1
c2	2
c3	3
c4	**5**
c5	**4**

Stack S
Ready vertex first edge idx

c1

4c) Update the horizon trees – UHT and LHT



Array of line equations E

$y = a_i x + b$

1	a_1	b_1
2	a_2	b_2
3	a_3	b_3
4	a_4	b_4
5	a_5	b_5

UHT array

Delimiting lines indices

1	$-\infty$	2
2	$-\infty$	5
3	$-\infty$	5
4	**5**	**6**
5	**4**	**6**
6	$-\infty$	$+\infty$

LHT array

Delimiting lines indices

1	$-\infty$	6
2	$-\infty$	1
3	$-\infty$	1
4	**5**	3
5	**4**	**3**
6	$+\infty$	$-\infty$

CUT edges N

Pairs of line indices delimiting the edge

c1	$-\infty$	2
c2	$-\infty$	1
c3	$-\infty$	5
c4	$-\infty$	4
c5	$-\infty$	5

CUT Lines C

Indexes of supporting lines

c1	1
c2	2
c3	3
c4	**5**
c5	**4**

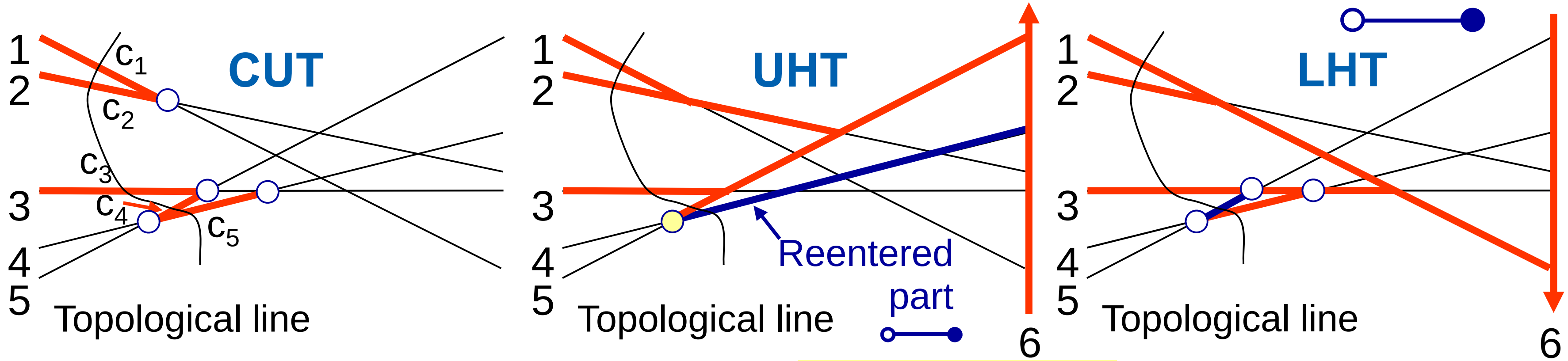
Stack S

Ready vertex upper edge id›

c1

Note: Edges are half open to prevent the tree after reinsertion

4d) Determine new cut edges endpoints – N



Array of line equations E
$y = a_i x + b$

1	a_1	b_1
2	a_2	b_2
3	a_3	b_3
4	a_4	b_4
5	a_5	b_5

UHT array
Delimiting lines indices

1	$-\infty$	2
2	$-\infty$	5
3	$-\infty$	5
4	**5**	**6**
5	**4**	**6**
6	$-\infty$	$+\infty$

LHT array
Delimiting lines indices

1	$-\infty$	6
2	$-\infty$	1
3	$-\infty$	1
4	**5**	**3**
5	**4**	**3**
6	$+\infty$	$-\infty$

CUT edges N
Pairs of line indices delimiting the edge

c1	$-\infty$	2
c2	$-\infty$	1
c3	$-\infty$	5
c4	**4**	**3**
c5	**5**	**3**

CUT Lines C
Indexes of sup-porting lines

c1	1
c2	2
c3	3
c4	**5**
c5	**4**

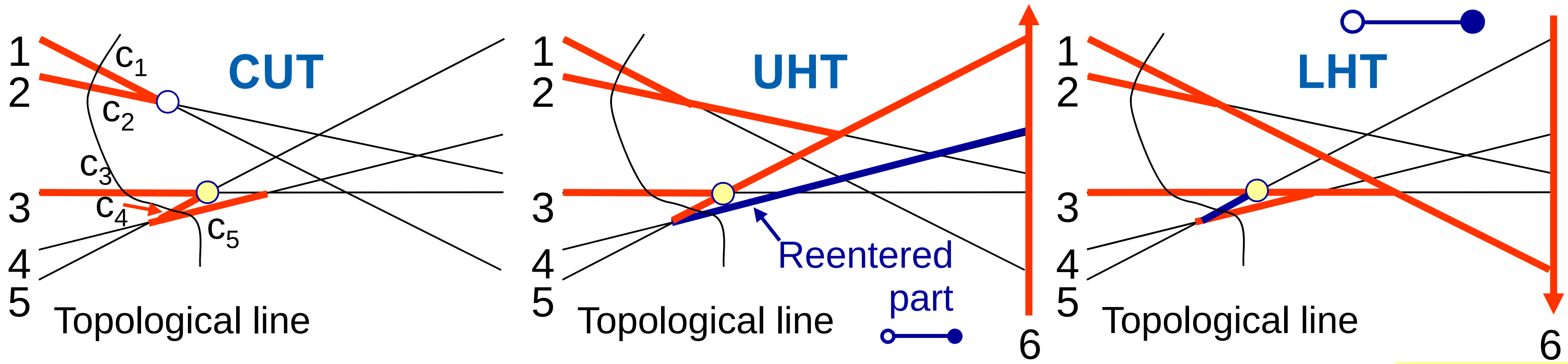
Stack S
Ready vertex upper edge id

c1

Intersect the trees – take the shorter edge

DCGI

4e) Intersect with neighbors – push into S



Array of line equations E
$y = a_i x + b$

1	a_1	b_1
2	a_2	b_2
3	a_3	b_3
4	a_4	b_4
5	a_5	b_5

UHT array
Delimiting lines indices

1	$-\infty$	2
2	$-\infty$	5
3	$-\infty$	5
4	**5**	**6**
5	**4**	**6**
6	$-\infty$	$+\infty$

LHT array
Delimiting lines indices

1	$-\infty$	6
2	$-\infty$	1
3	$-\infty$	1
4	**5**	3
5	**4**	**3**
6	$+\infty$	$-\infty$

CUT edges N
Pairs of line indices delimiting the edge

c1	$-\infty$	2
c2	$-\infty$	1
c3	$-\infty$	5
c4	**4**	**3**
c5	**5**	**3**

CUT Lines C
Indexes of supporting lines

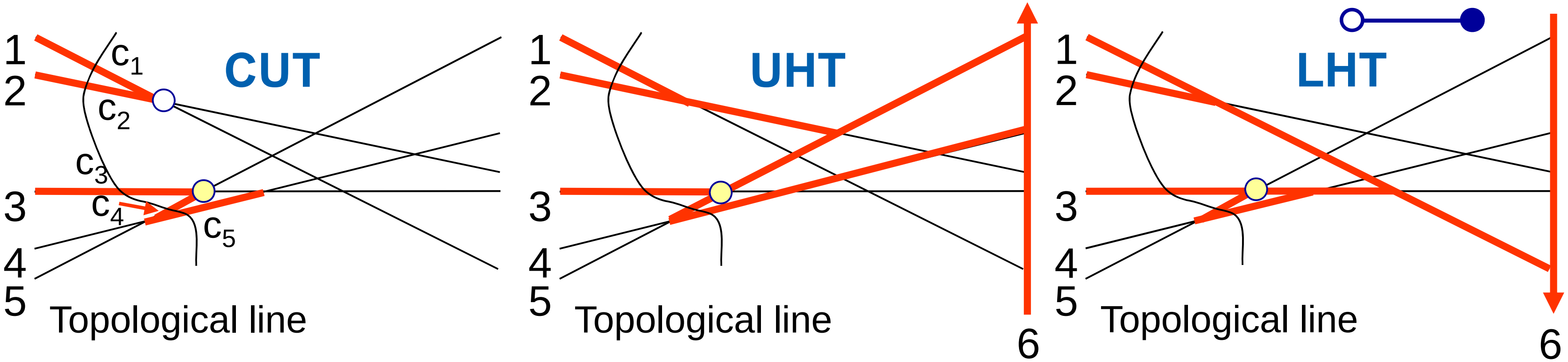
c1	1
c2	2
c3	3
c4	**5**
c5	**4**

Stack S
Ready vertex upper edge idx

c3
c1

Intersections of neighbors - into stack

4a) Pop ready vertex from S – process c3



Array of line equations E $y = a_i x + b$		
1	a_1	b_1
2	a_2	b_2
3	a_3	b_3
4	a_4	b_4
5	a_5	b_5

UHT array – Delimiting lines indices

1	$-\infty$	2
2	$-\infty$	5
3	$-\infty$	5
4	5	6
5	4	6
6	$-\infty$	$+\infty$

LHT array – Delimiting lines indices

1	$-\infty$	6
2	$-\infty$	1
3	$-\infty$	1
4	5	3
5	4	3
6	$+\infty$	$-\infty$

CUT edges N – Pairs of line indices delimiting the edge

c1	$-\infty$	2
c2	$-\infty$	1
c3	$-\infty$	5
c4	4	3
c5	5	3

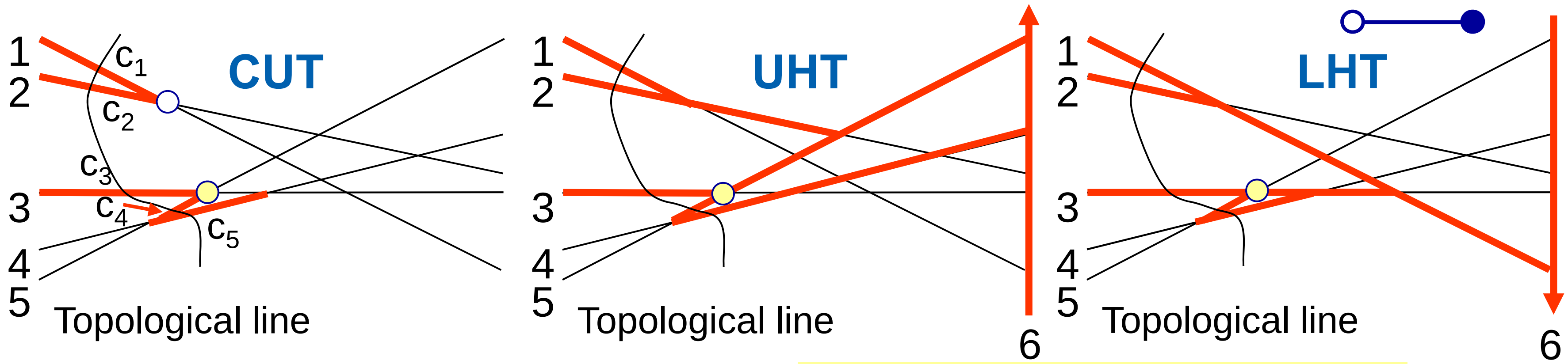
CUT Lines C – Indexes of sup-porting lines

c1	1
c2	2
c3	**3**
c4	**5**
c5	4

Stack S – Ready vertex first edge idx

c3
c1

4b) Swap lines c4 and c5 – swap 4 and 5



Array of line equations E
$y = a_i x + b$

1	a_1	b_1
2	a_2	b_2
3	a_3	b_3
4	a_4	b_4
5	a_5	b_5

UHT array
Delimiting lines indices

1	$-\infty$	2
2	$-\infty$	5
3	$-\infty$	5
4	5	6
5	4	6
6	$-\infty$	$+\infty$

LHT array
Delimiting lines indices

1	$-\infty$	6
2	$-\infty$	1
3	$-\infty$	1
4	5	3
5	4	3
6	$+\infty$	$-\infty$

CUT edges N
Pairs of line indices delimiting the edge

c1	$-\infty$	2
c2	$-\infty$	1
c3	4	3
c4	$-\infty$	5
c5	5	3

CUT Lines C
Indexes of supporting lines

c1	1
c2	2
c3	**5**
c4	**3**
c5	4

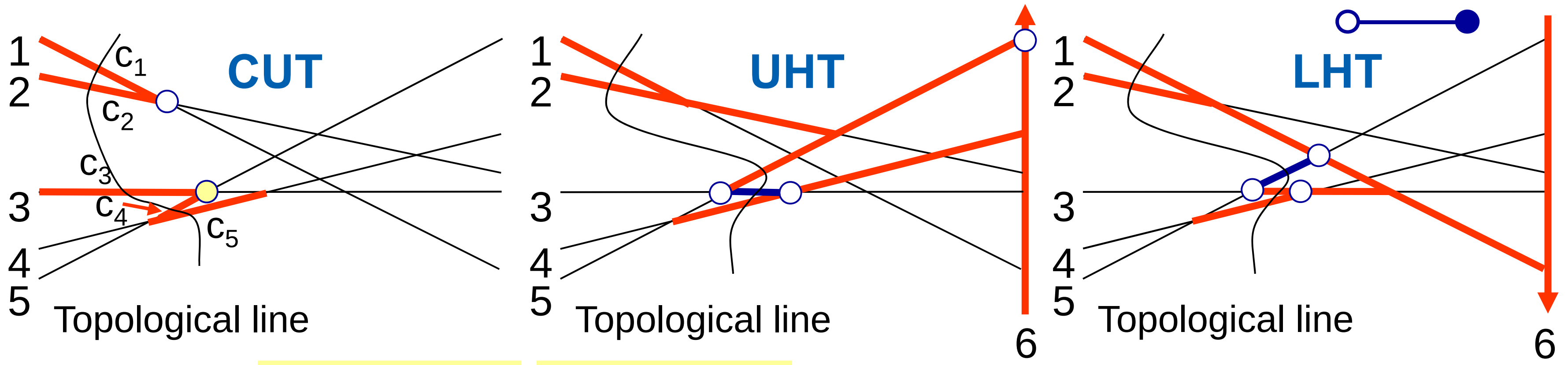
Stack S
Ready vertex first edge idx

c1

Swapped invalidated

Swapped

4c) Update the horizon trees – UHT and LHT



Array of line equations E
$y = a_i x + b$

1	a_1	b_1
2	a_2	b_2
3	a_3	b_3
4	a_4	b_4
5	a_5	b_5

UHT array
Delimiting lines indices

1	$-\infty$	2
2	$-\infty$	5
3	**5**	**4**
4	5	6
5	**3**	6
6	$-\infty$	$+\infty$

LHT array
Delimiting lines indices

1	$-\infty$	6
2	$-\infty$	1
3	**5**	1
4	5	3
5	**3**	**1**
6	$+\infty$	$-\infty$

CUT edges N
Pairs of line indices delimiting the edge

c1	$-\infty$	2
c2	$-\infty$	1
c3	4	3
c4	$-\infty$	5
c5	5	3

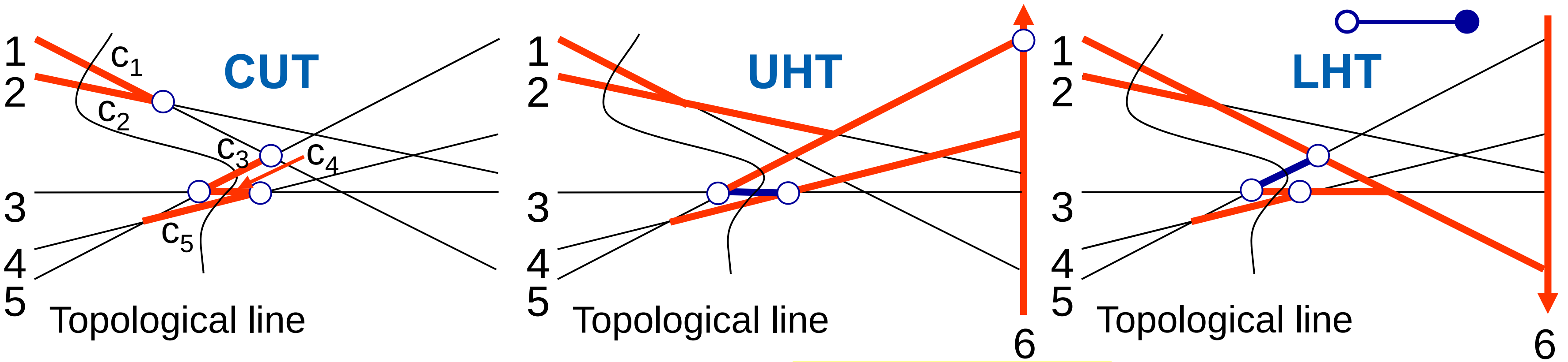
CUT Lines C
Indexes of supporting lines

c1	1
c2	2
c3	**5**
c4	**3**
c5	4

Stack S
Ready vertex first edge idx

c1

4d) Determine new cut edges endpoints



Array of line equations E

$y = a_i x + b$

1	a_1	b_1
2	a_2	b_2
3	a_3	b_3
4	a_4	b_4
5	a_5	b_5

UHT array

Delimiting lines indices

1	$-\infty$	2
2	$-\infty$	5
3	**5**	**4**
4	5	6
5	**3**	6
6	$-\infty$	$+\infty$

LHT array

Delimiting lines indices

1	$-\infty$	6
2	$-\infty$	1
3	**5**	1
4	5	3
5	**3**	**1**
6	$+\infty$	$-\infty$

CUT edges N

Pairs of line indices delimiting the edge

c1	$-\infty$	2
c2	$-\infty$	1
c3	3	1
c4	5	4
c5	5	3

CUT Lines C

Indexes of supporting lines

c1	1
c2	2
c3	5
c4	3
c5	4

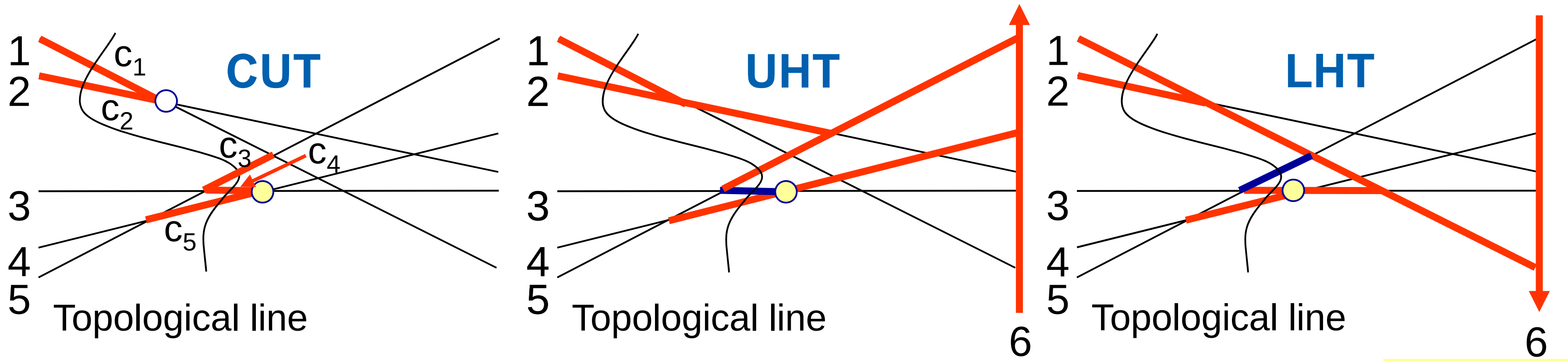
Stack S

Ready vertex first edge idx

c1

Intersect the trees – take the shorter edge

4e) Intersect with neighbors – push into S



Array of line equations E
$y = a_i x + b$

1	a_1	b_1
2	a_2	b_2
3	a_3	b_3
4	a_4	b_4
5	a_5	b_5

UHT array
Delimiting lines indices

1	$-\infty$	2
2	$-\infty$	5
3	**5**	**4**
4	5	6
5	**3**	6
6	$-\infty$	$+\infty$

LHT array
Delimiting lines indices

1	$-\infty$	6
2	$-\infty$	1
3	**5**	1
4	5	3
5	**3**	**1**
6	$+\infty$	$-\infty$

CUT edges N
Pairs of line indices delimiting the edge

c1	$-\infty$	2
c2	$-\infty$	1
c3	3	1
c4	5	4
c5	5	3

CUT Lines C
Indexes of supporting lines

c1	1
c2	2
c3	**5**
c4	**3**
c5	4

Stack S
Ready vertex first edge idx

c4
c1

Topological sweep algorithm

Slope

TopoSweep(*L*)

Input: Set of **lines *L* sorted by slope (-90° to 90°)**, simple, not vertical

Output: All parts of an **Arrangement *A*(*L*)** detected and then destroyed

1. Let *C* be the initial (leftmost) cut – lines in increasing order of slope
2. Create the initial UHT and LHT incrementally:
   a) UHT by inserting lines in decreasing order of slope
   b) LHT by inserting lines in increasing order of slope
3. By consulting UHT and LHT
   a) Determine the right endpoints N of all edges of the initial cut C
   b) Store neighboring lines with common endpoint into stack *S* (initial set of *ready vertices*)
4. Repeat until stack not empty
   a) Pop next ready vertex from stack *S* (its upper edge c_i)
   b) Swap these lines within the cut *C* (c_i <-> c_{i+1})
   c) Update the horizon trees UHT and LHT (reenter edge parts)
   d) Consulting UHT and LHT determine new cut edges endpoints N
   e) If new neighboring edges share an endpoint -> push them on *S*

4d) Determining cut edges from UHT and LHT

- for lines *i* = 1 to n
  - Compare UHT and LHT edges on line *i*
  - Set the cut lying on edge *i* to the shorter edge of these
- Order of the cuts along the sweep line
  - Order changes only at the intersection *v* (neighbors)
  - Order of remaining cuts not incident with intersection *v* does not change
- After changes of the order, test the new neighbors for intersections
  - Store intersections right from sweep line into the stack

Complexity

- $O(n^2)$ intersections
  => $O(n^2)$ events (elementary steps)
- $O(1)$ amortized time for one step – 4c)

  => $O(n^2)$ time for the algorithm

Amortized time

= even though a single elementary step can take more than $O(1)$ time, the total time needed to perform $O(n^2)$ elementary steps is $O(n^2)$, hence the average time for each step is $O(1)$.

References

[Berg] Mark de Berg, Otfried Cheong, Marc van Kreveld, Mark Overmars: Computational Geometry: Algorithms and Applications, Springer-Verlag, 3rd rev. ed. 2008. 386 pages, 370 fig. ISBN: 978-3-540-77973-5, Chapters 8., http://www.cs.uu.nl/geobook/

[Mount] Mount, D.: *Computational Geometry Lecture Notes for Fall 2016*, University of Maryland, Lectures 14, 15, and 27. http://www.cs.umd.edu/class/fall2016/cmsc754/Lects/cmsc754-fall16-lects.pdf

[Edelsbrunner] Edelsbrunner and Guibas. Topologically sweeping an arrangement. TR 9, 1986, Digital www.hpl.hp.com/techreports/Compaq-DEC/SRC-RR-9.pdf

[Rafalin] E. Rafalin, D. Souvaine, I. Streinu, "Topological Sweep in Degenerate cases", in Proceedings of the 4th international workshop on Algorithm Engineering and Experiments, ALENEX 02, in LNCS 2409, Springer-Verlag, Berlin, Germany, pages 155-156. http://www.cs.tufts.edu/research/geometry/other/sweep/paper.pdf

[Agarwal] Pankaj K. Agarwal and Mica Sharir. Arrangements and Their Applications, 1998, http://www.math.tau.ac.il/~michas/arrsurv.pdf