DEPARTMENT OF COMPUTER GRAPHICS AND INTERACTION

VORONOI DIAGRAM PART II

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https://cw.felk.cvut.cz/doku.php/courses/a4m39vg/start

Based on [Berg], [Reiberg] and [Nandy]

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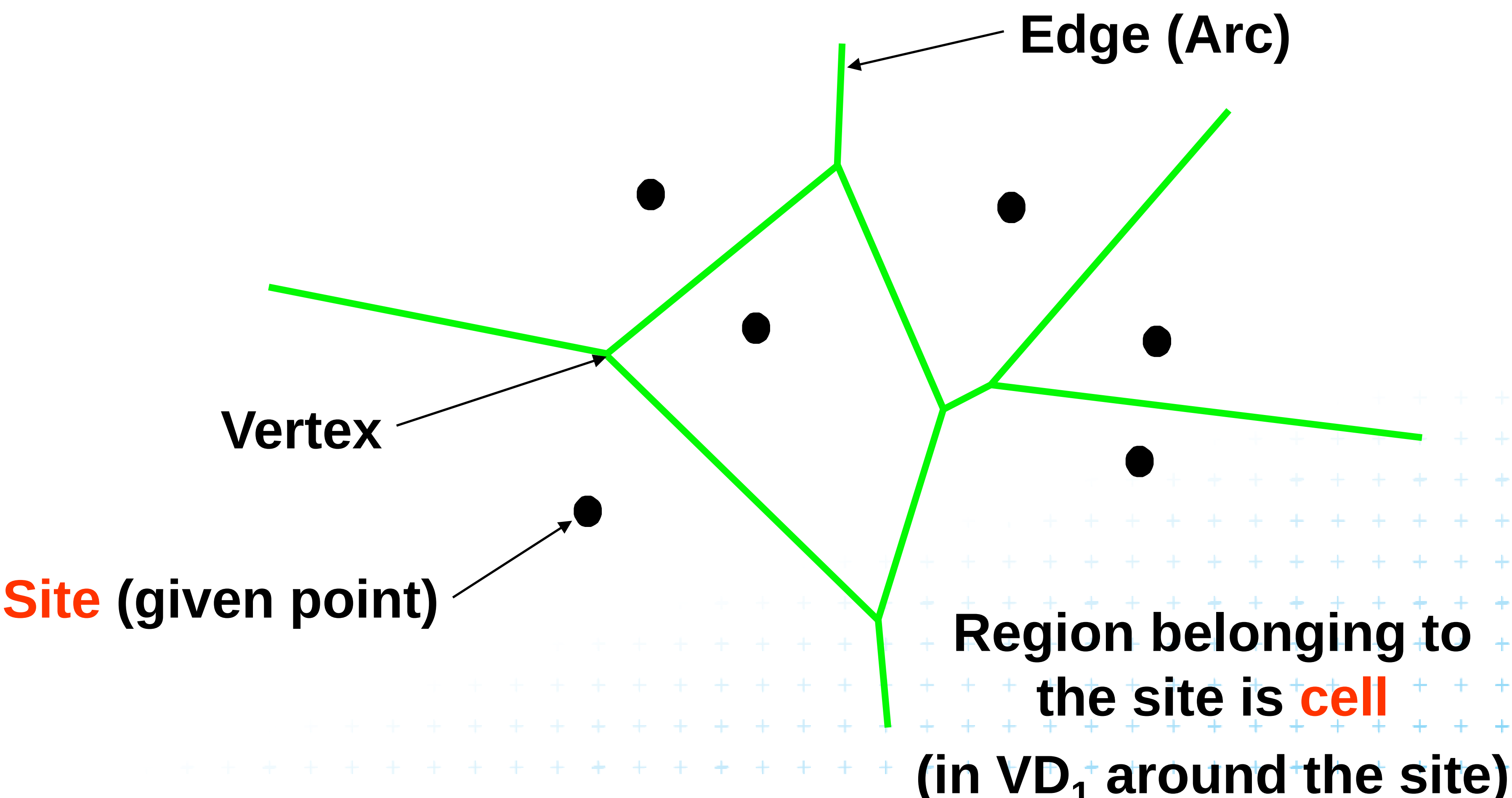
Talk overview

- Incremental construction
- Voronoi diagram of line segments
- VD of order k
- Farthest-point VD

Summary of the VD terms

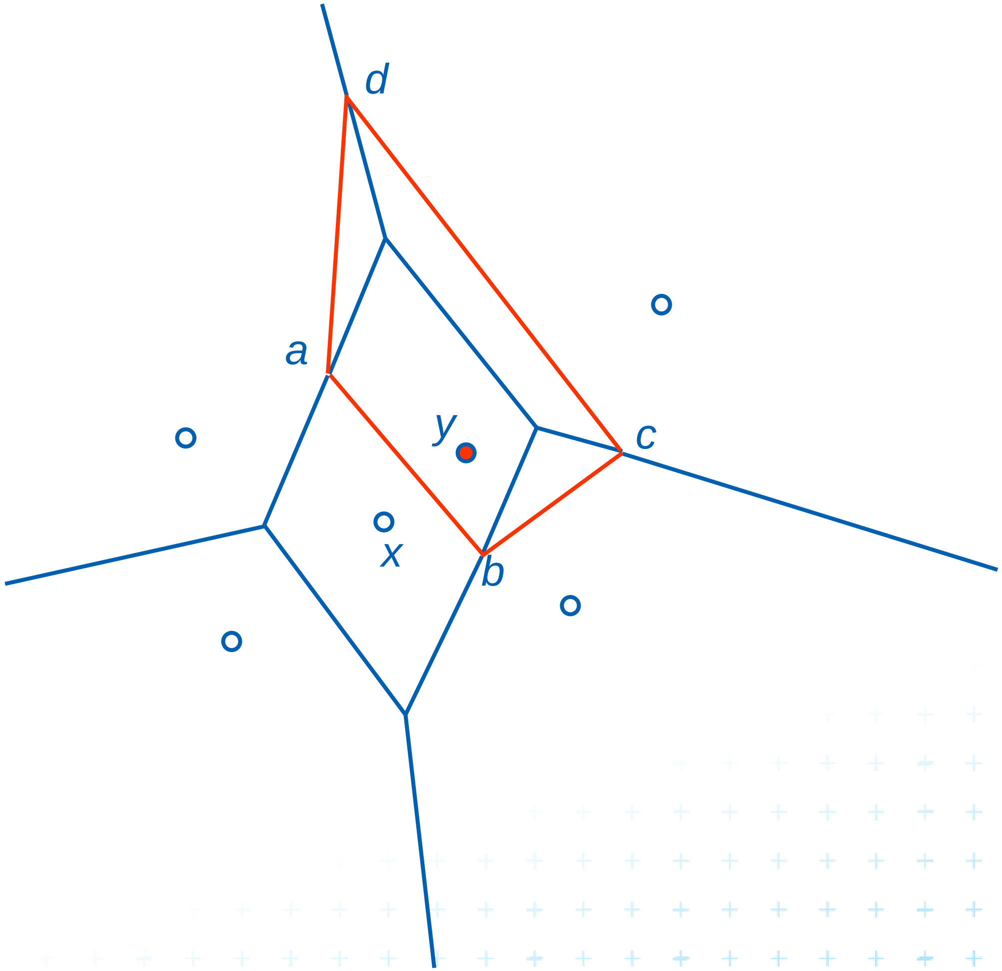
- Site = input point, line segment, …
- Cell = area belonging to the site,
  in VD_1 locus of points nearest to the site
- Edge, arc = part of Voronoi diagram
  (border between cells)
- Vertex = intersection of VD edges

Summary of the VD terms

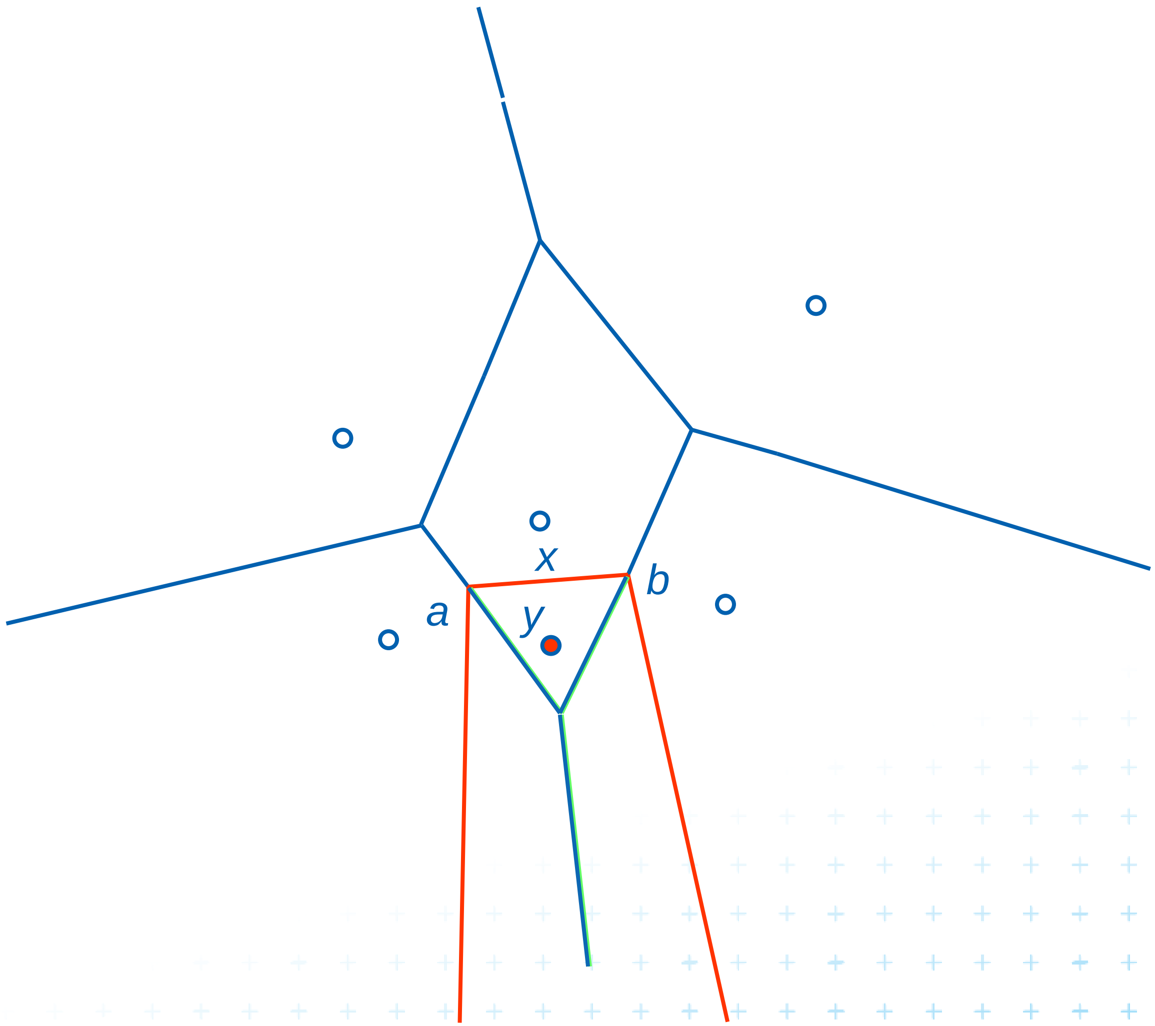


Incremental construction

Incremental construction – bounded cell



Incremental construction – unbounded cell



Incremental construction algorithm

InsertPoint(*S*, Vor(*S*), y) **… y = a new site**

Input: Point set *S*, its Voronoi diagram, and inserted point $y \notin S$

Output: VD after insertion of ***y***

1. Find the site *x* in which cell point ***y*** falls, …O(log *n*)
2. Detect the intersections {*a*,*b*} of bisector *L*(*x*,***y***) with cell *x* boundary
   => create the first edge *e* = *ab* on the border of site *x* …O(*n*)
3. site *z* = neighbor site across the border with intersection *b* …O(1)
4. Set start intersection point *p* = *b*, set new intersection *c* = undef
5. **while**(exists(*p*) *and* ***c*** $\neq$ *a*) // trace the bisectors from *b* in one direction
   a. Detect intersection ***c*** of *L*(***y***,*z*) with border of cell *z*
   b. Report Voronoi edge *pc*
   c. *p* = *c*, *z* = neighbor site across border with intersec. *c*

   …O(n^2) (for steps a–c)
5. **if**(*c* $\neq$ *a*) **then** // open site → trace the bisectors from *a* in other direction
   a. *p* = *a*
   b. *Similarly as in steps 3,4,5 with a*

O(n^2) worst-case, O(*n*) expected time for some distributions

DCGI

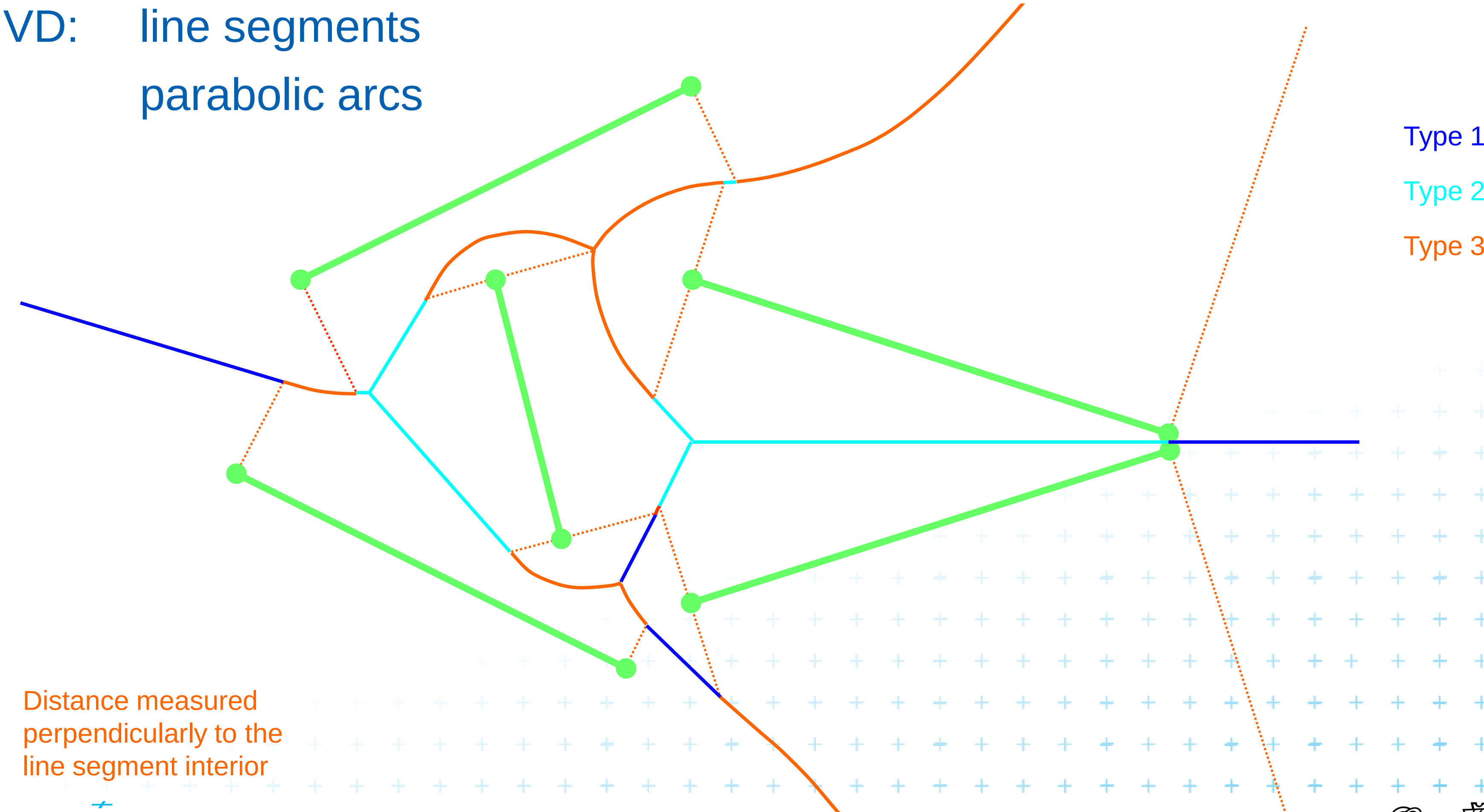
Voronoi diagram of line segments

Voronoi diagram of line segments

Input: $S = \{s_1, \ldots, s_n\}$ = set of n disjoint line segments (sites)

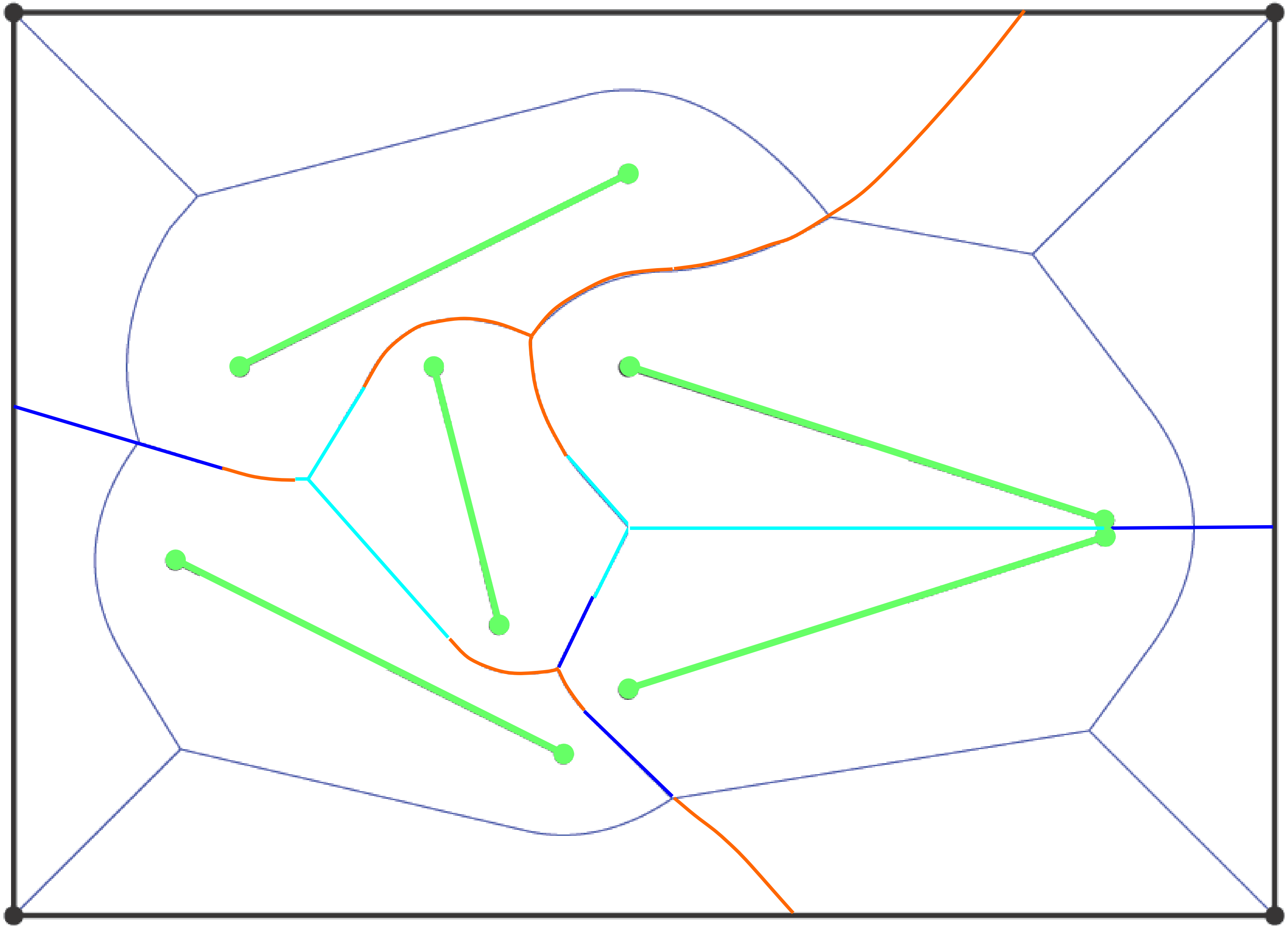
VD: line segments

parabolic arcs



[Berg]

VD of line segments with bounding box



BBOX
=>
standard
DCEL

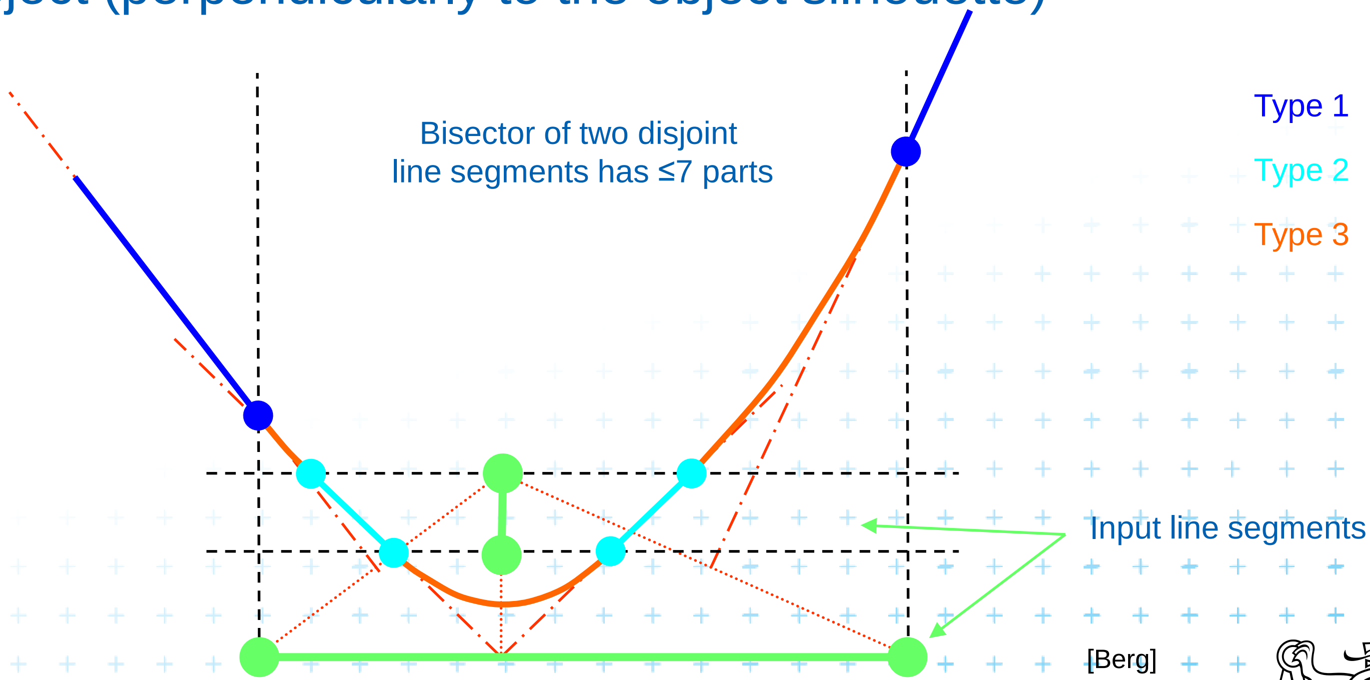
[Berg]

VD of 2 line-segments in detail

VD consists of line segments and parabolic arcs

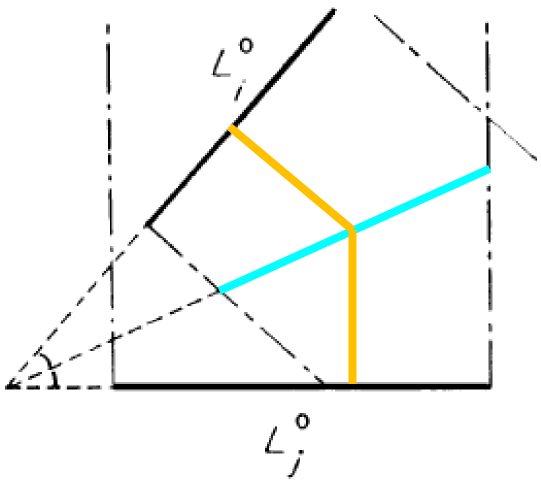
- Line segment – bisector of end-points(1) or of interiors(2)
- Parabolic arc – of point and interior(3) of a line segment

Distance from point-to-object (line segment) is measured to the closest point on the object (perpendicularly to the object silhouette)

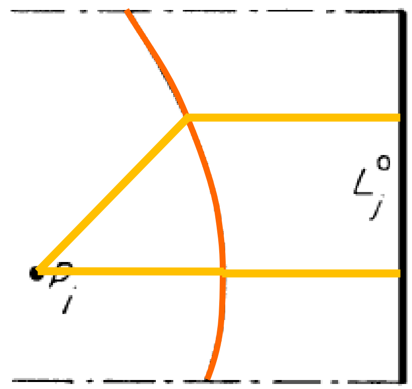


VD in greater details

Type 2



Type 3



[Reiberg]

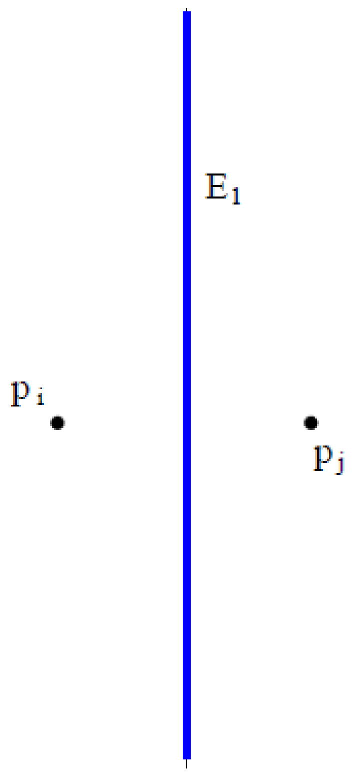
Bisector of two
line segment interiors
(in intersection of perpendicular slabs only)

Bisector of (end-)point and
line segment interior

DCGI

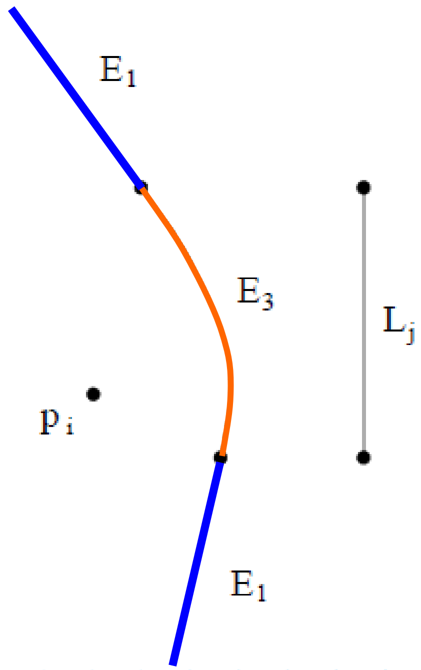
VD of points and line segments examples

2 points



Type 1

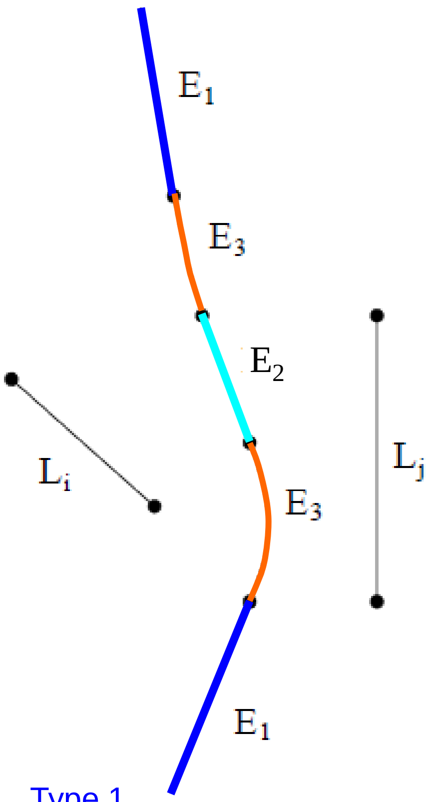
Point & segment



Type 1

Type 3

2 line segments



Type 1

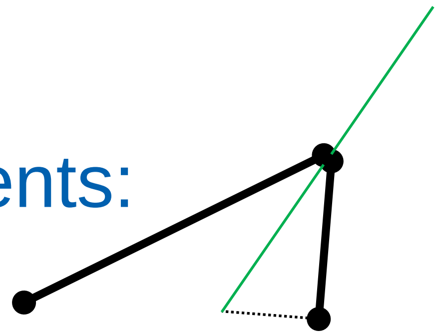
Type 2

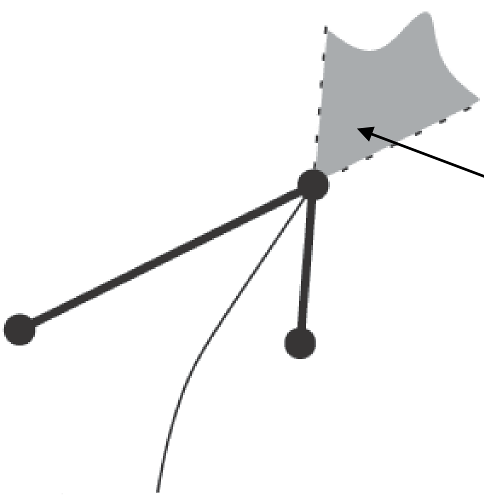
Type 3

[Reiberg]

Voronoi diagram of line segments

- Has more complex bisectors of line segments
  - VD contains line segments and parabolic arcs
- Still $O(n)$ combinatorial complexity
- Assumptions on the input line segments:
  - non-crossing
  - strictly disjoint end-points (slightly shorten the segm.)



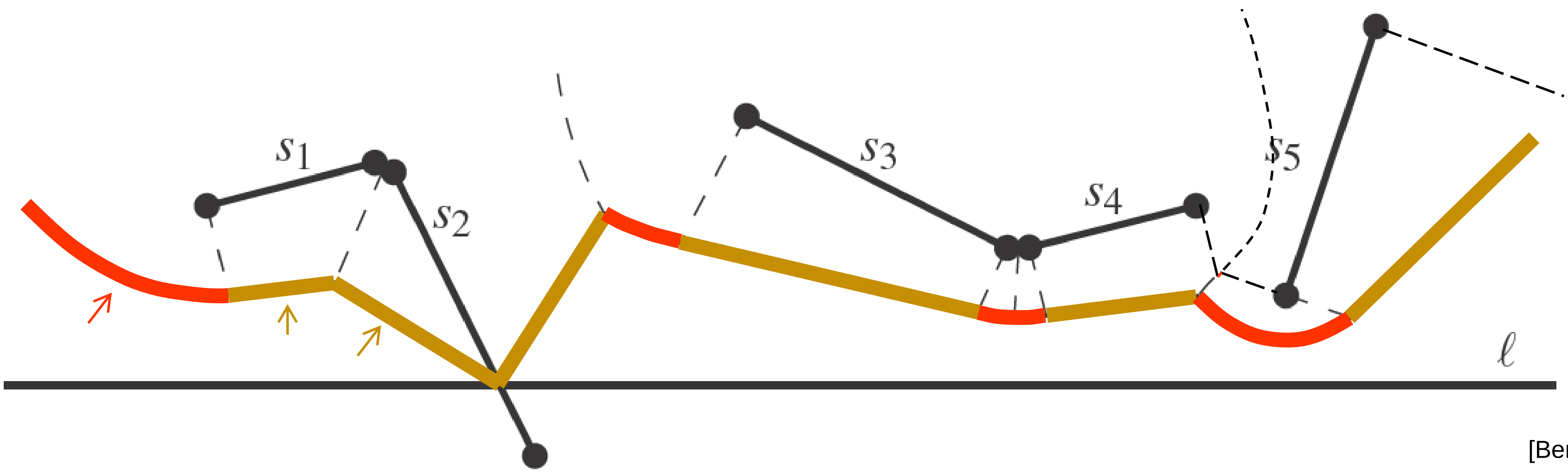


if(we allow touching segments)

Shared endpoints cause complication:
The whole region is equally close
to two line segments

Fortune's algorithm for line segments

Shape of beach line for line segments



[Berg]

Beach line = points with distance to the closest site above sweep line *l* equal to the distance to *l*

Beach line contains

- *parabolic arcs* when closest to a site end-point
- *straight line segments* when closest to a site interior
  (or just the part of the site interior above *l* if the site *s* intersects *l*)

(This is the shape of the beach line)

Beach line **breakpoints types** site = line segment

Breakpoint *p* on the beach line is equidistant from *l* and equidistant and closest to:

points

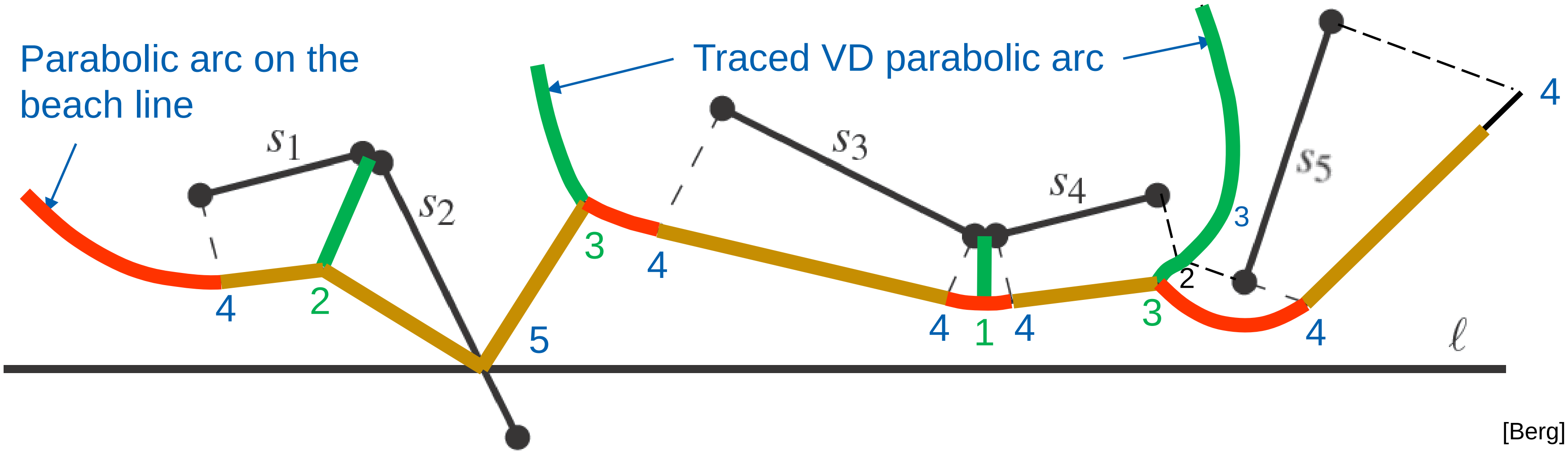
1. two site end-points => *p* traces a VD line segment

segments

2. two site interiors => *p* traces a VD line segment
3. end-point and interior => *p* traces a VD parabolic arc
4. one site end-point => *p* traces a line segment (border of the slab perpendicular to the site)
5. site interior intersects the scan line *l* => *p* = intersection, traces the input line segment

Cases 4 and 5 involve only one site and therefore do not form a Voronoi diagram edge (are used by alg.only)

Breakpoints types - what they trace on VD



[Berg]

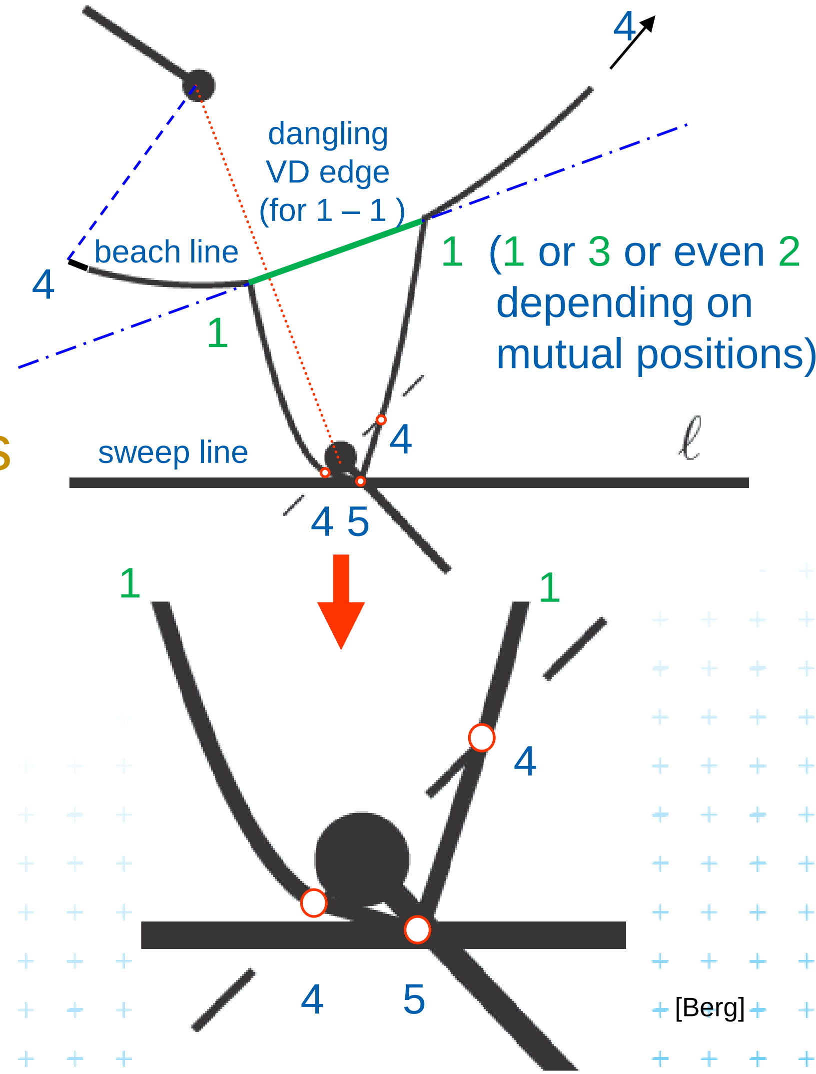
- 1,2 trace a Voronoi line segment (part of VD edge) DRAW
- 3 traces a Voronoi parabolic arc (part of VD edge) DRAW
- 4,5 trace a line segment (used only by the algorithm) MOVE
  - 4 limits the slab perpendicular to the line segment
  - 5 traces the intersection of input segment with a sweep line

(This is the shape of the traced VD arcs)

Site event – sweep line reaches an endpoint

I. At upper endpoint of

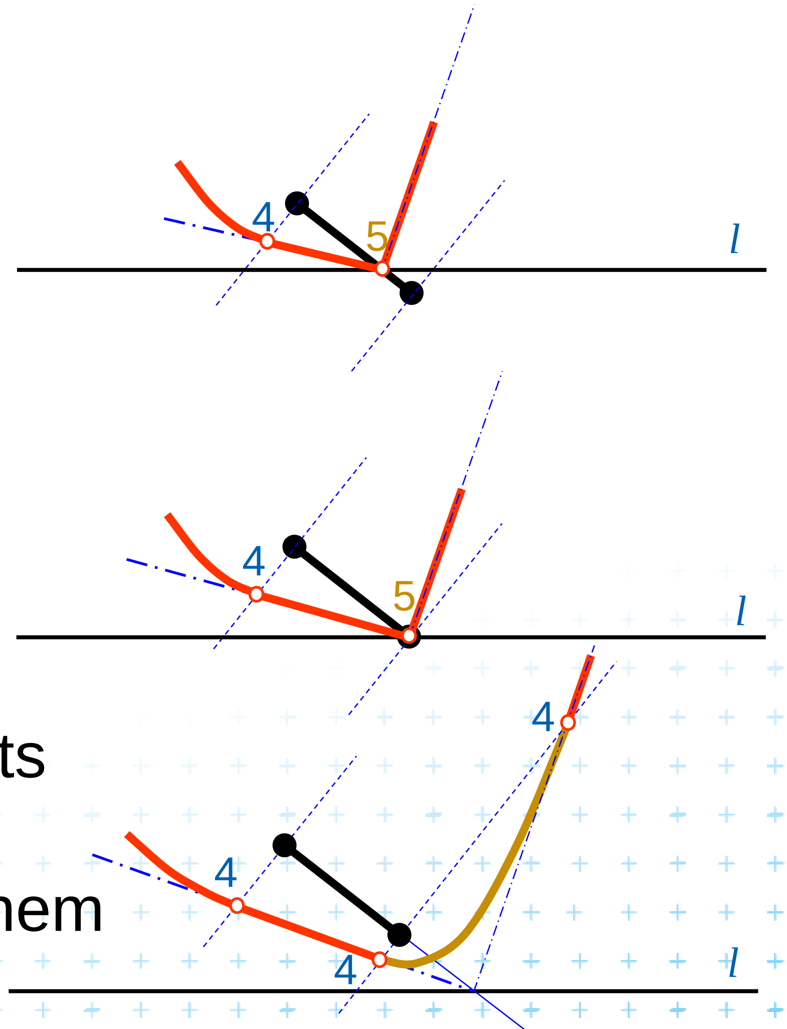
- Arc above is split into two 1-1
- four new arcs are created (2 segments + 2 parabolas) — 4-5, 5-4 above "segments"; 1-4, 4-1 above "parabolas"
- Breakpoints for two segments are of type 4-5-4
- Breakpoints for parabolas depend on the surrounding sites
  - Type 1 for two end-points
  - Type 3 for endpoint and interior
  - etc…



Site event – sweep line reaches an endpoint

II. At lower endpoint of

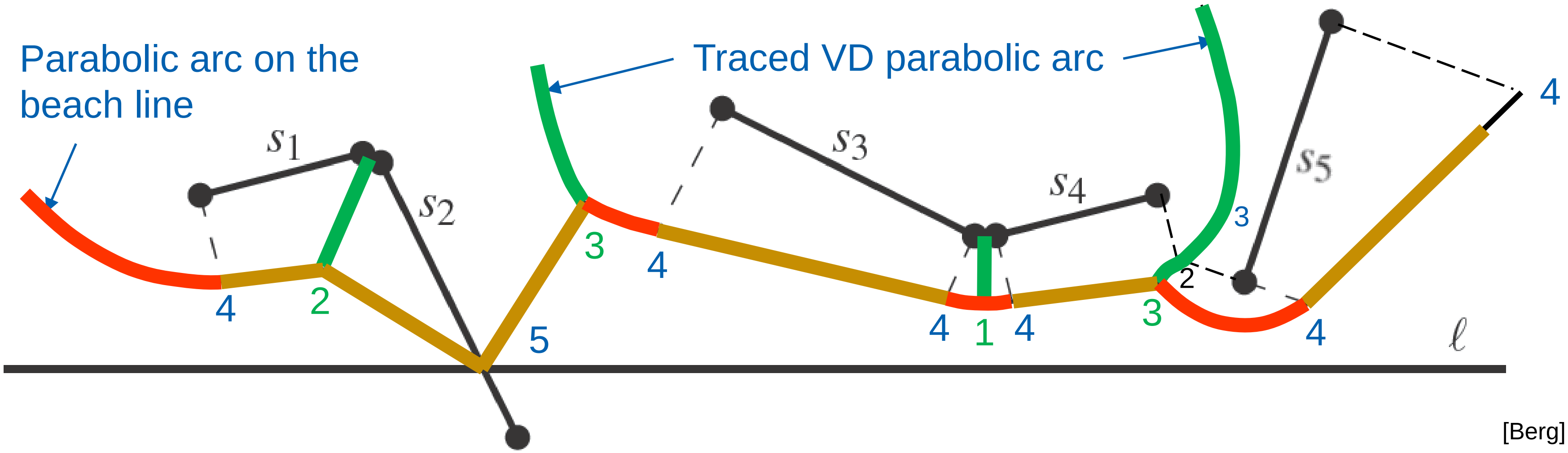
- Intersection with interior (breakpoint of type 5)
- is replaced by two breakpoints (of type 4) with parabolic arc between them



Circle event – lower point of circle of 3 sites

- Two breakpoints meet (on the beach-line)
- Solution depends on their type
  - Any of first three types (1,2,or 3) meet (circle event)
    - 3 sites involved – Voronoi vertex created
  - Type 4 (*segment interiors*) with something else
    - two sites involved – breakpoint changes its type
    - Voronoi vertex not created (Voronoi edge may change its shape)
  - Type 5 (*on segment*) with something else
    - never happens for disjoint segments (meet with type 4 happens before)

Breakpoints types - what they trace on VD



[Berg]

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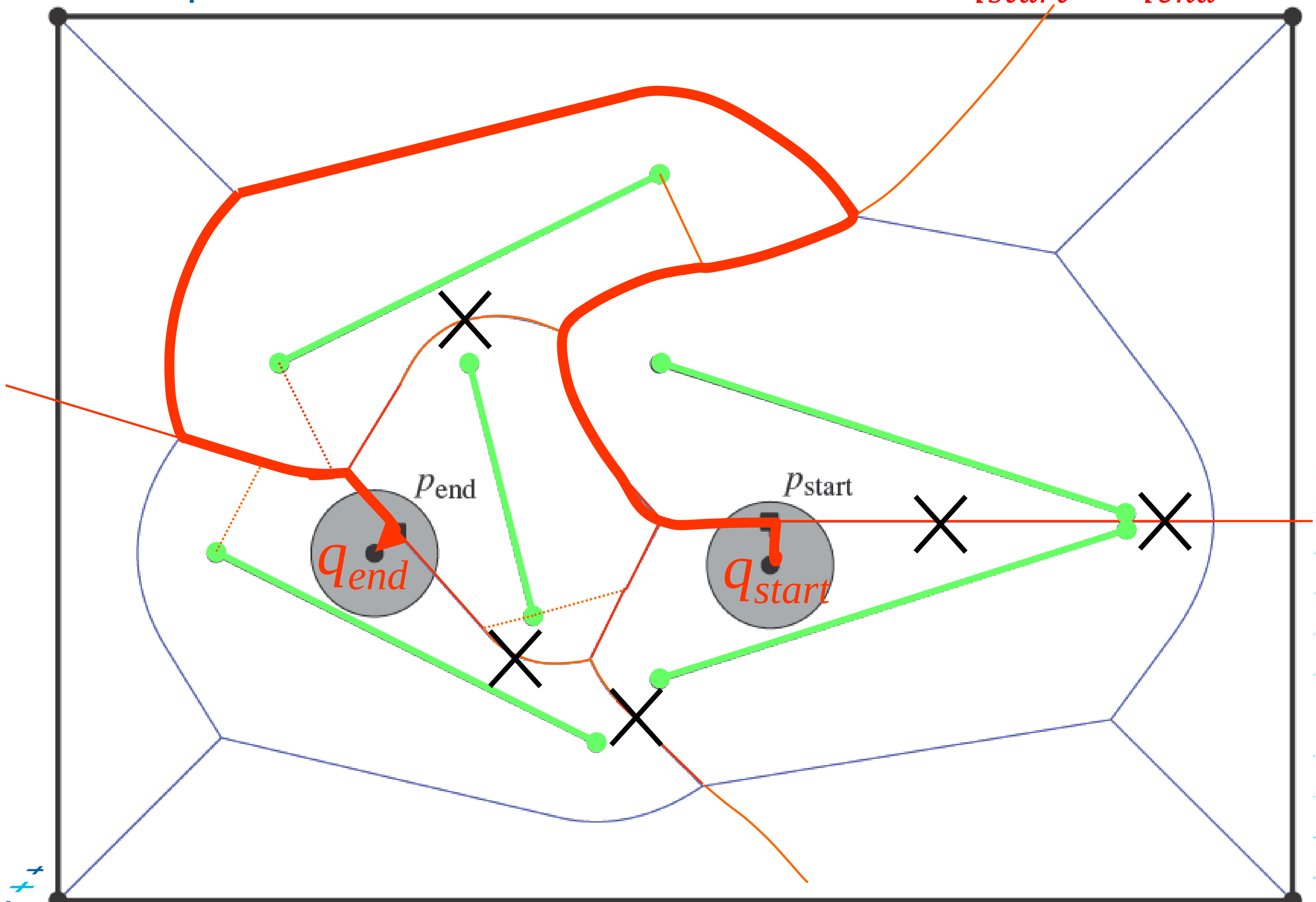
- 4,5 trace a line segment (used only by the algorithm) MOVE
  - 4 limits the slab perpendicular to the line segment
  - 5 traces the intersection of input segment with a sweep line

(This is the shape of the traced VD arcs)

Motion planning example

Motion planning example - retraction Rušení hran

Find path for a circular robot of radius *r* from q_{start} to q_{end}



[Berg]

Motion planning example - retraction Rušení hran

Find path for a circular robot of radius *r* from q_{start} to q_{end}

- Create Voronoi diagram of line segments, take it as a graph
- Project q_{start} *and* q_{end} to P_{start} and P_{end} *on the VD*
- Remove segments with distance to sites smaller than radius *r* of a robot
- Depth first search if path from P_{start} to P_{end} *exists*
- Report path $q_{start}\ P_{start} \dots$ *path* $\dots\ P_{end}\ q_{end}$

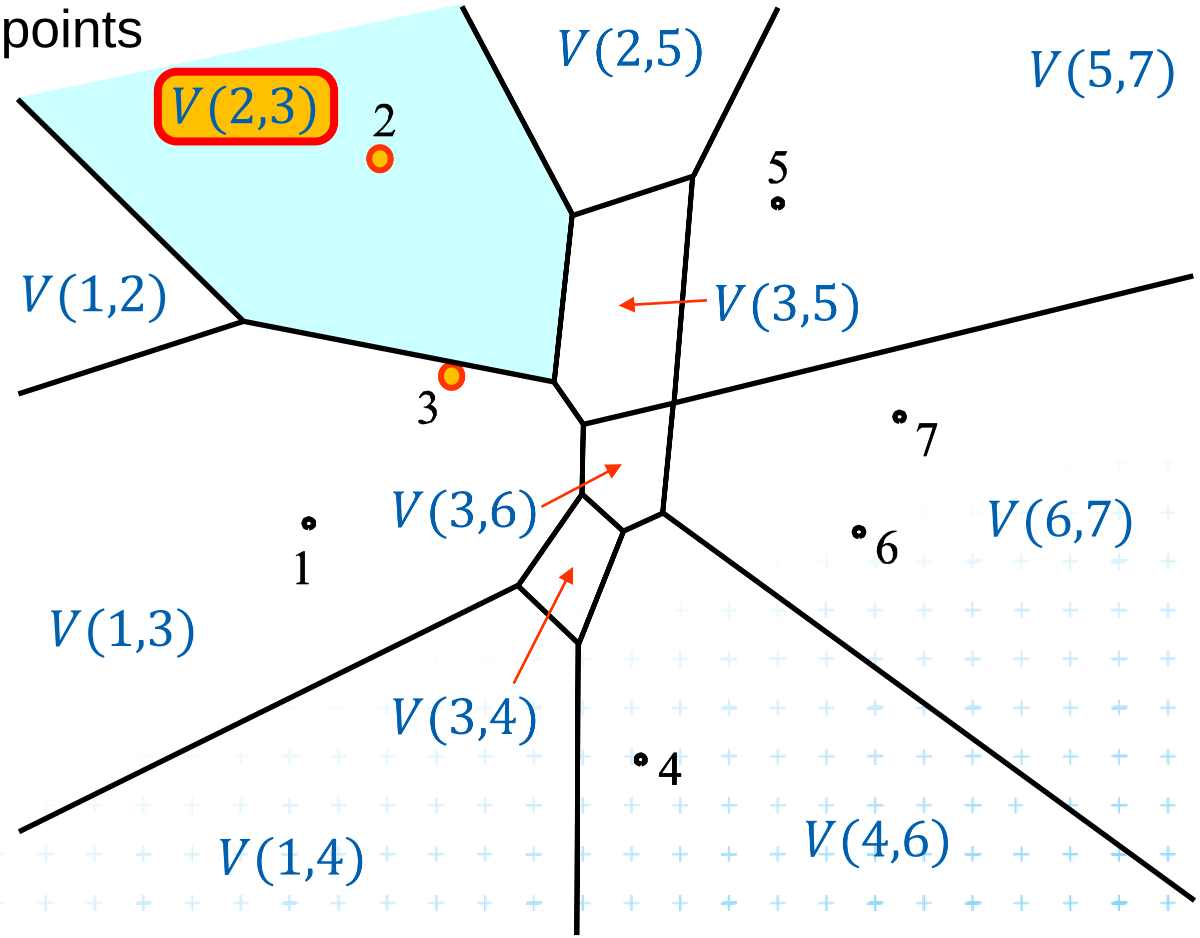
- $O(n \log n)$ time using $O(n)$ storage

Higher order VD

Order-2 Voronoi diagram (nearest to two sites)

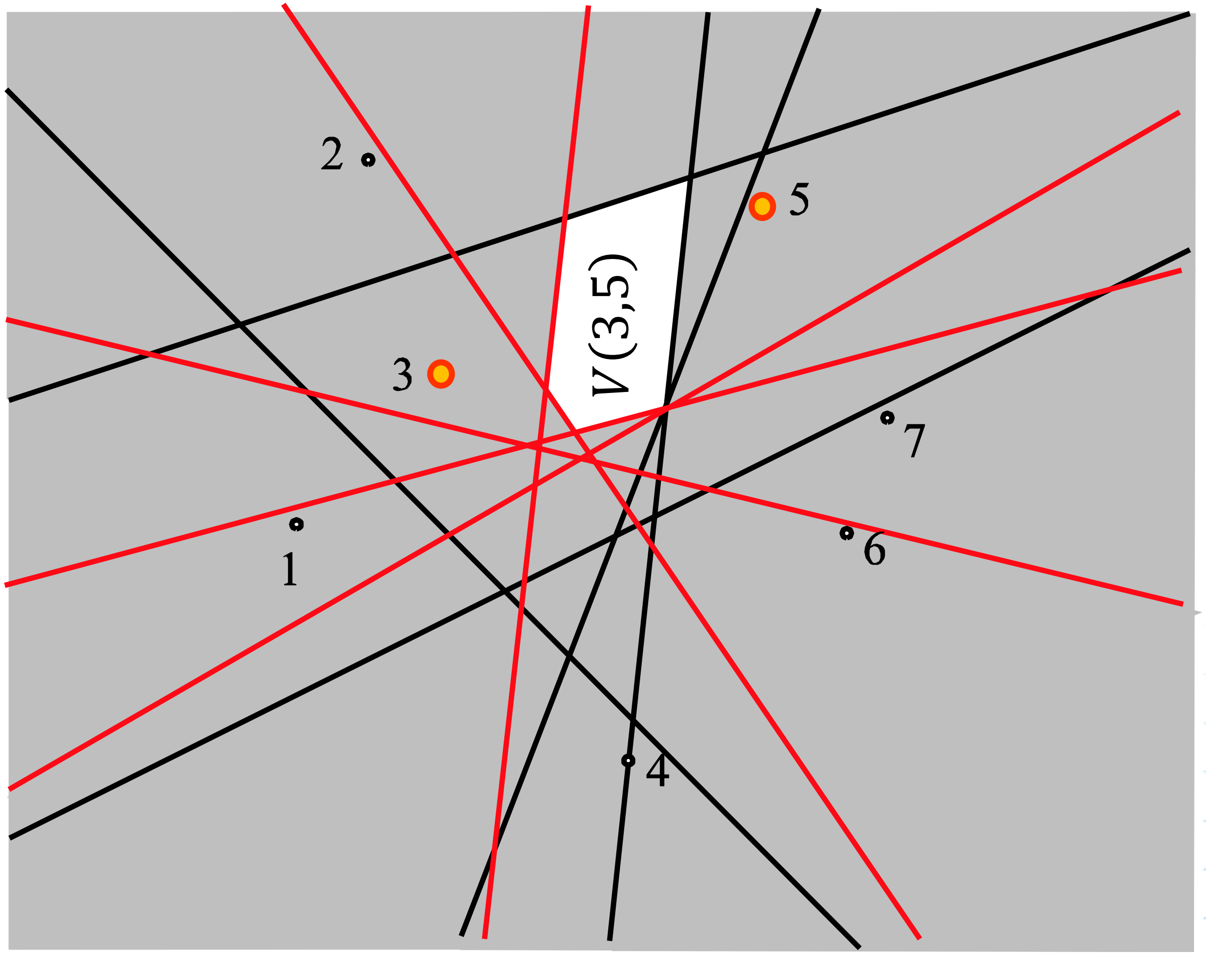
Cell $V(p_i, p_j)$: the set of points of the plane closer to each of p_i and p_j than to any other site

Property
The order-2 Voronoi regions are convex



[Nandy]

Construction of V(3,5) = V(5,3)



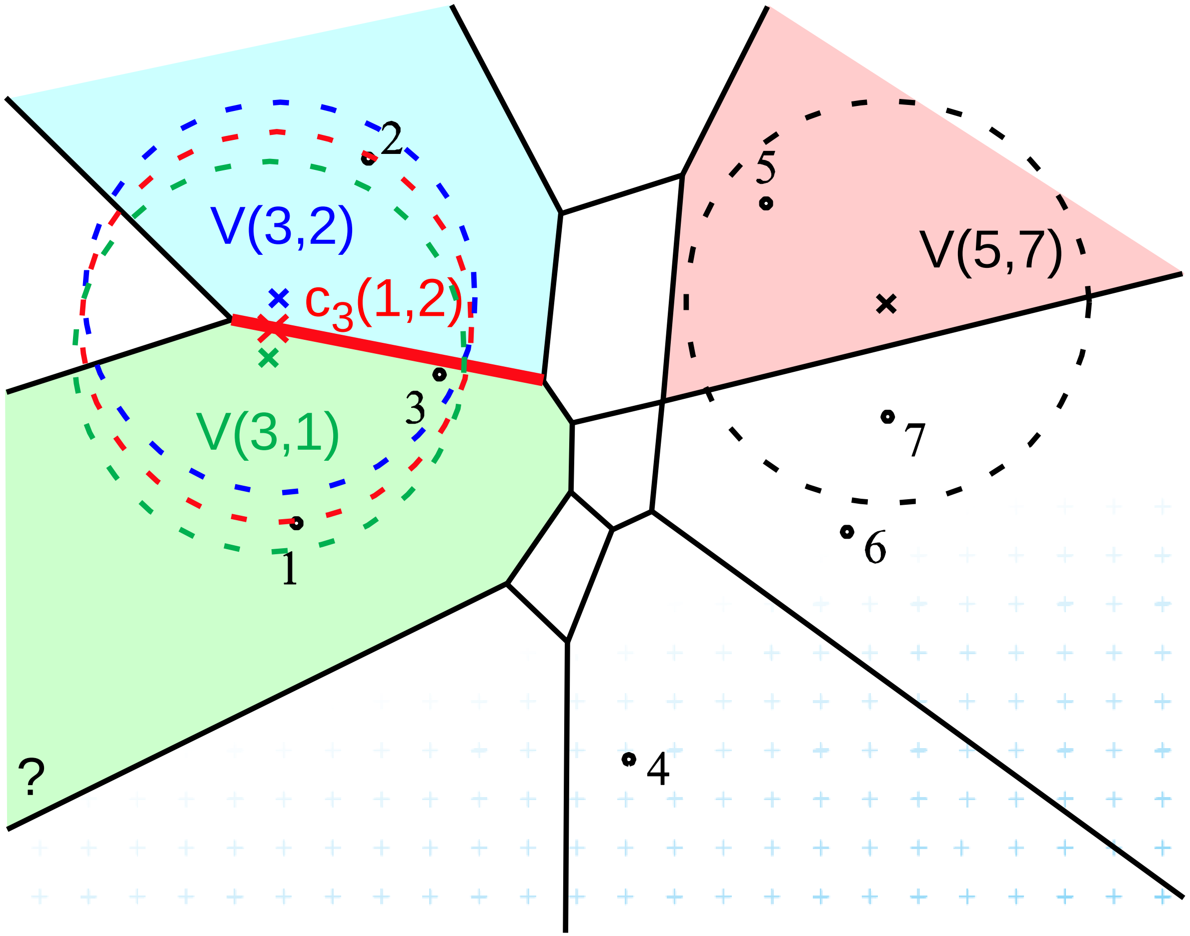
[Nandy]

Intersection of all halfplanes except $h(3,5)$ and $h(5,3)$:

$$V(3,5) = \bigcap_{x \neq 5} h(3,x) \cap \bigcap_{x \neq 3} h(5,x)$$

Order-2 Voronoi edges

- edge : set of centers of circles passing through 2 sites *s* and *t* and containing site *p*
  - => $c_p(s,t)$
  - (Edge splits the cell for *p*)



Question

Which are the regions on both sides of $c_p(s,t)$?

=> cells V(p,s) and V(p,t)

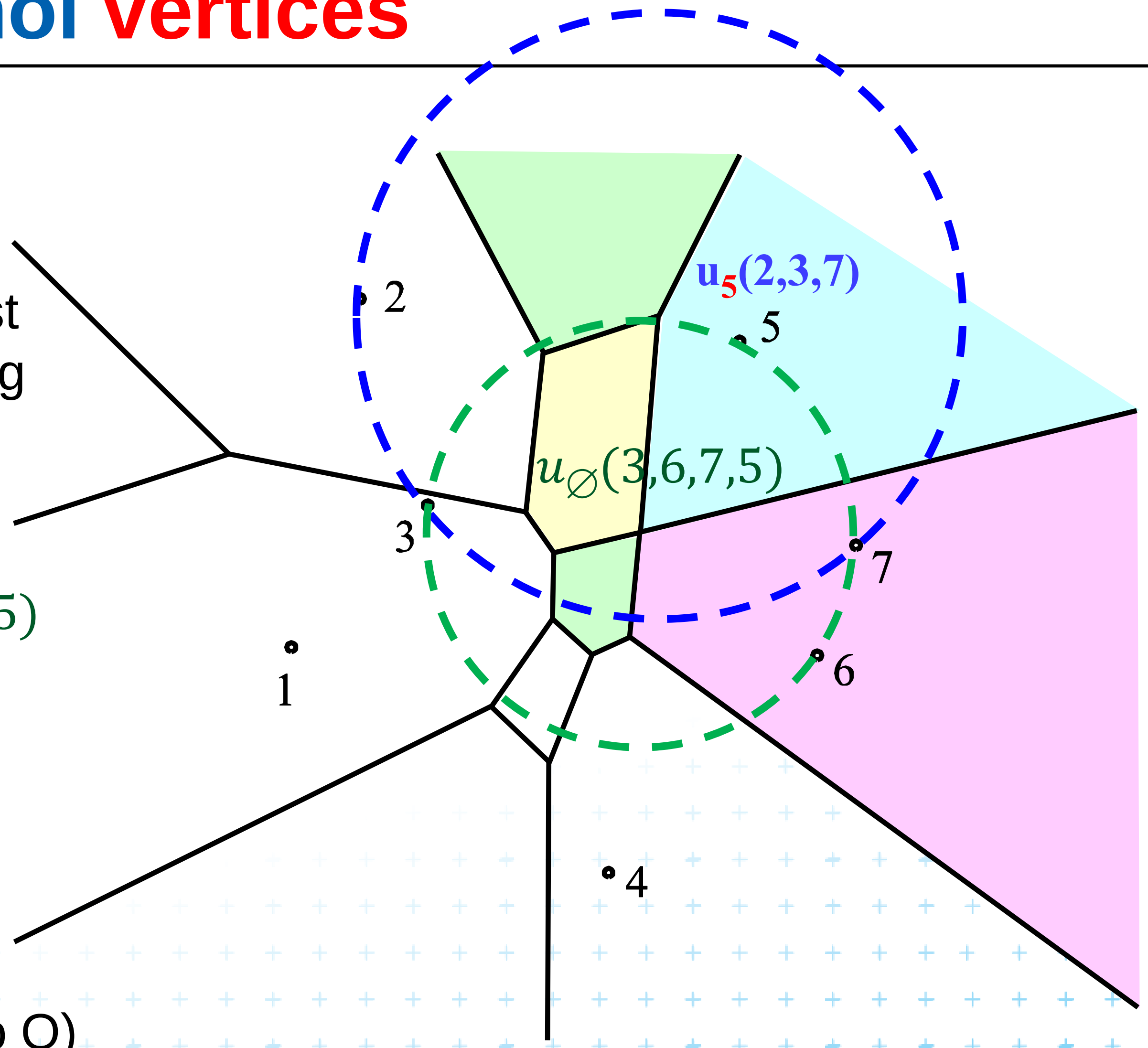
[Nandy]

Order-2 Voronoi vertices

vertex : center of a circle passing through at least 3 sites Q and containing either site p or nothing

$\Rightarrow u_p(Q)$ or $u_\emptyset(Q \cup p)$

$u_5(2,3,7)$, $u_\emptyset(3,6,7,5)$



(circle circumscribed to Q)

Order-2 Voronoi vertex $u_p(Q)$

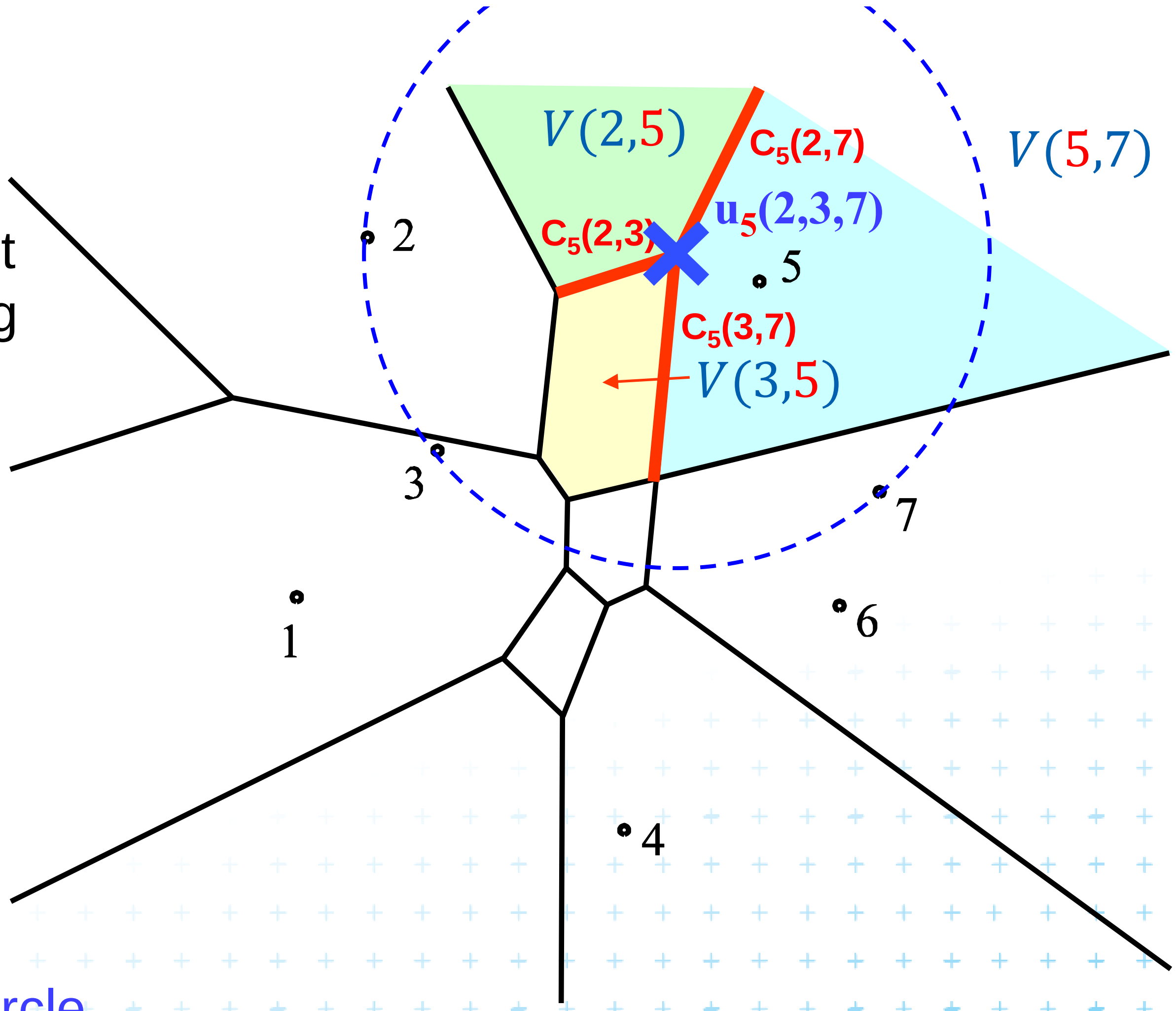
vertex : center of a circle passing through at least 3 sites Q and containing either site p or nothing

Case $u_p(Q)$
$u_5(2,3,7)$

Cell 5 is inside for all incident edges:

$C_5(2,3)$
$C_5(2,7)$
$C_5(3,7)$

=> 5 is inside for the circle with center in Voronoi vertex

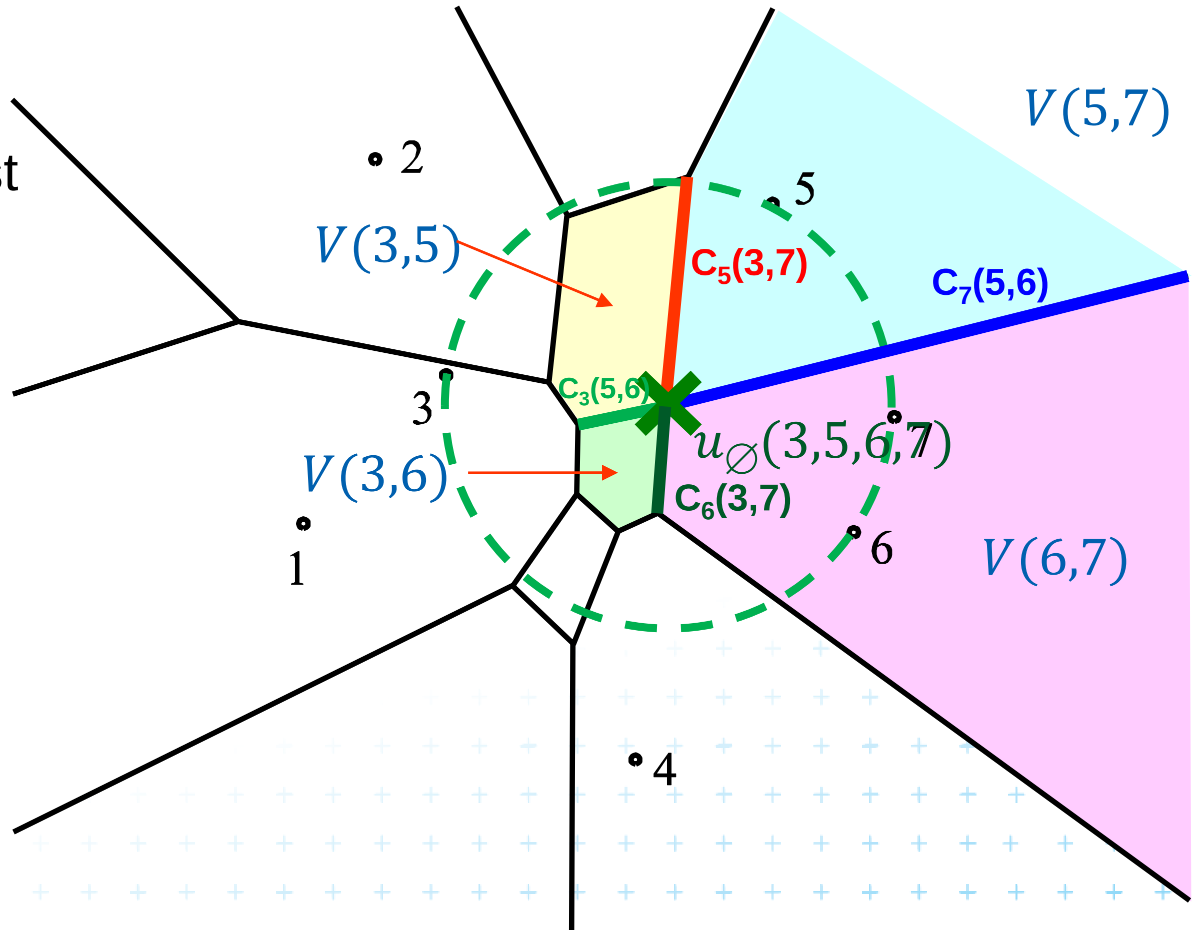


Order-2 Voronoi vertex $u_{\emptyset}(Q \cup p)$

vertex : center of a circle passing through at least 3 sites and containing either site p or nothing

Case $u_{\emptyset}(Q \cup p)$

$u_{\emptyset}(3,5,6,7)$



Cell 5 is not inside for all incident edges:

$C_5(3,7)$

$C_6(3,7)$

$C_3(5,6)$

$C_7(5,6)$

=> 5 is on circle with center in Voronoi vertex

Order-k Voronoi Diagram

Single step $V_k \rightarrow V_{k+1}$

The order-k diagram can be constructed from the order-$(k-1)$ diagram in $O(kn \log n)$ time

Globally

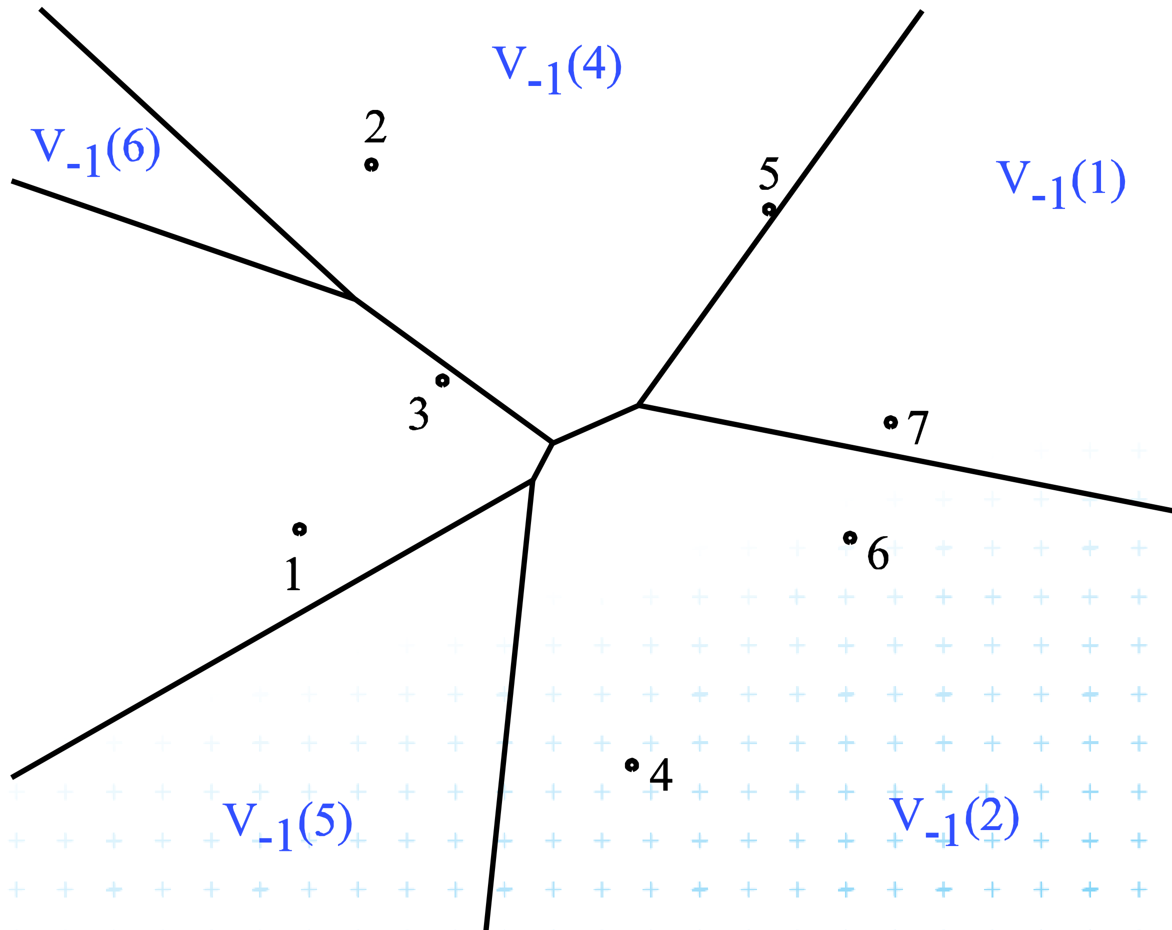
$$\sum_{i=1}^{k-1} O(in \log n) = O(k^2 n \log n)$$

From $V_1 \rightarrow V_k$

The order-k diagram can be iteratively constructed in $O(k^2 n \log n)$ time from the pointset of size n

[Preparata]

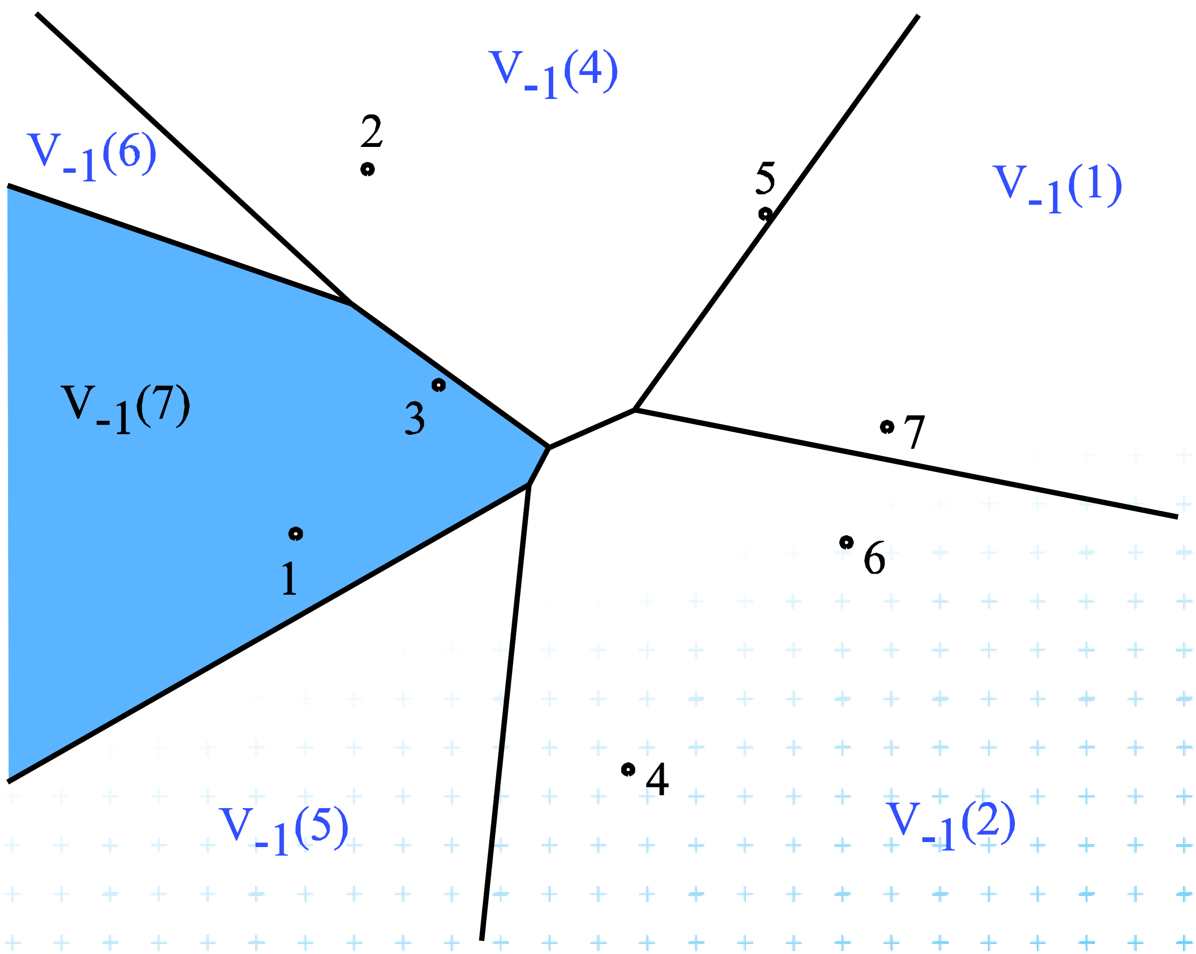
Order n-1 VD
(Farthest-point Voronoi diagram)


$V_{-1}(4)$
$V_{-1}(6)$
$V_{-1}(1)$
2
5
3
7
1
6
4
$V_{-1}(5)$
$V_{-1}(2)$

Farthest-point Voronoi diagram

$V_{-1}(p_i)$ cell
= set of points in the plane farther from p_i than from any other site

$Vor_{-1}(P)$ diagram
= partition of the plane formed by the farthest point Voronoi regions, their edges, and vertices



[Nandy]

Farthest-point Voronoi region (cell)

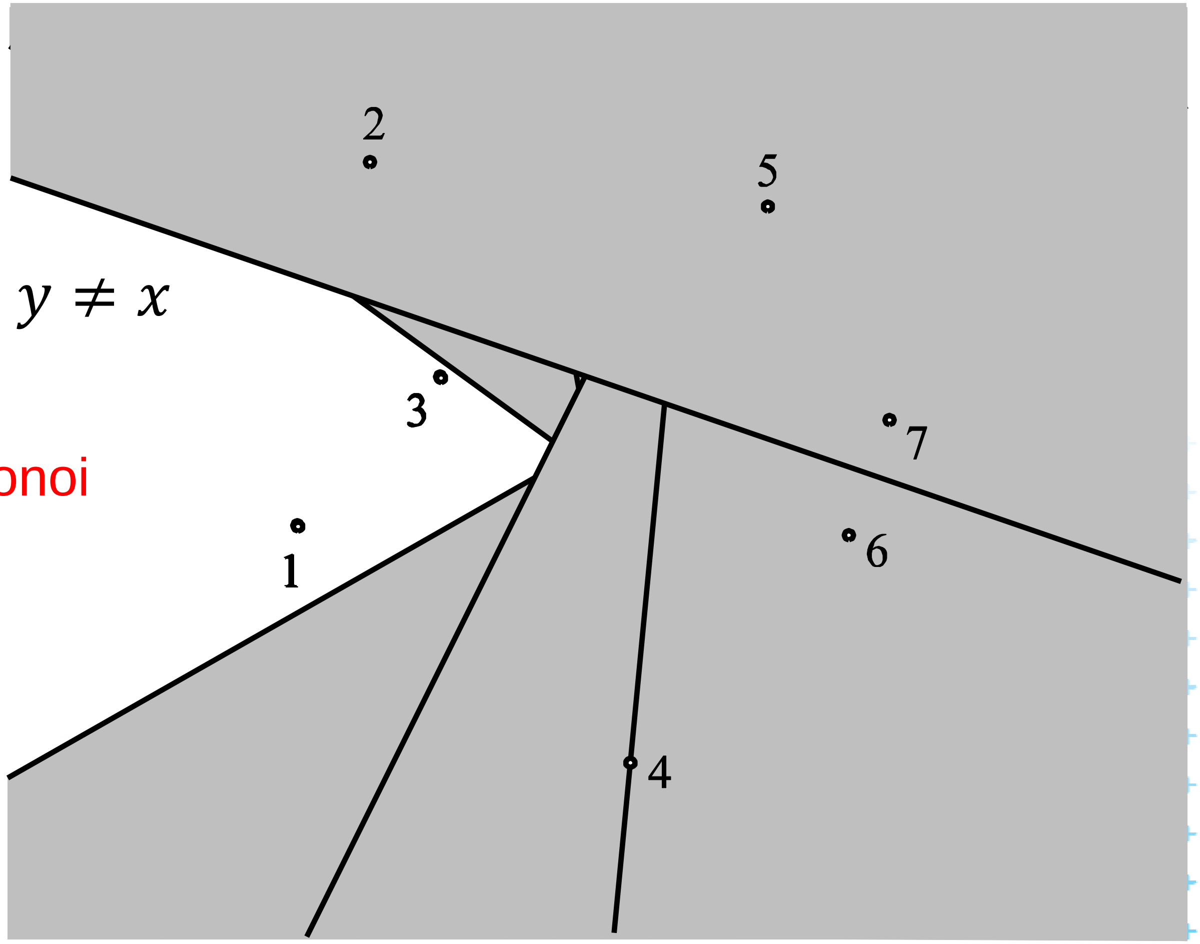
Computed as intersection of halfplanes, but we take "other sides" of bisectors

Construction of $V_{-1}(7)$

$$V_{-1}(y) = \bigcap_{x=1}^{n} h(y, x),\ y \neq x$$

Property

The farthest point Voronoi regions are convex and unbounded

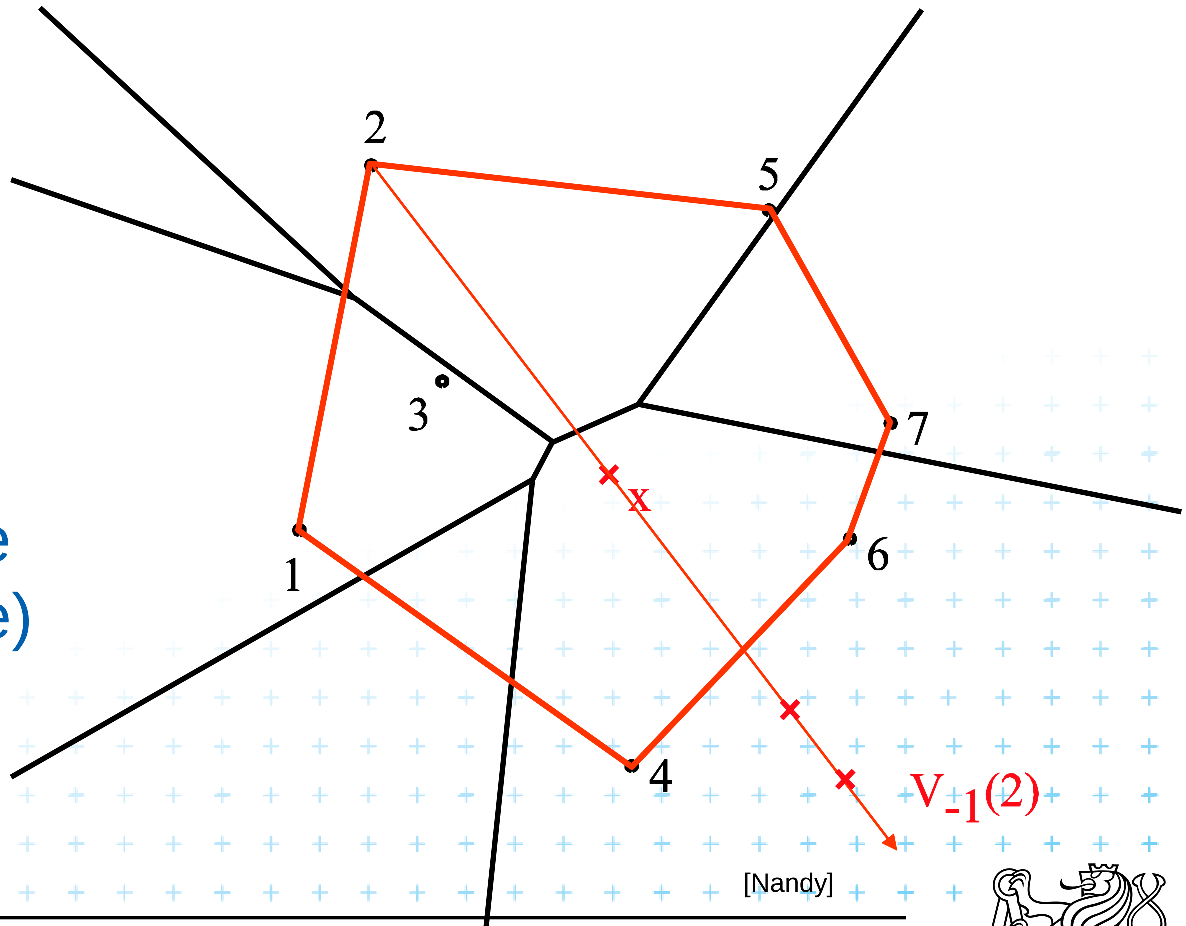


[Nandy]

Farthest-point Voronoi region

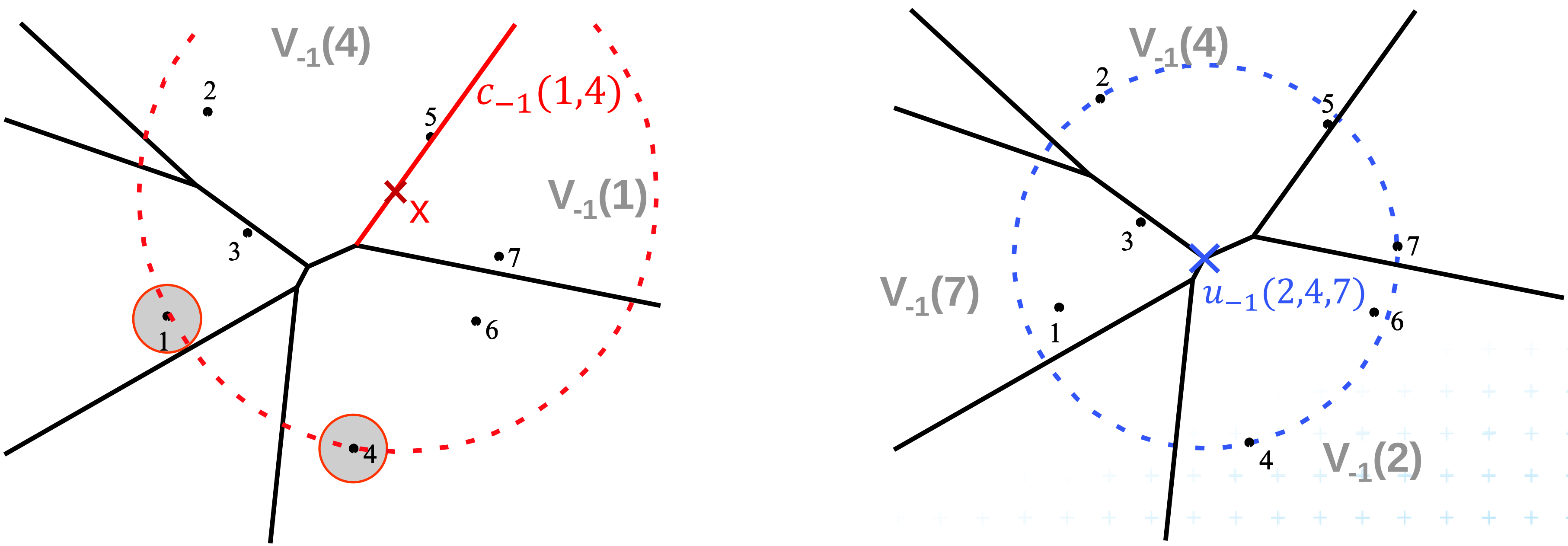
Properties:

- Only vertices of the convex hull have their cells in farthest Voronoi diagram
- The farthest point Voronoi regions are unbounded
- The farthest point Voronoi edges and vertices form a tree (in the graph sense)



[Nandy]

Farthest point Voronoi edges and vertices



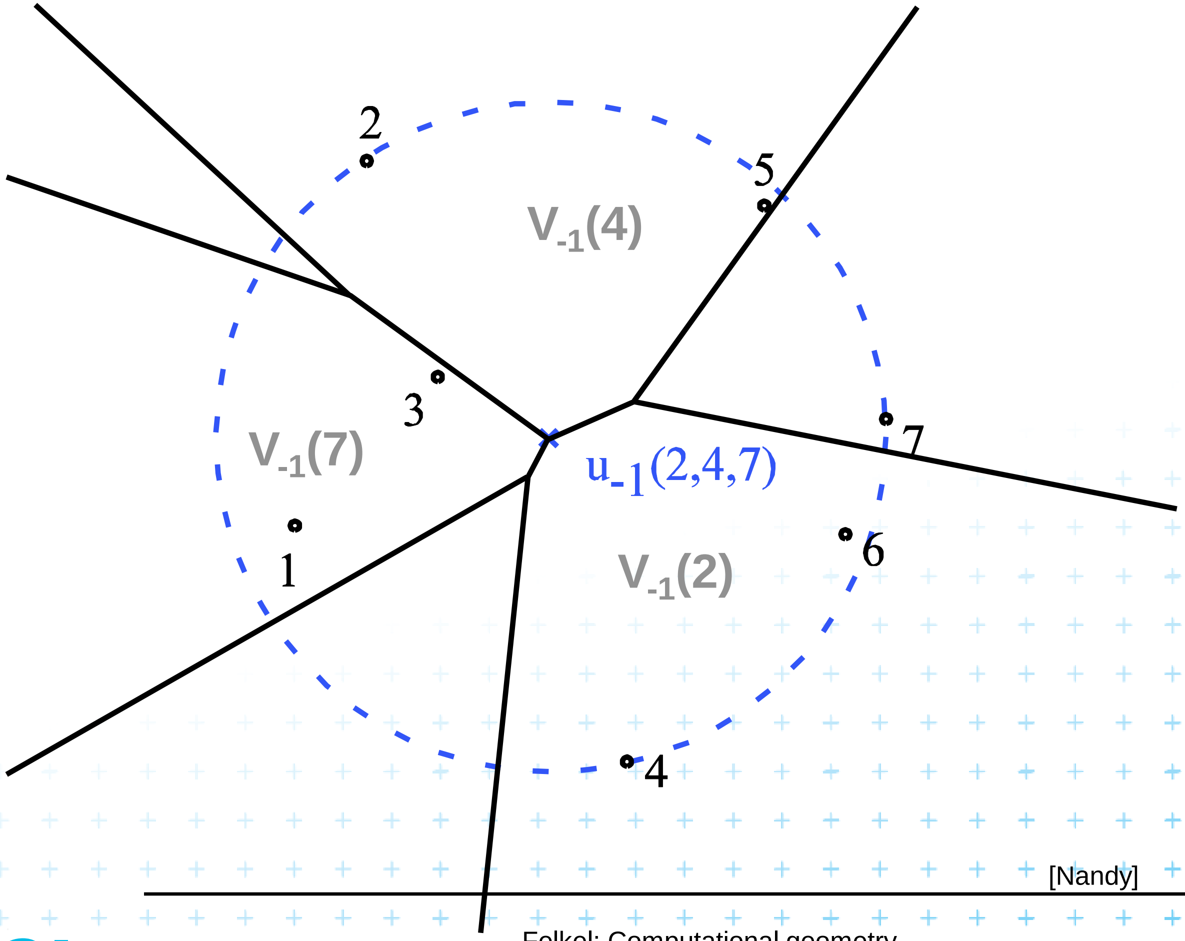
edge : set of points equidistant from 2 sites and closer to all the other sites

vertex : point equidistant from at least 3 sites and closer to all the other sites – Enclosing circle

[Nandy]

Application of $Vor_{-1}(P)$: Smallest enclosing circle

- Construct $Vor_{-1}(P)$ and find minimal circle with center in $Vor_{-1}(P)$ vertices or on edges



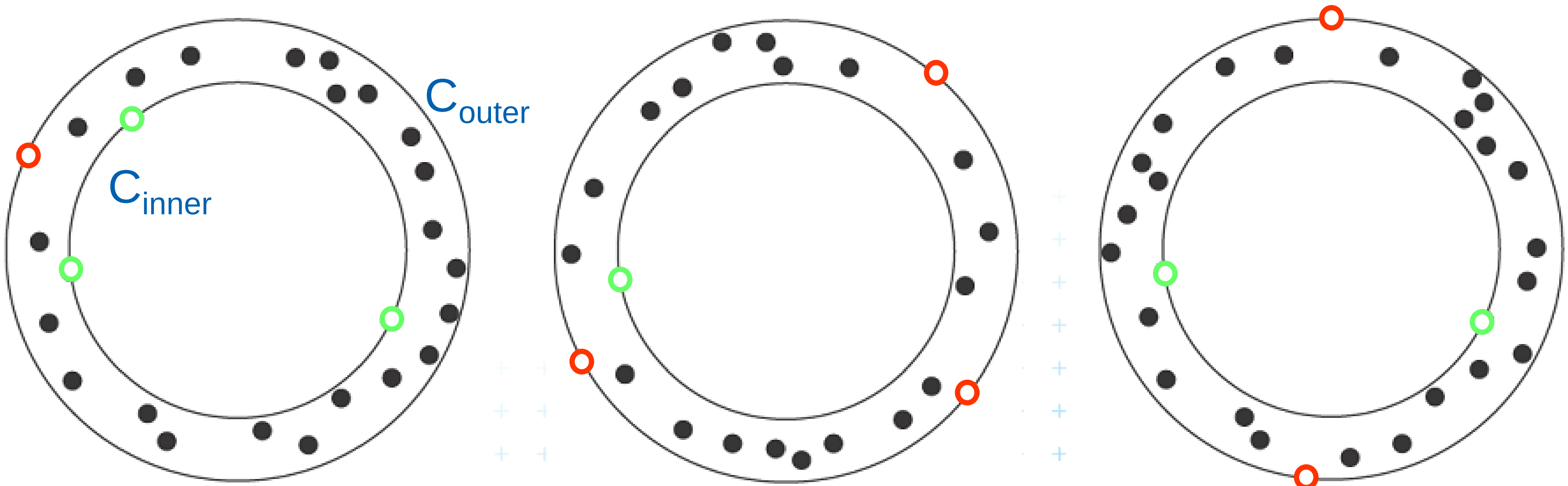
[Nandy]

Farthest-point Voronoi diagrams example

Roundness of manufactured objects

- Input: set of measured points in 2D
- Output: width of the smallest-width annulus mezikruží s nejmenší šířkou (region between two concentric circles C_{inner} and C_{outer})

Three cases to test – one will win:



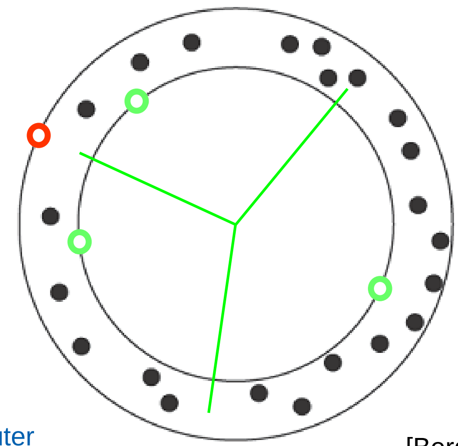
a) 3 in – 1 out b) 1 point in – 3 out c) 2 in – 2 out

[Berg]

Smallest width annulus – cases with 3 pts

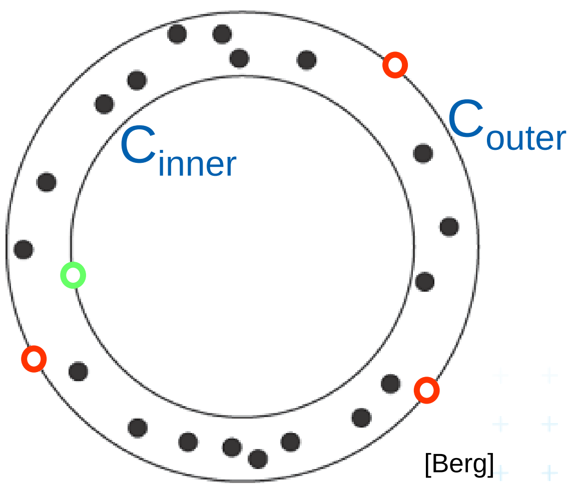
a) C_{inner} contains at least 3 points

- Center is the *vertex of normal Voronoi diagram* (1st order VD)
- The remaining point on C_{outer} in O(n) for each vertex
  - ⇒ not the largest (inscribed) empty circle - as discussed on seminar as we must test all VD vertices in combination with point on C outer
  - ⇒ $O(n^2)$



[Berg]

3 in – 1 out



[Berg]

1 point in – 3 out

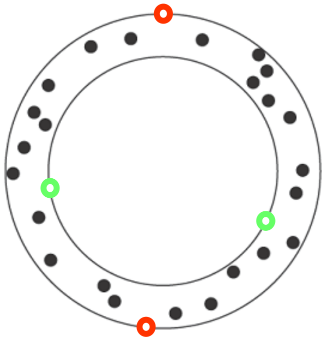
b) C_{outer} contains at least 3 points

- Center is the *vertex of the farthest Voronoi diagram*
- The remaining point on C_{inner} in O(n)
  - ⇒ not the smallest enclosing circle - as discussed on seminar as we must test all vertices **in combination** with point on C inner
  - ⇒ $O(n^2)$

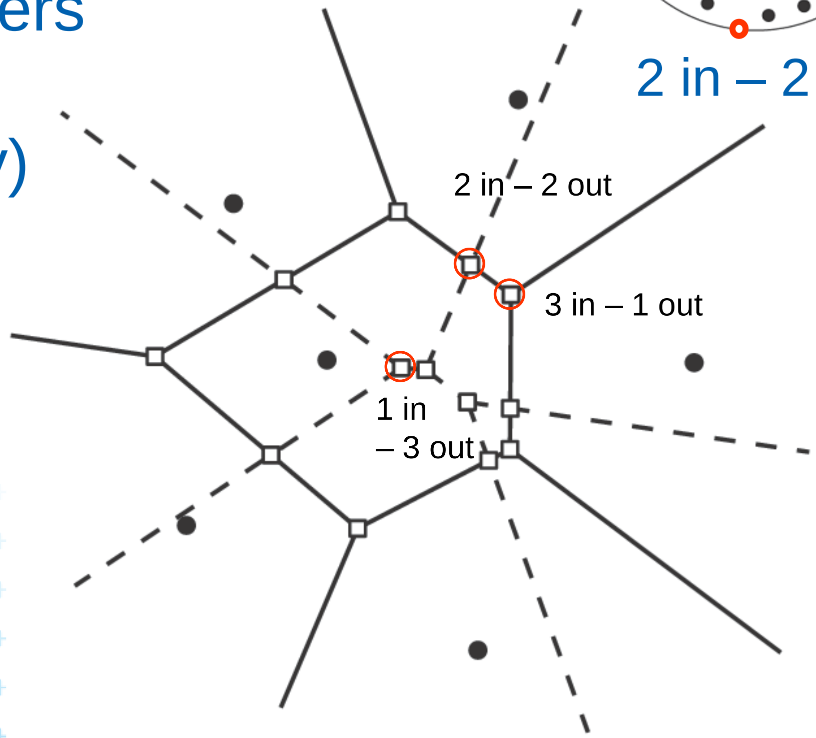
Smallest width annulus – case with 2+2 pts

c) C_{inner} and C_{outer} contain 2 points each

- Generate vertices of overlay of Voronoi (___) and farthest-point Voronoi (- - -) diagrams
  => $O(n^2)$ candidates for centers
  (we need only vertices, not the complete overlay)
- annulus computed in O(1) from center and 4 points (same for all 3 cases)
- $O(n^2)$



2 in – 2 out



[Berg]

Smallest width annulus

Smallest-Width-Annulus

Input: Set *P* of *n* points in the plane

Output: Smallest width annulus center and radii r and R (roundness)

1. Compute Voronoi diagram Vor(*P*)
   and farthest-point Voronoi diagram $Vor_{-1}(P)$ of *P*
2. For each vertex of Vor(*P*) (*r*) determine the *farthest point* (*R*) from *P*
   => $O(n)$ sets of four points defining candidate annuli – case a)
3. For each vertex of $Vor_{-1}(P)$ (*R*) determine the *closest point* (*r*) from *P*
   => $O(n)$ sets of four points defining candidate annuli – case b)
4. For every pair of edges Vor(*P*) and $Vor_{-1}(P)$ test if they intersect
   => another set of four points defining candidate annulus – c)
5. For all candidates of all three types
   chose the smallest-width annulus

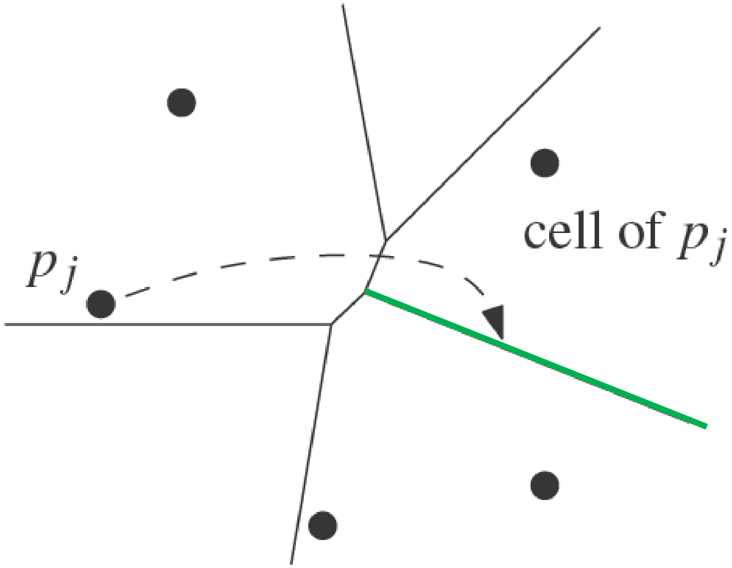
$O(n^2)$ time using $O(n)$ storage

1. $O(n \log n)$
2. $O(n^2)$
3. $O(n^2)$
4. $O(n^2)$
5. $O(n^2)$

Order n-1 VD construction

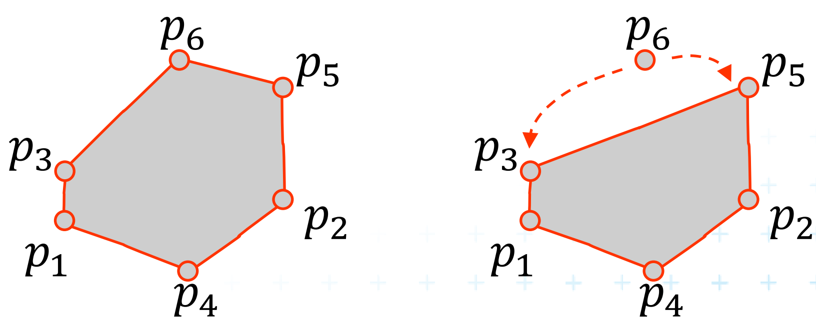
Modified DCEL for farthest-point Voronoi d

- Half-infinite edges -> we adapt DCEL
- Half-edges with origin in infinity
  - Special vertex-like record for origin in infinity
  - Store direction instead of coordinates
  - Next(e) or Prev(e) pointers undefined
- For each inserted site p_j
  - store a pointer to the most CCW half-infinite half-edge of its cell in DCEL



Idea of the algorithm

1. Create the convex hull
   and number the CH points randomly
2. Remove the points starting in the last of this random order and store *cw*(p_i) and *ccw*(p_i) points at the time of removal.
3. Include the points back and compute V_{-1}



p_i	$ccw(p_i)$	$cw(p_i)$
p_6	p_3	p_5
p_5	p_3	p_2
…		

Farthest-point Voronoi d. construction

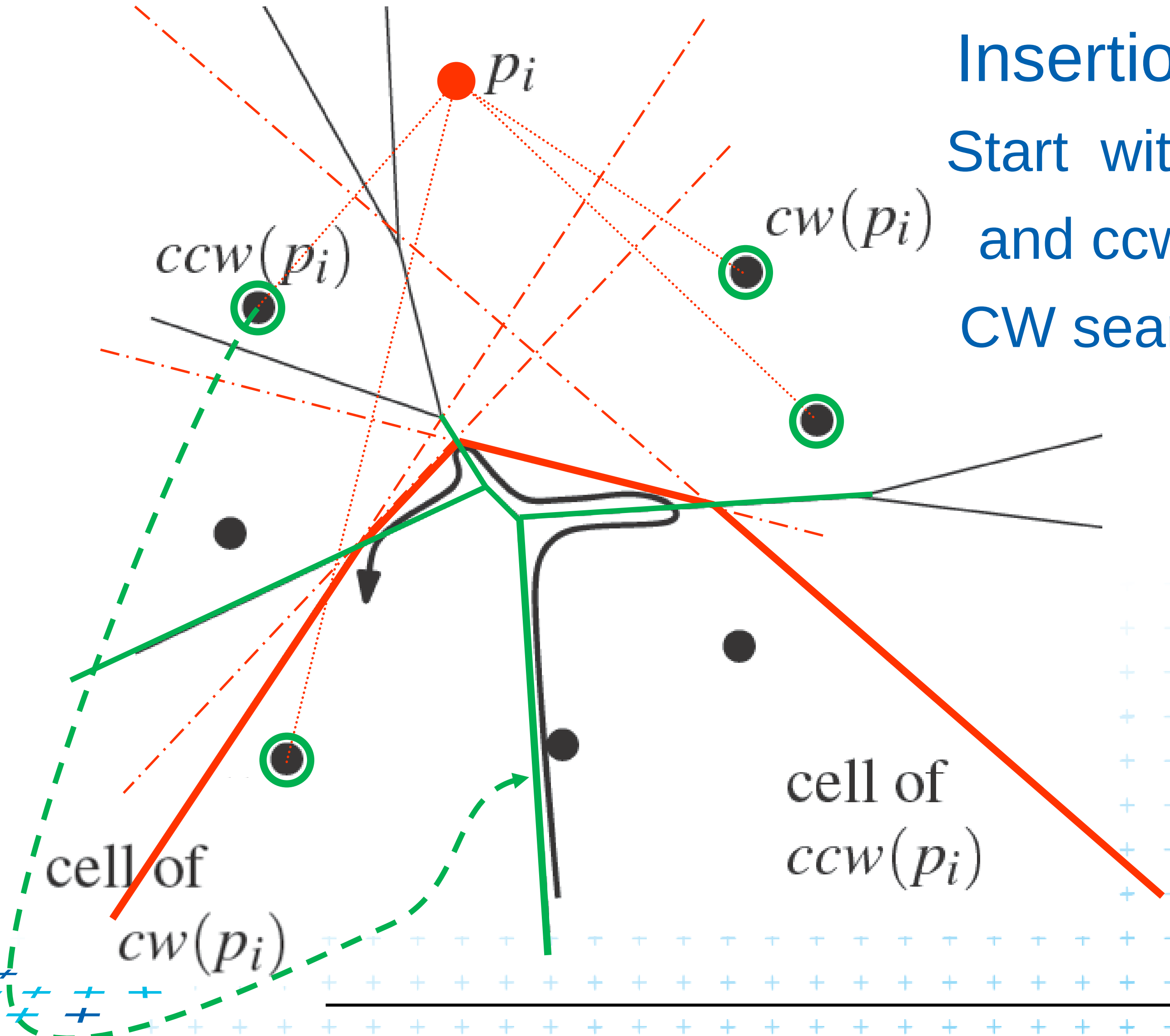
Farthest-pointVoronoi $O(n \log n)$ expected time in $O(n)$ storage

Input: Set of points *P* in plane

Output: Farthest-point VD $\text{Vor}_{-1}(P)$

1. Compute convex hull of *P*
2. Put points in CH(*P)* of *P* in random order $p_1, \dots, p_h$
3. Remove $p_h, \dots, p_4$ from the cyclic order (around the CH). When removing p_i, store the neighbors: *cw*(p_i) and *ccw*(p_i) at the time of removal. (This is done to know the neighbors needed in step 6.)
4. Compute $\text{Vor}_{-1}(\{p_1, p_2, p_3\})$ as init
5. **for** i = 4 **to** *h* **do**
6. Add site p_i to $\text{Vor}_{-1}(\{p_1, p_2, \dots, p_{i-1}\})$ between site *cw*(p_i) and *ccw*(p_i)
7. - start at most CCW edge of the cell *ccw*(p_i)
8. - continue CW to find intersection with bisector(*ccw*(p_i), p_i)
9. - trace borders of Voronoi cell p_i in CCW order, add edges
10. - remove invalid edges inside of Voronoi cell p_i

Farthest-point Voronoi d. construction



Insertion of site p_i

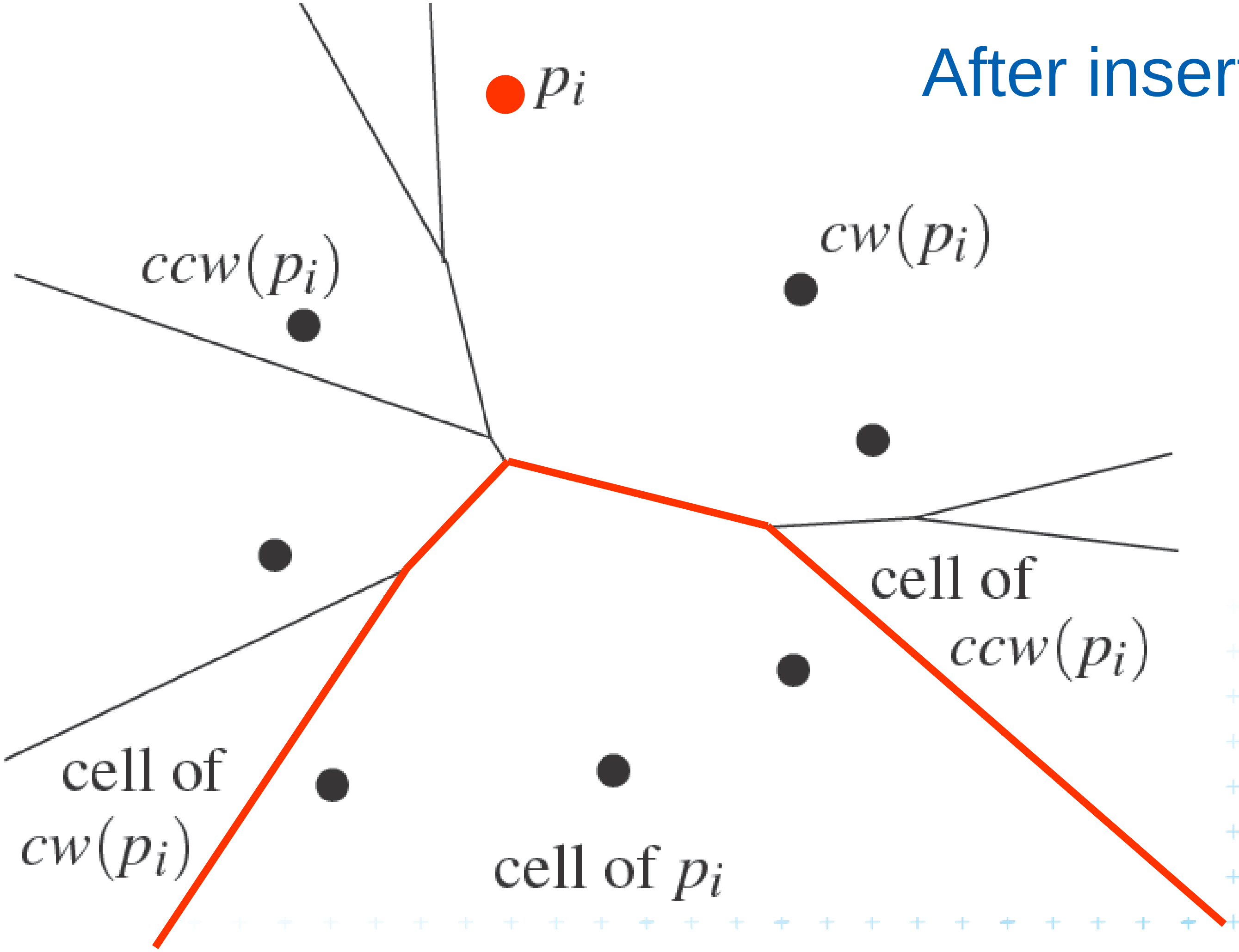
Start with site ccw(p_i)

and ccw edge of its cell

CW search of intersection

Farthest-point Voronoi d. construction

After insertion of site p_i



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