DEPARTMENT OF COMPUTER GRAPHICS AND INTERACTION

VORONOI DIAGRAM PART II

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https://cw.felk.cvut.cz/doku.php/courses/a4m39vg/start

Based on [Berg], [Reiberg] and [Nandy]

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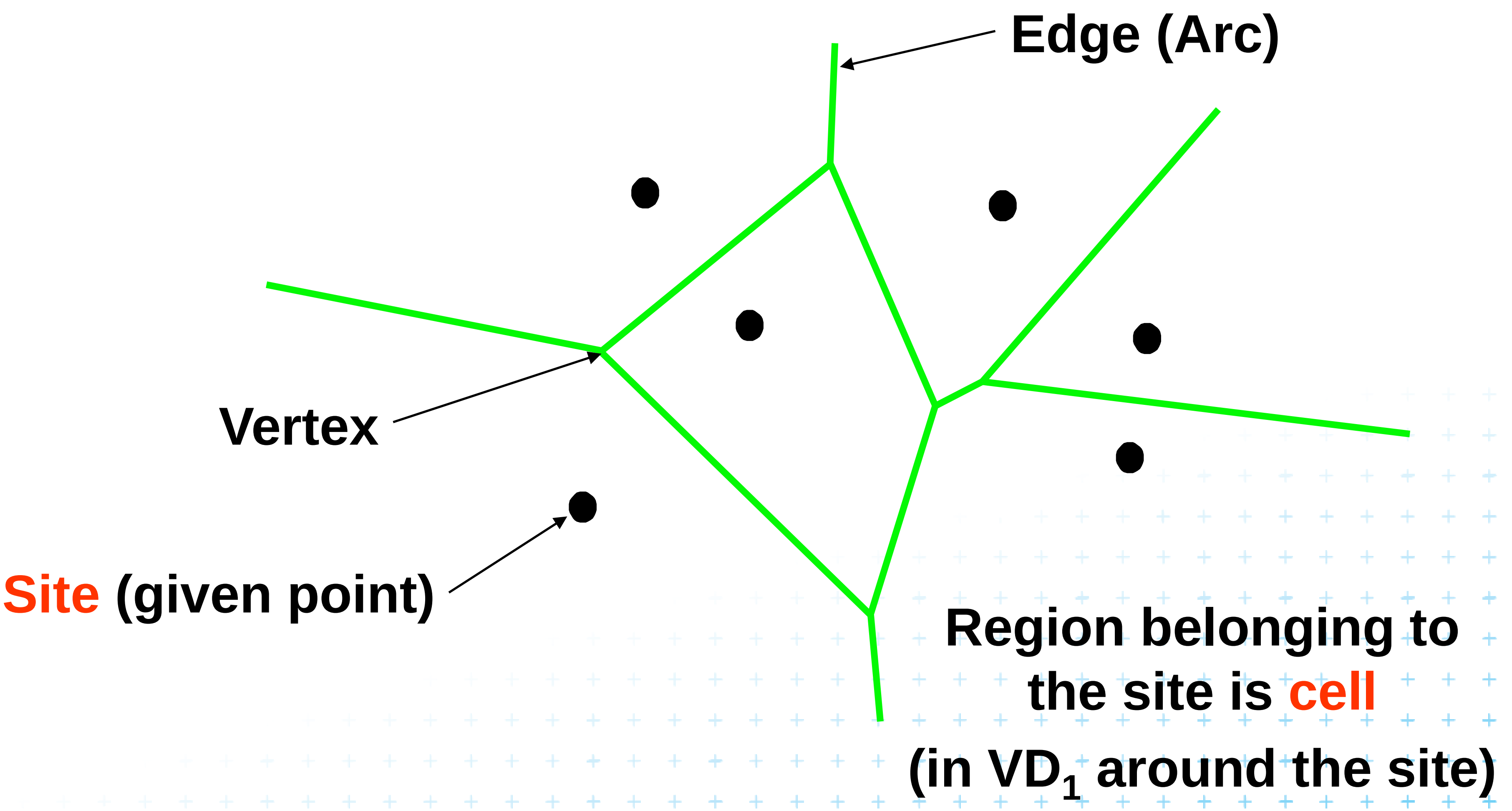
Talk overview

- Incremental construction
- Voronoi diagram of line segments
- VD of order k
- Farthest-point VD

Summary of the VD terms

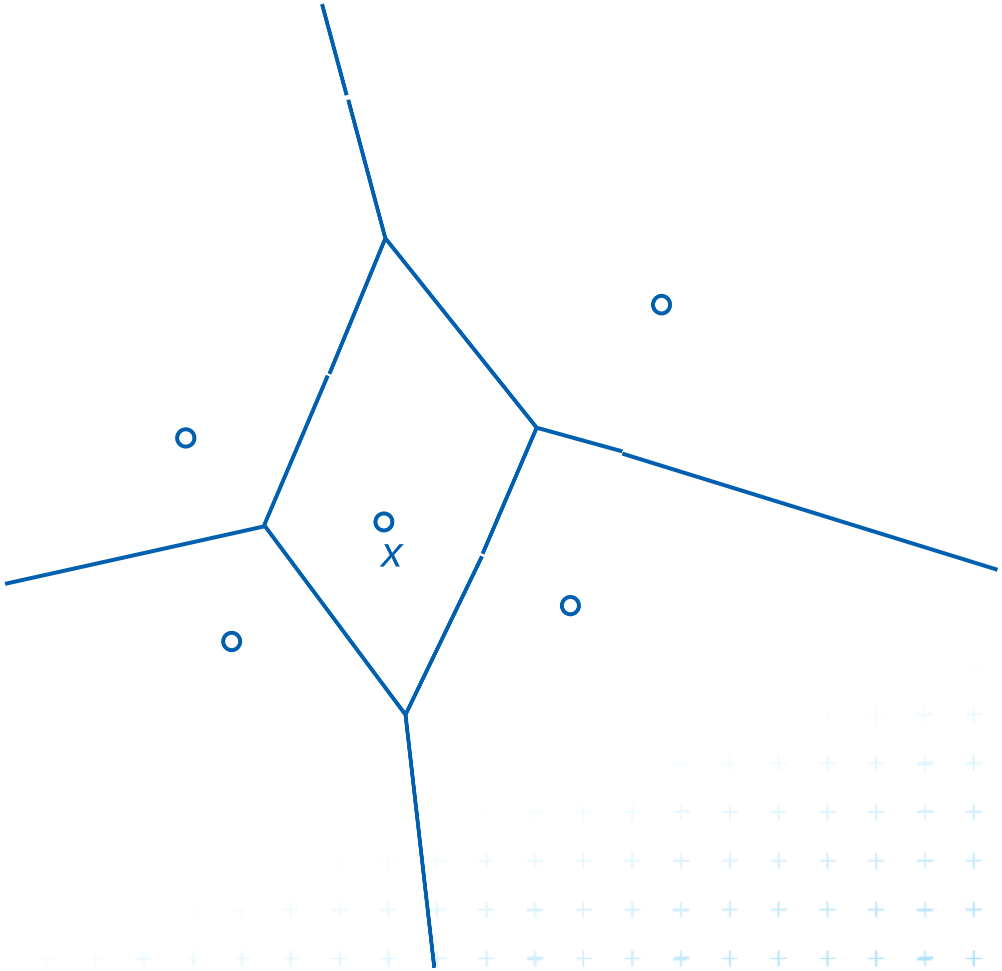
- Site = input point, line segment, …
- Cell = area belonging to the site,
  in VD_1 locus of points nearest to the site
- Edge, arc = part of Voronoi diagram
  (border between cells)
- Vertex = intersection of VD edges

Summary of the VD terms



Incremental construction

Incremental construction – bounded cell



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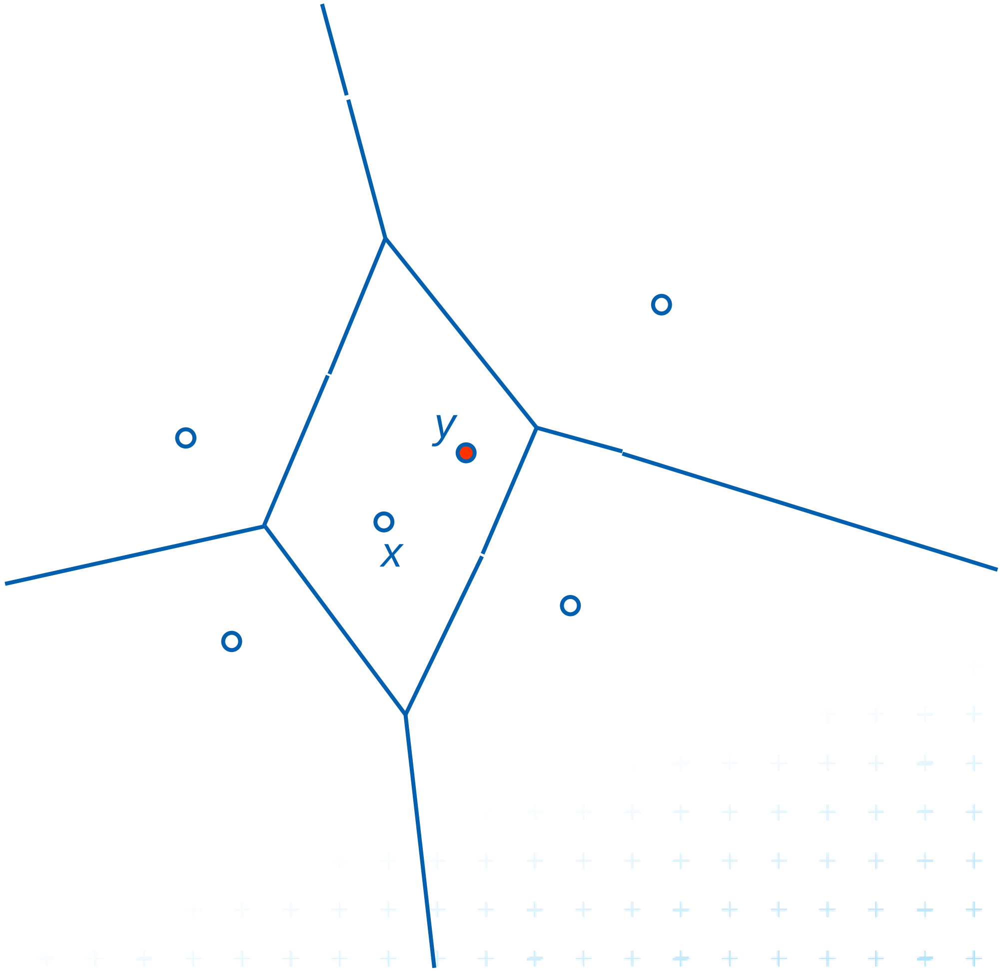

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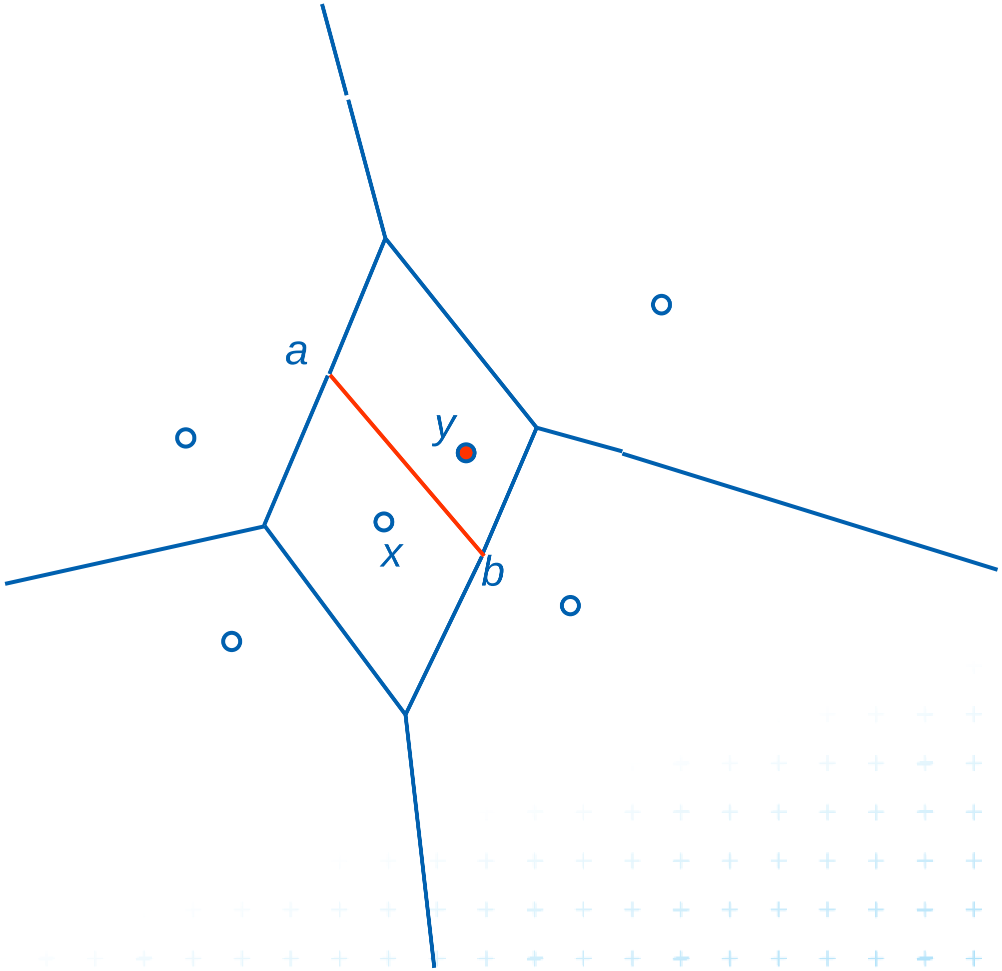

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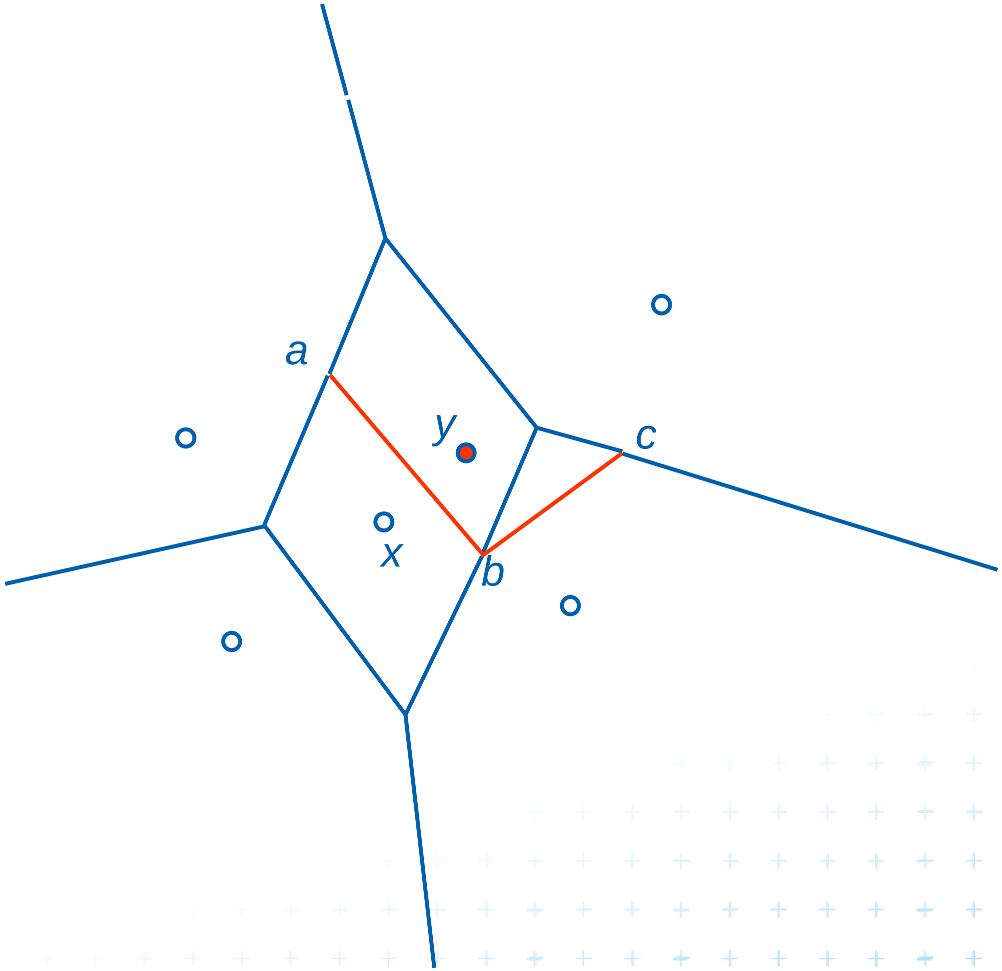

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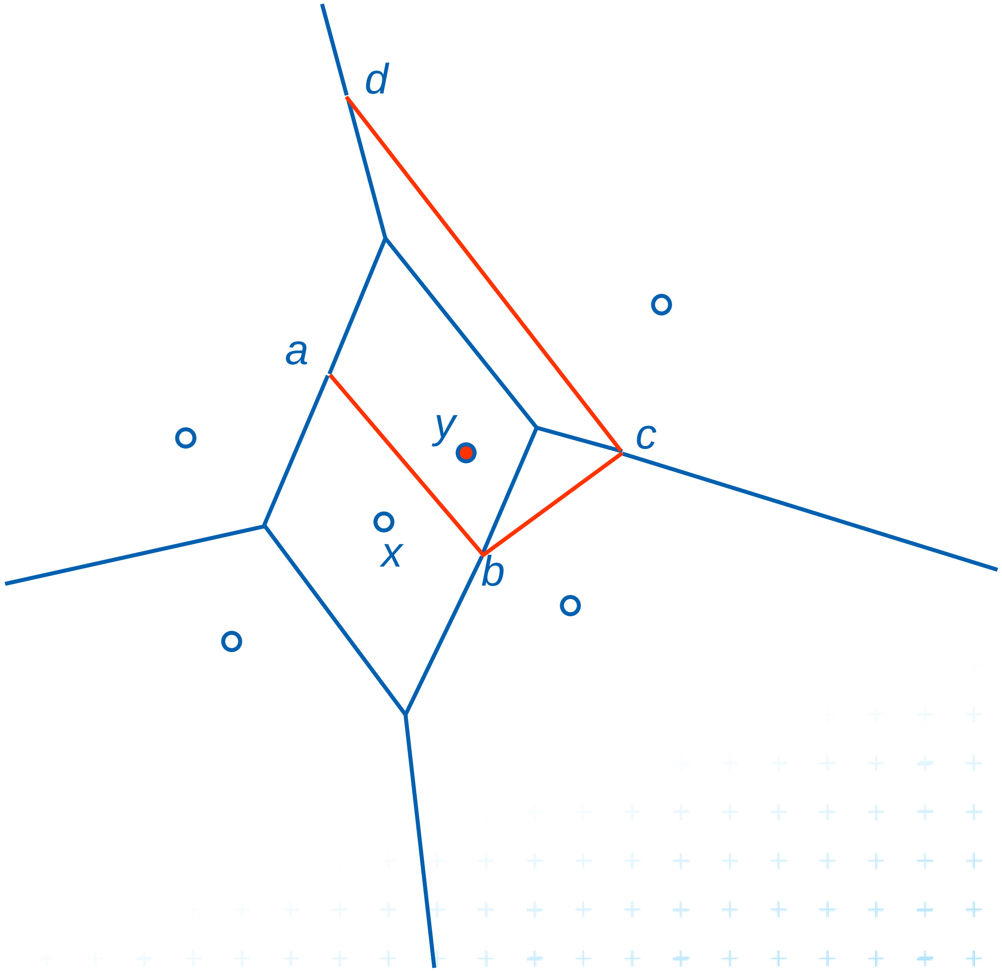

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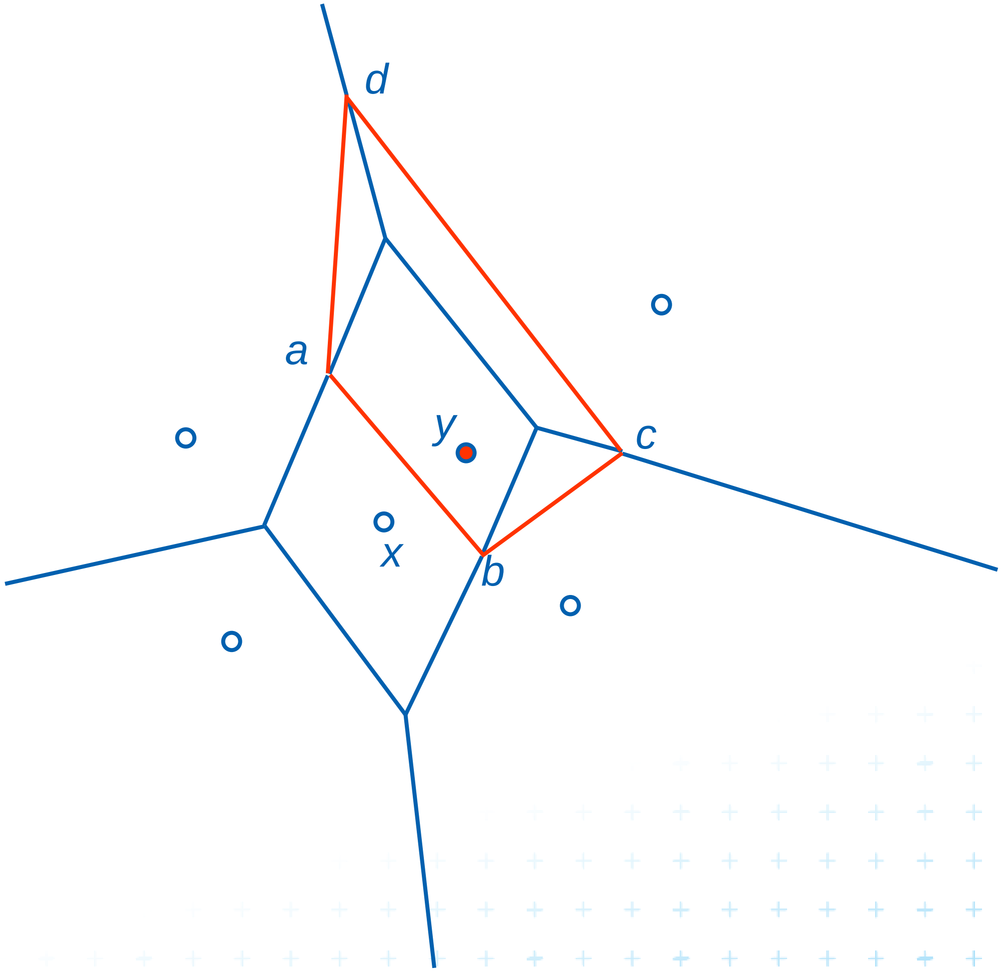

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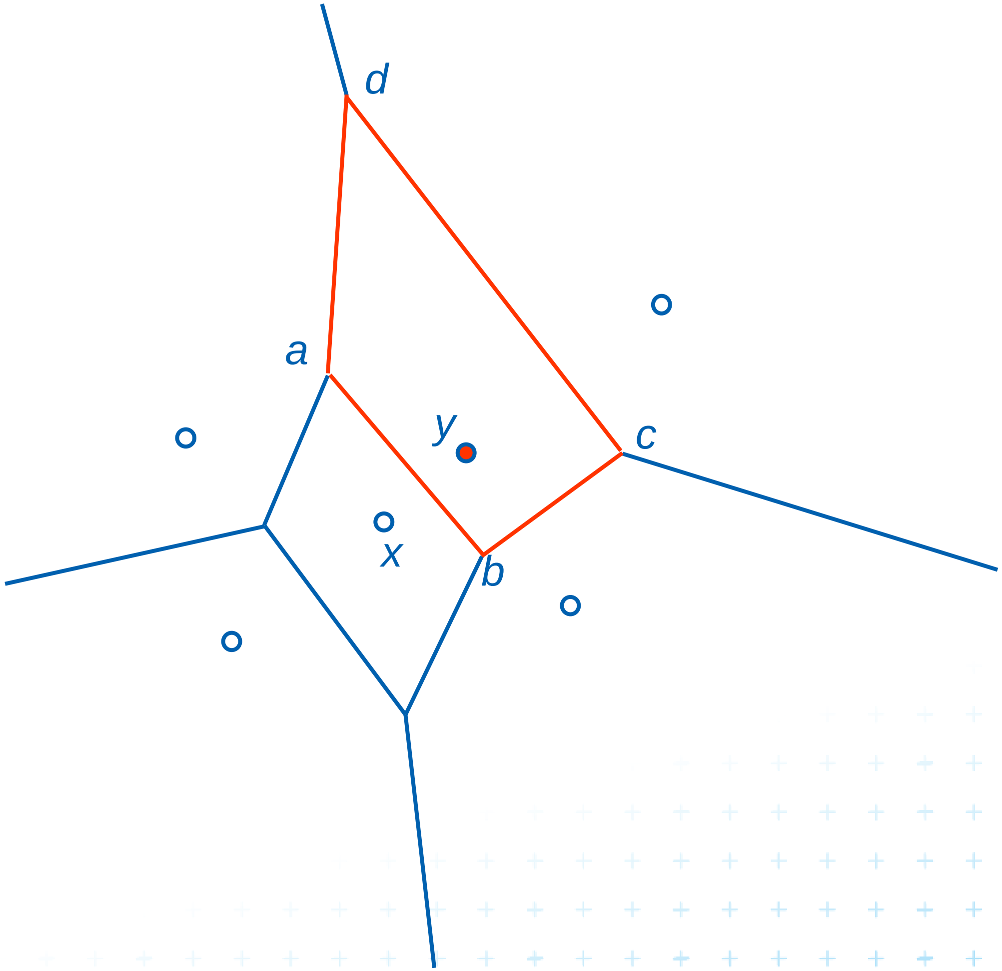

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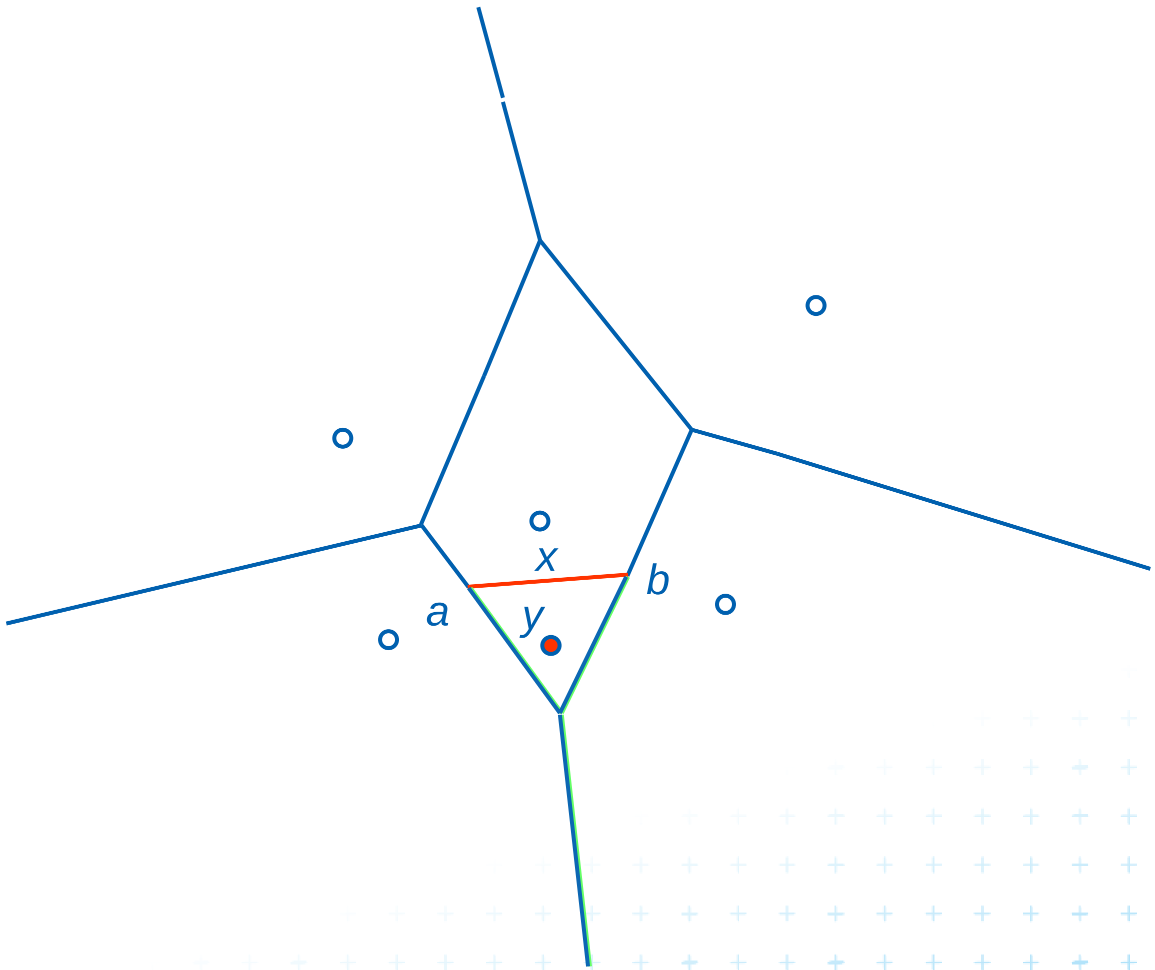

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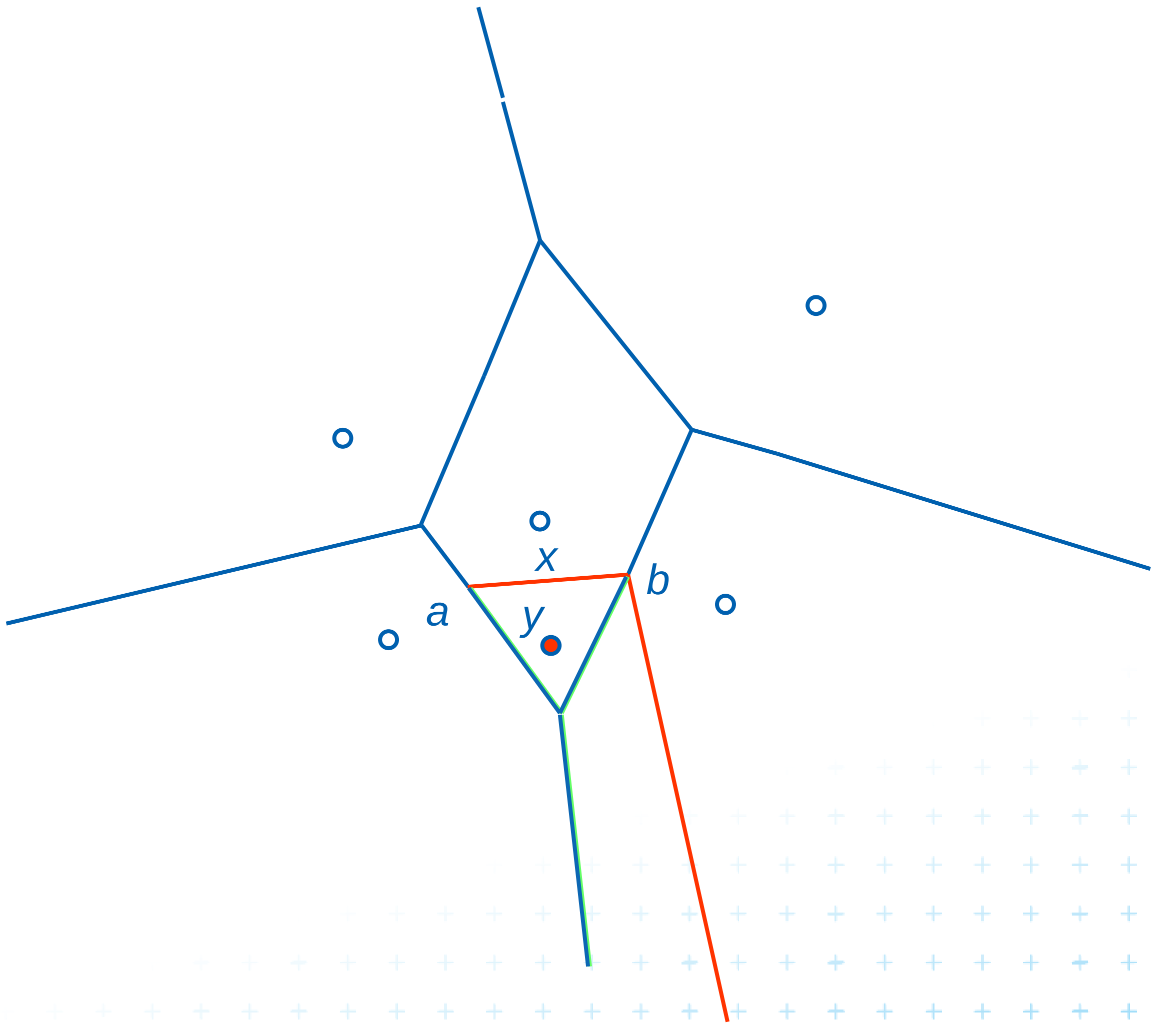

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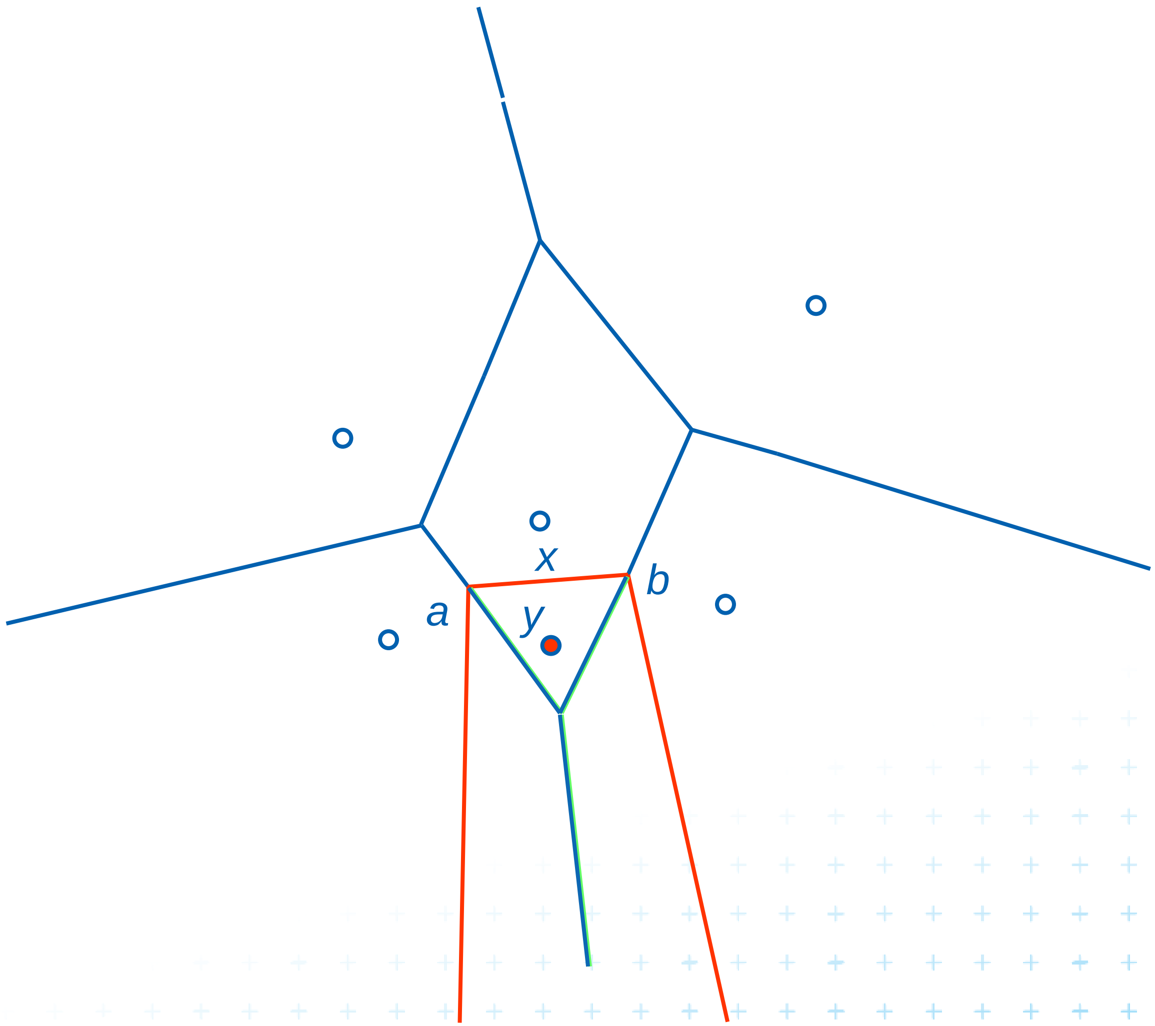

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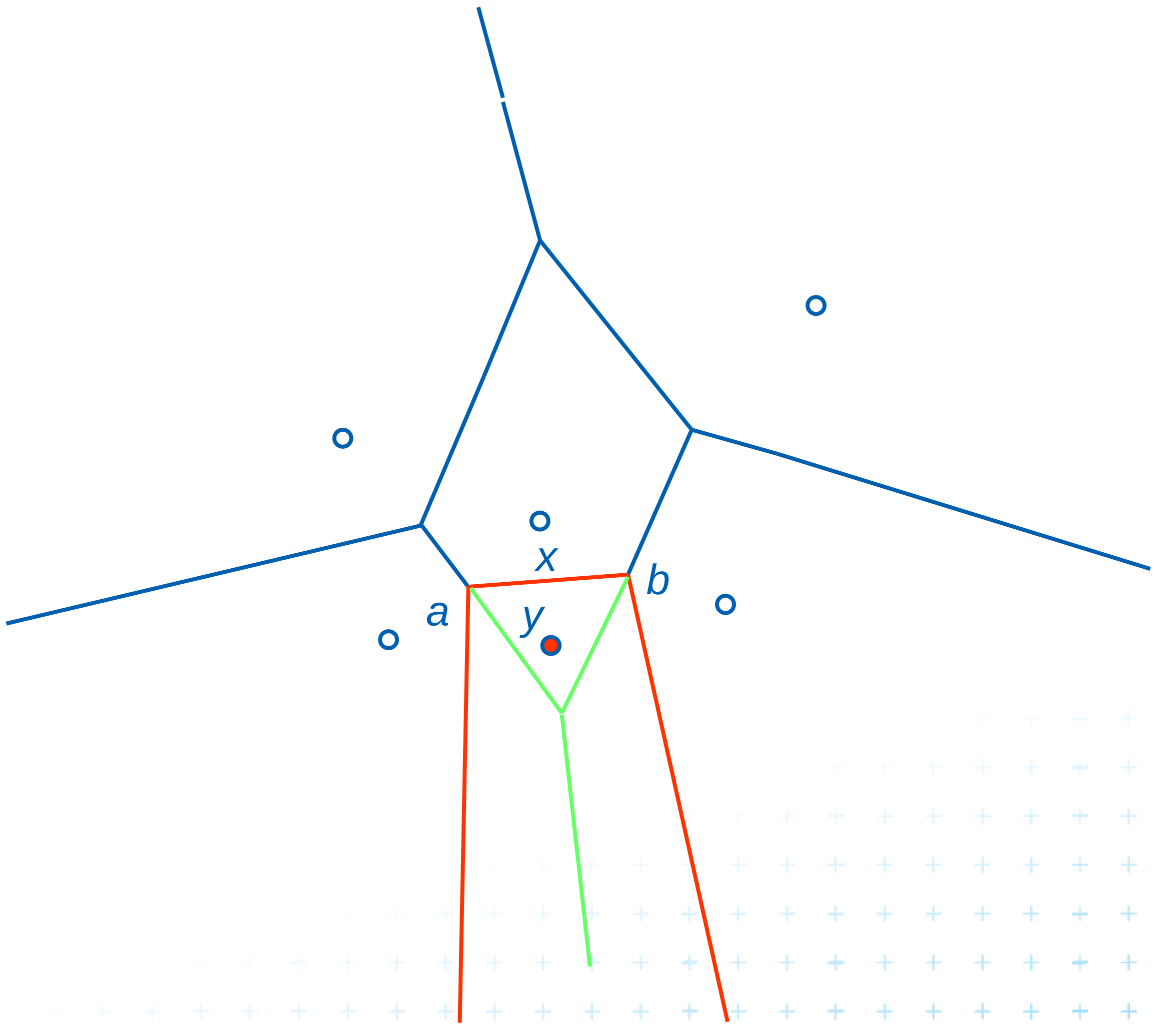

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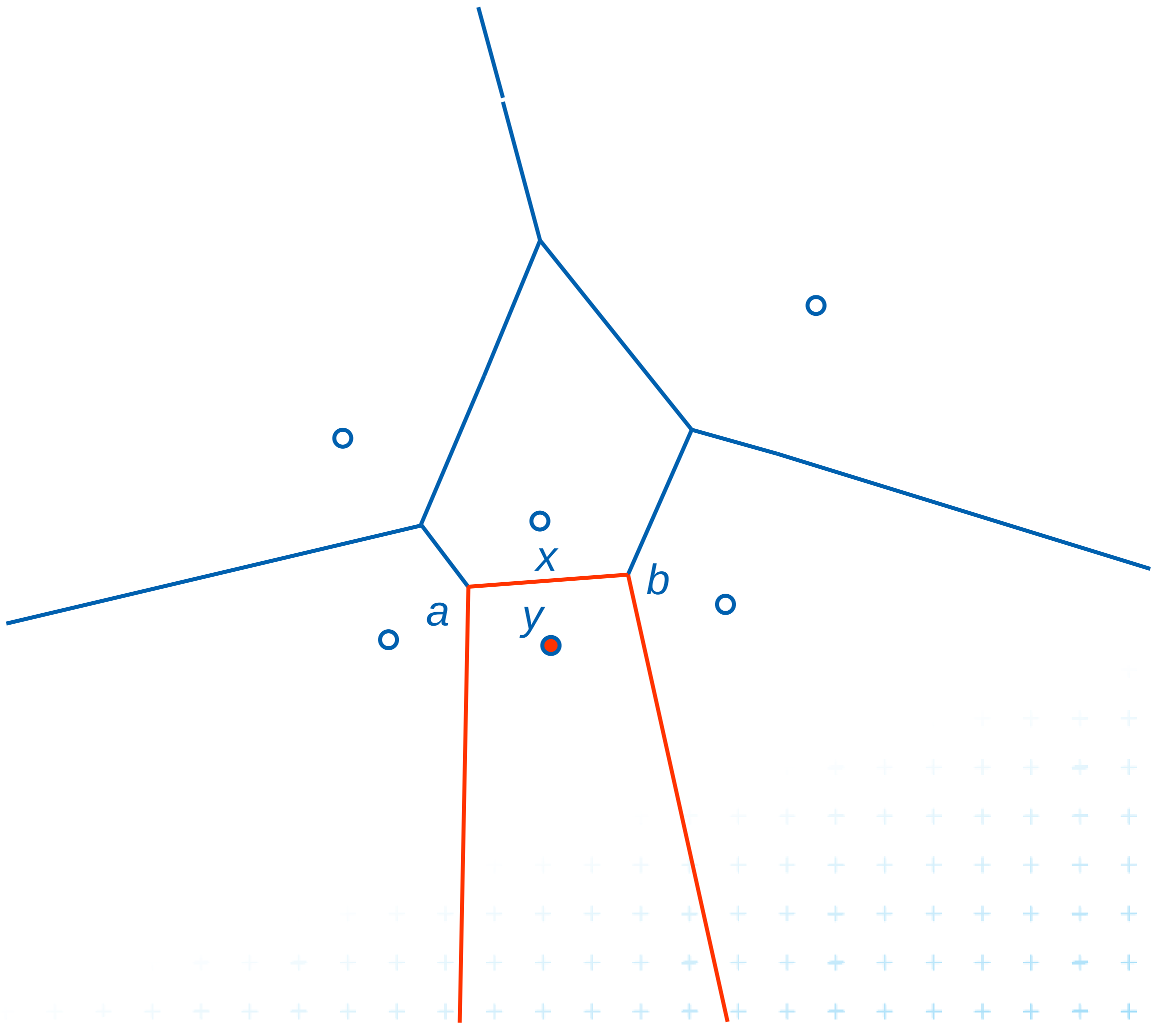

Incremental construction algorithm

InsertPoint(*S*, Vor(*S*), y) … **y = a new site**

Input: Point set *S*, its Voronoi diagram, and inserted point $y \notin S$

Output: VD after insertion of ***y***

1. Find the site *x* in which cell point ***y*** falls, …O(log *n*)
2. Detect the intersections {*a*,*b*} of bisector *L*(*x*,***y***) with cell *x* boundary
   => create the first edge *e* = *ab* on the border of site *x* …O(*n*)
3. site *z* = neighbor site across the border with intersection *b* …O(1)
4. Set start intersection point *p* = *b*, set new intersection *c* = undef
5. **while**(exists(*p*) *and* *c* ≠ *a*) // trace the bisectors from *b* in one direction
   a. Detect intersection *c* of *L*(***y***,*z*) with border of cell *z*
   b. Report Voronoi edge *pc*
   c. *p* = *c*, *z* = neighbor site across border with intersec. *c*

   (steps a–c: …O(n^2))
5. **if**(*c* ≠ *a*) **then** // open site → trace the bisectors from *a* in other direction
   a. *p* = *a*
   b. *Similarly as in steps 3,4,5 with a*

O(n^2) worst-case, O(*n*) expected time for some distributions

DCGI

Voronoi diagram of line segments

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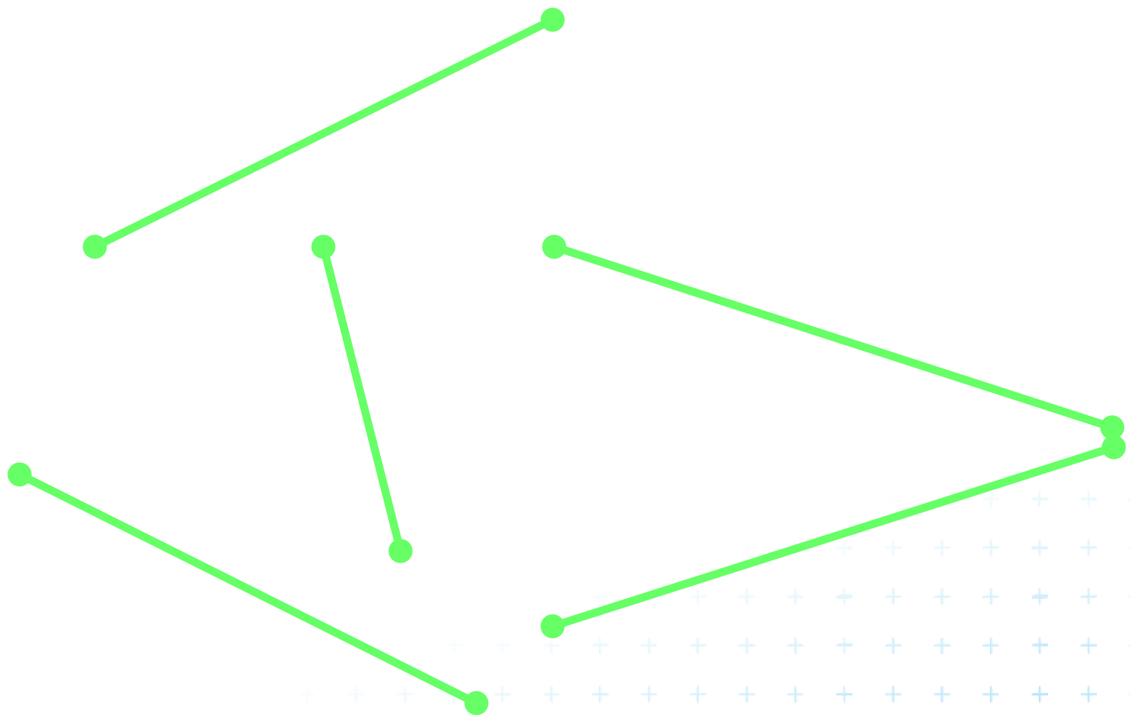
Input: $S = \{s_1, \ldots, s_n\}$ = set of n disjoint line segments (sites)



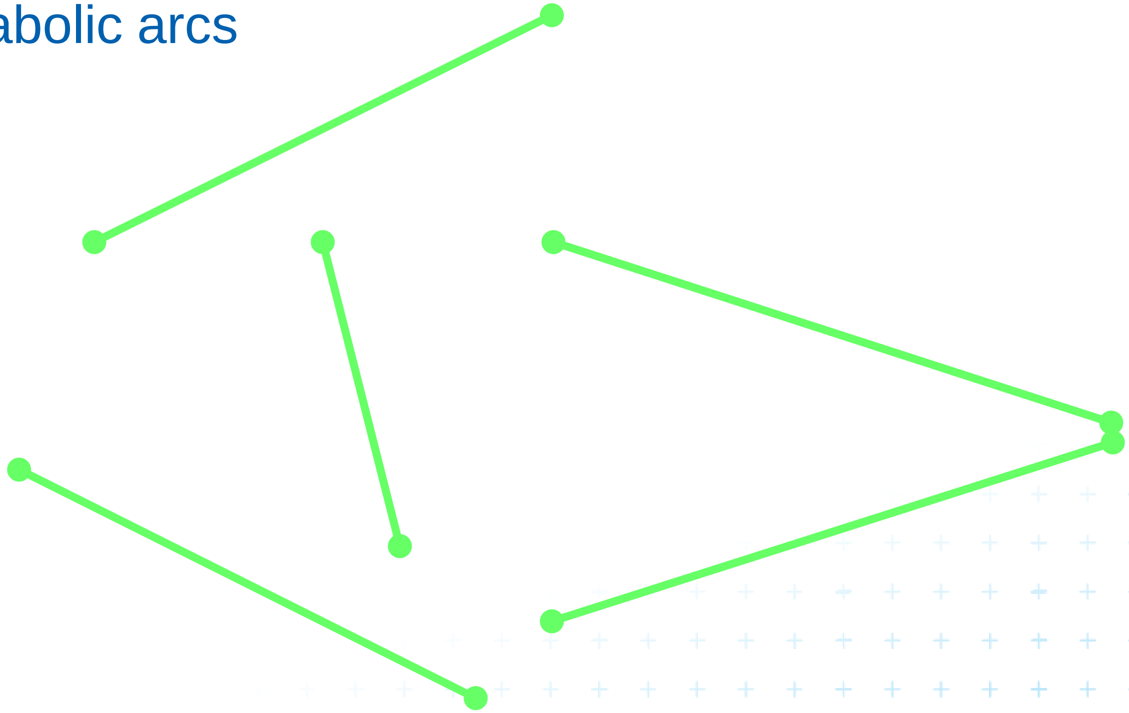
[Berg]

Voronoi diagram of line segments

Input: $S = \{s_1, \ldots, s_n\}$ = set of n disjoint line segments (sites)

VD: line segments

parabolic arcs



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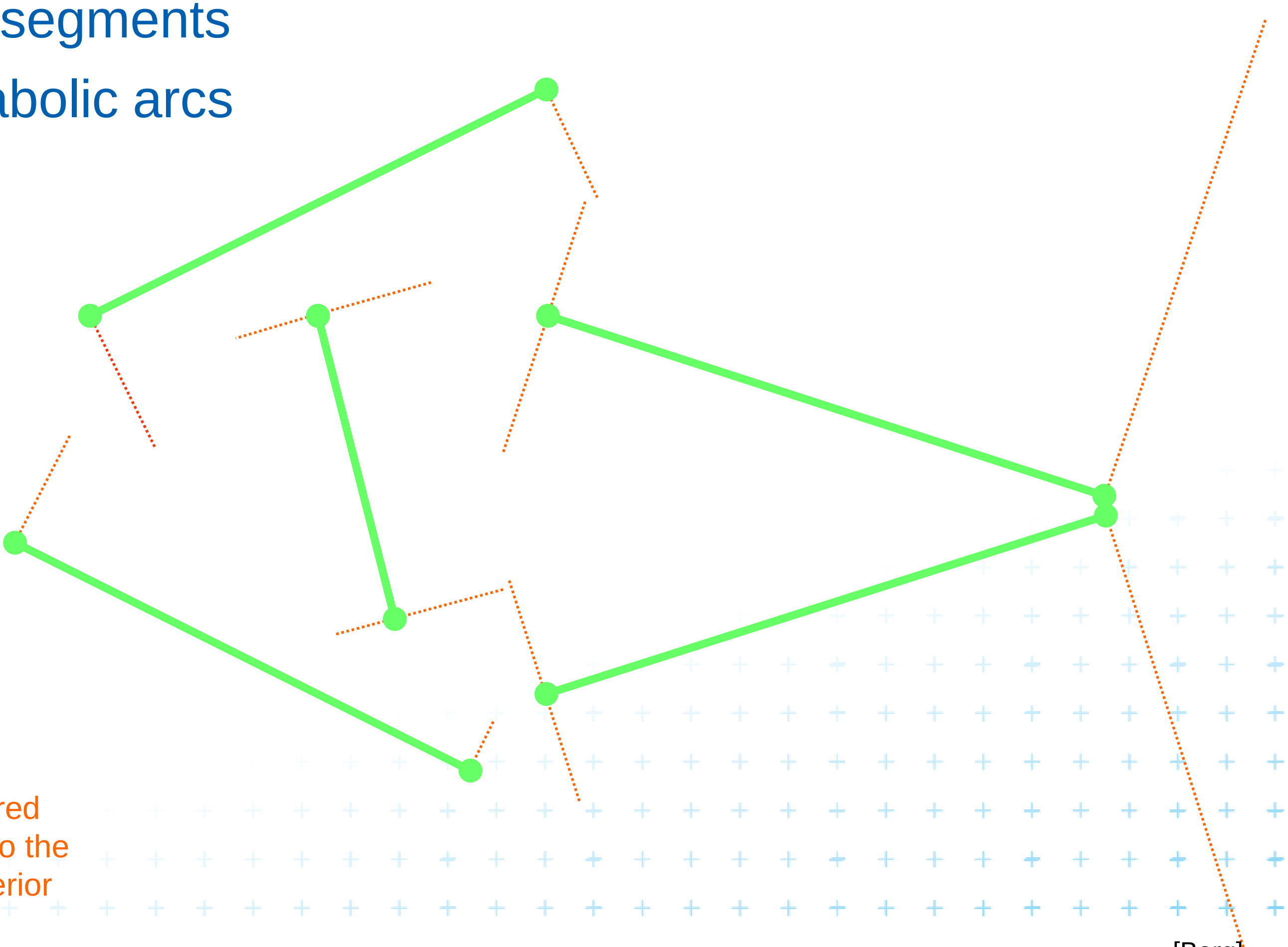

Distance measured perpendicularly to the line segment interior

[Berg]

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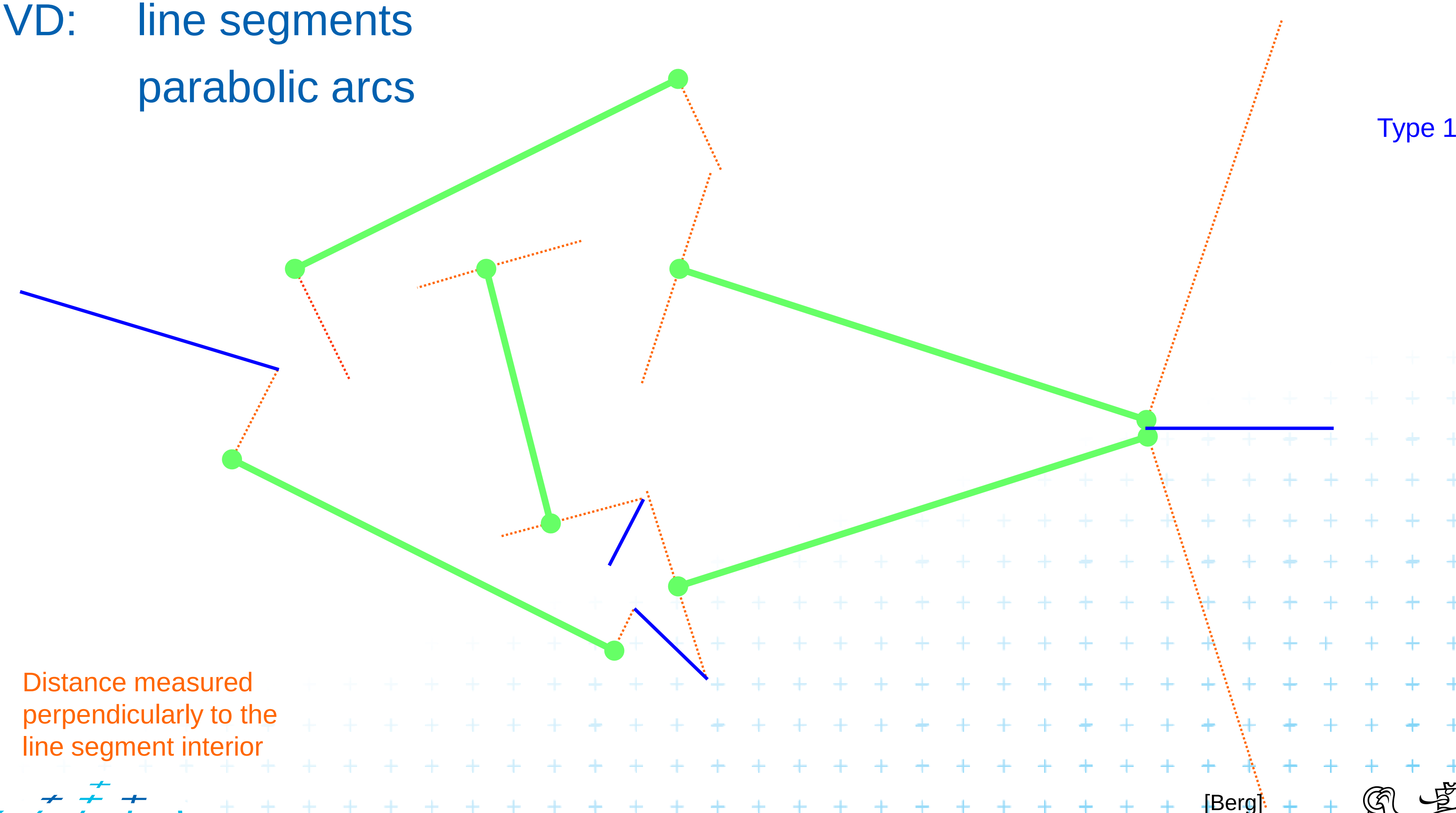

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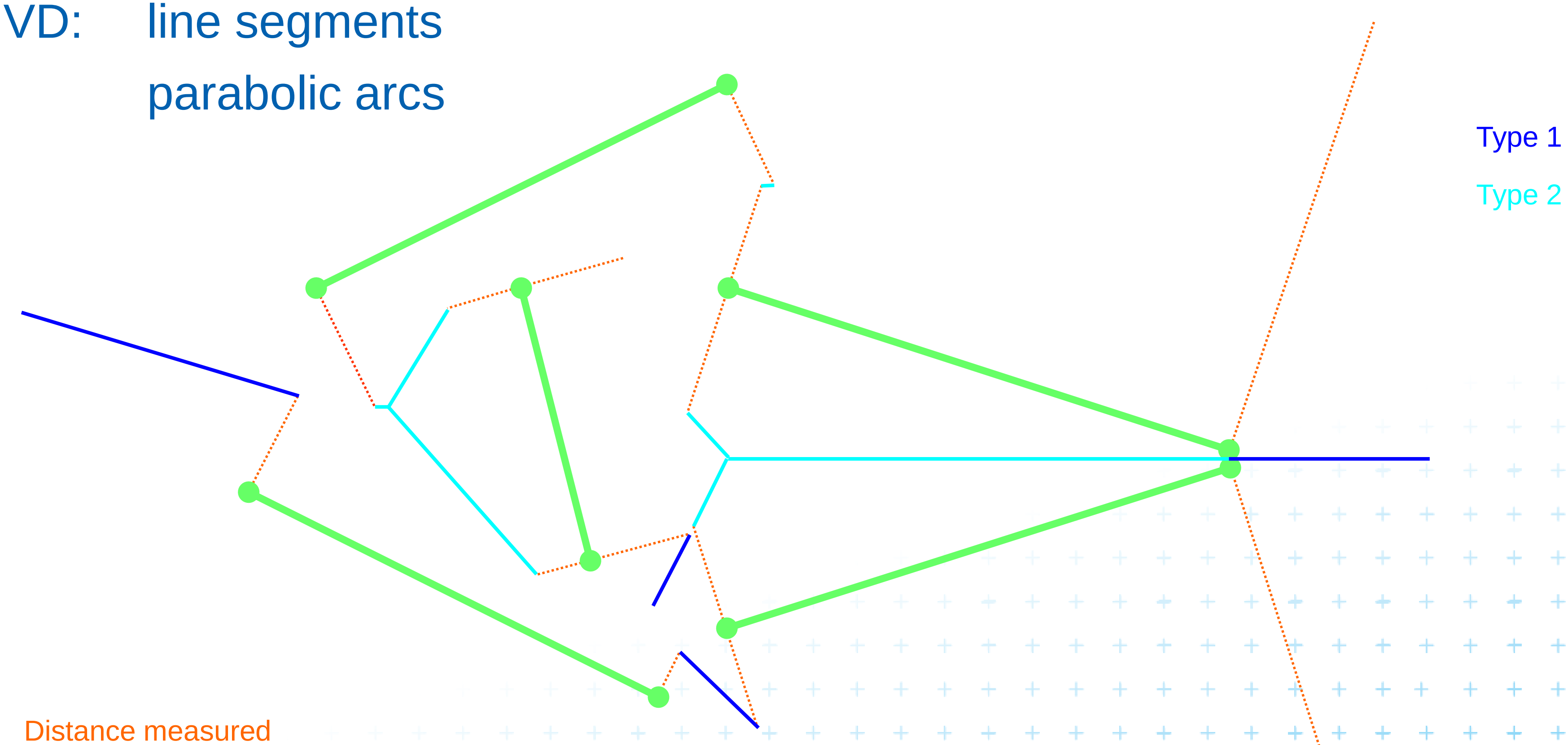

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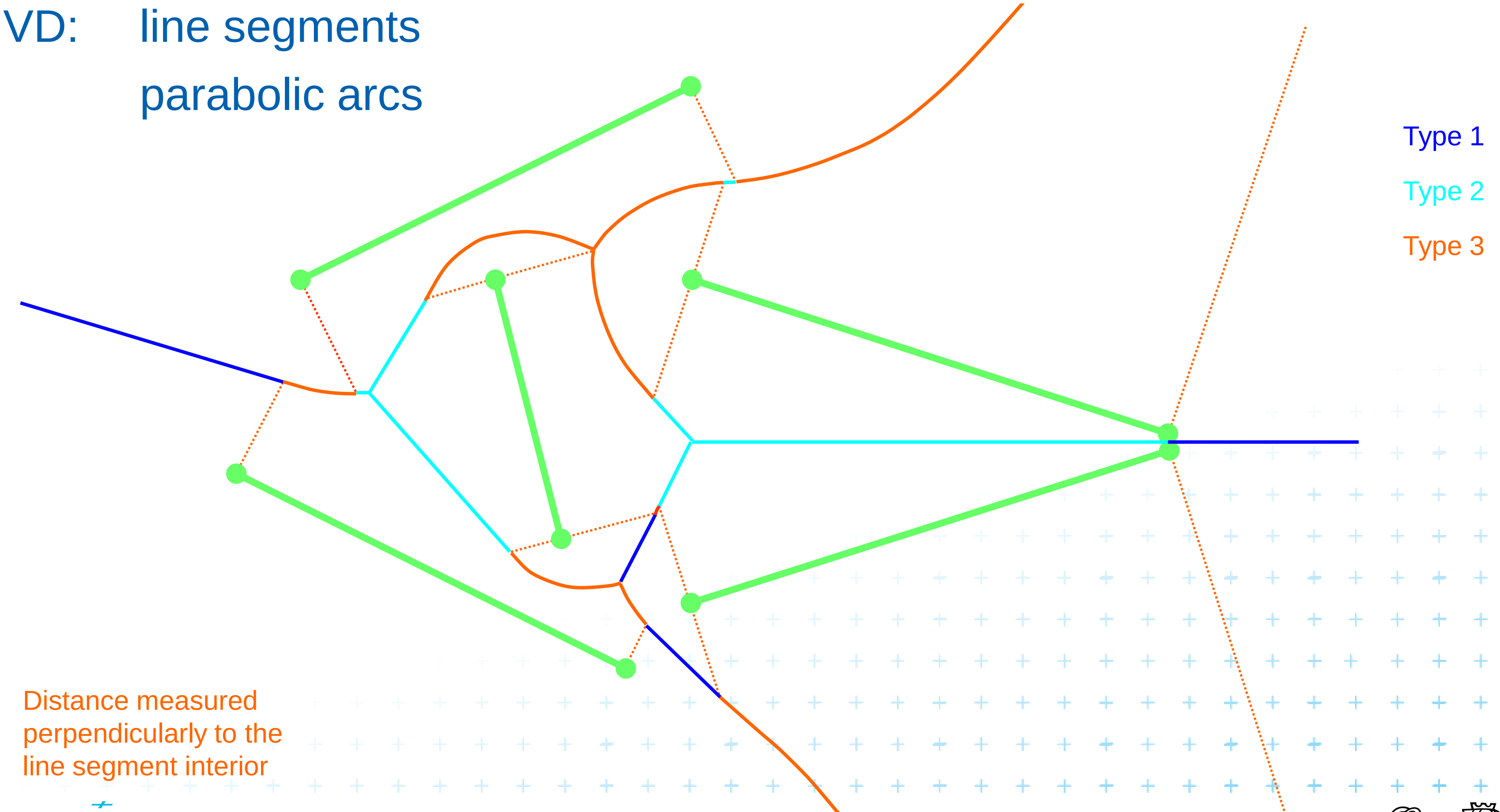

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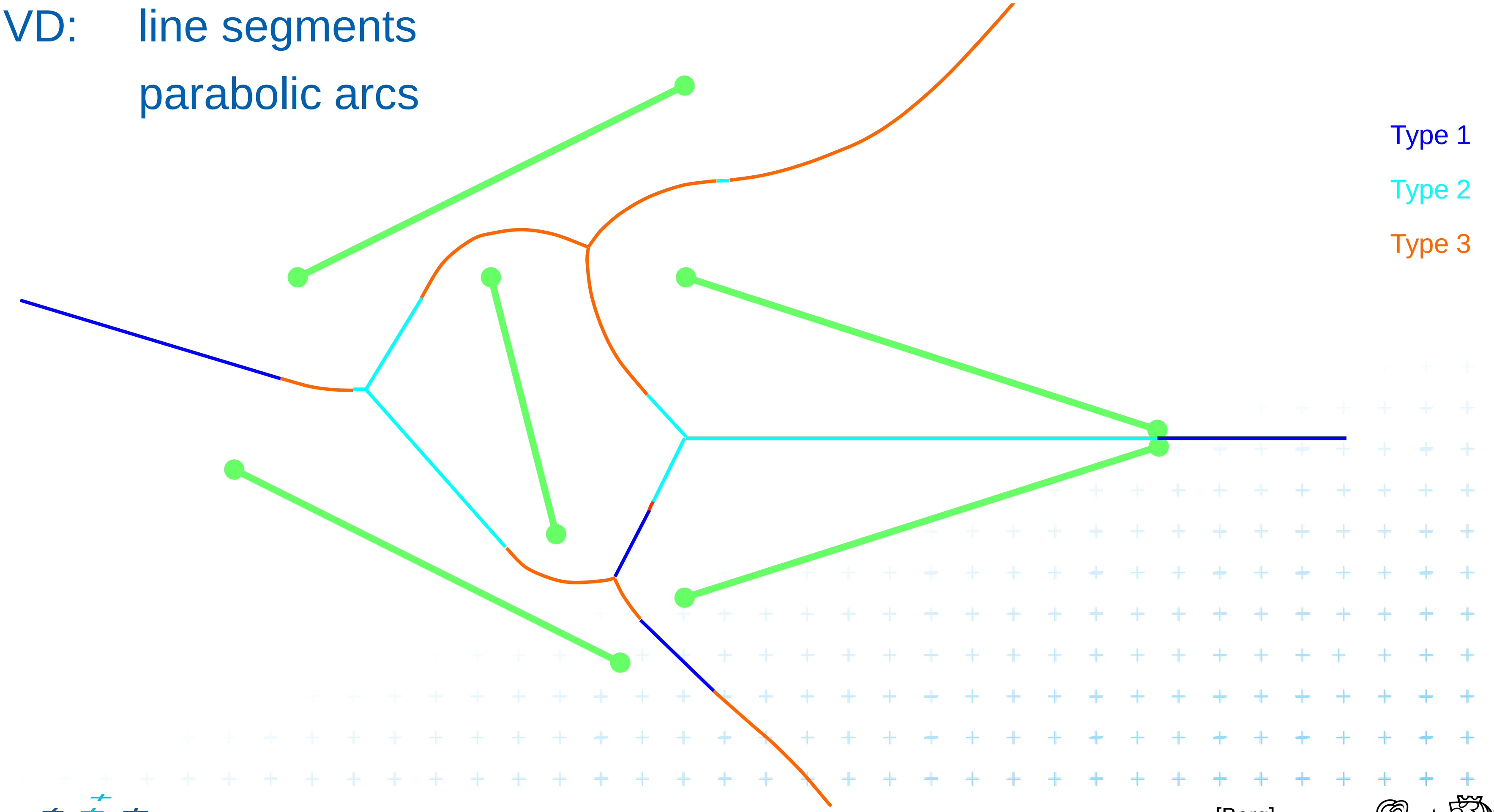
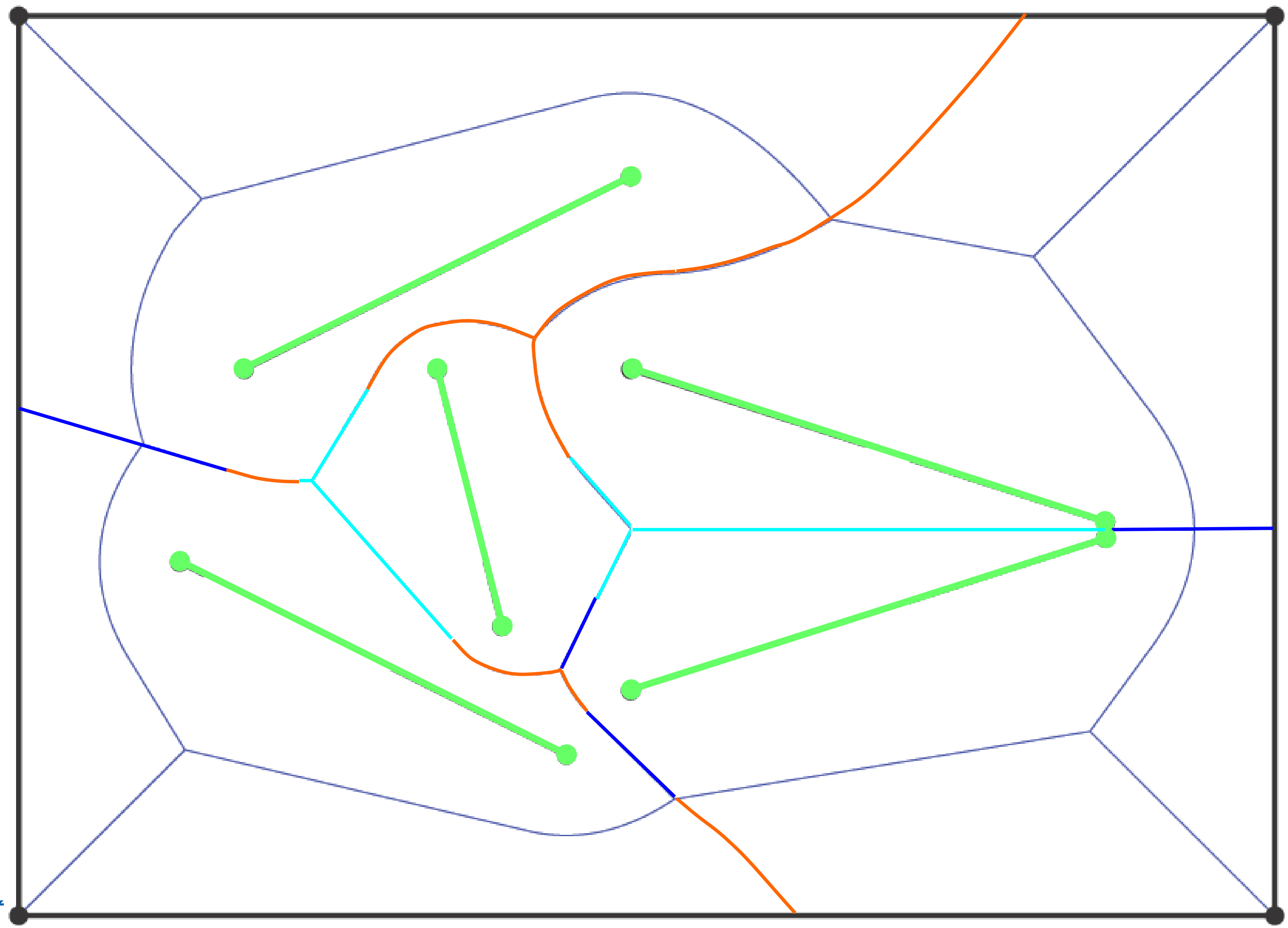


[Berg]

VD of line segments with bounding box



BBOX
=>
standard
DCEL

[Berg]

VD of 2 line-segments in detail

VD consists of line segments and parabolic arcs

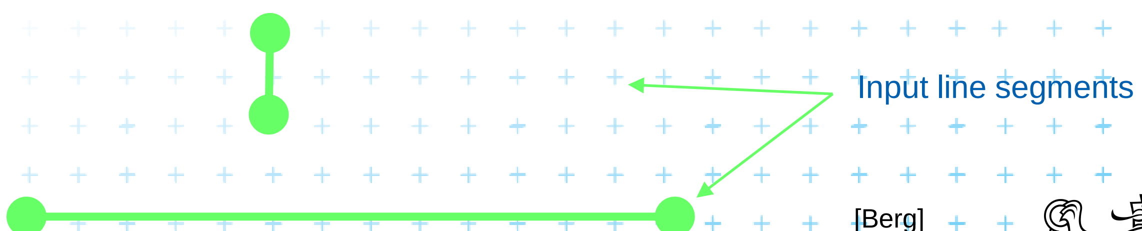
- Line segment – bisector of end-points(1) or of interiors(2)
- Parabolic arc – of point and interior(3) of a line segment

Distance from point-to-object (line segment) is measured to the closest point on the object (perpendicularly to the object silhouette)

Type 1

Type 2

Type 3



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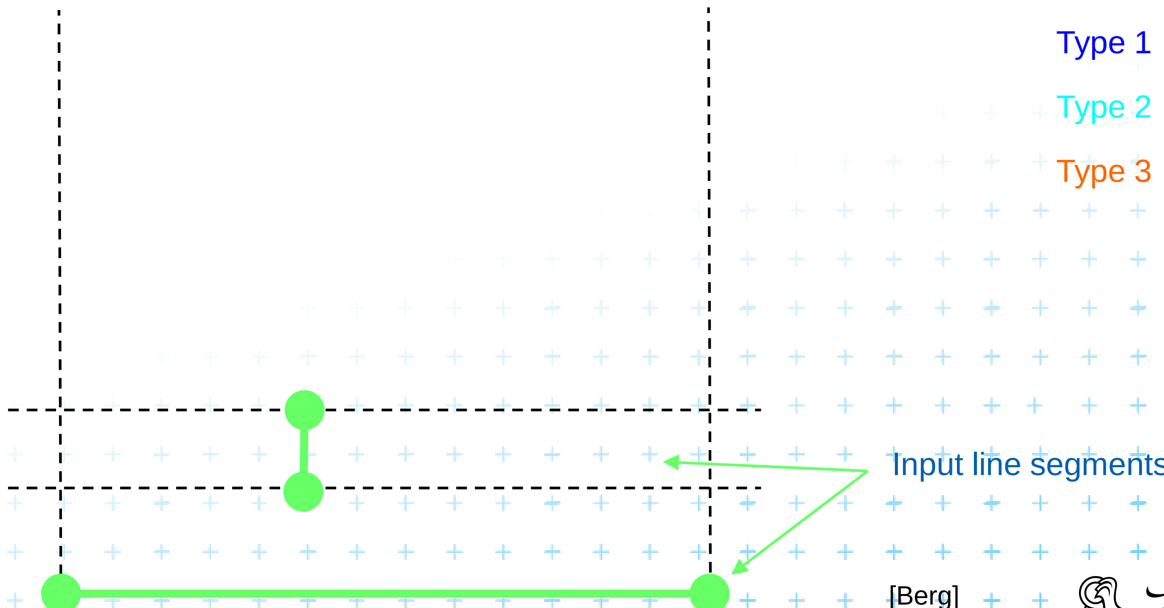

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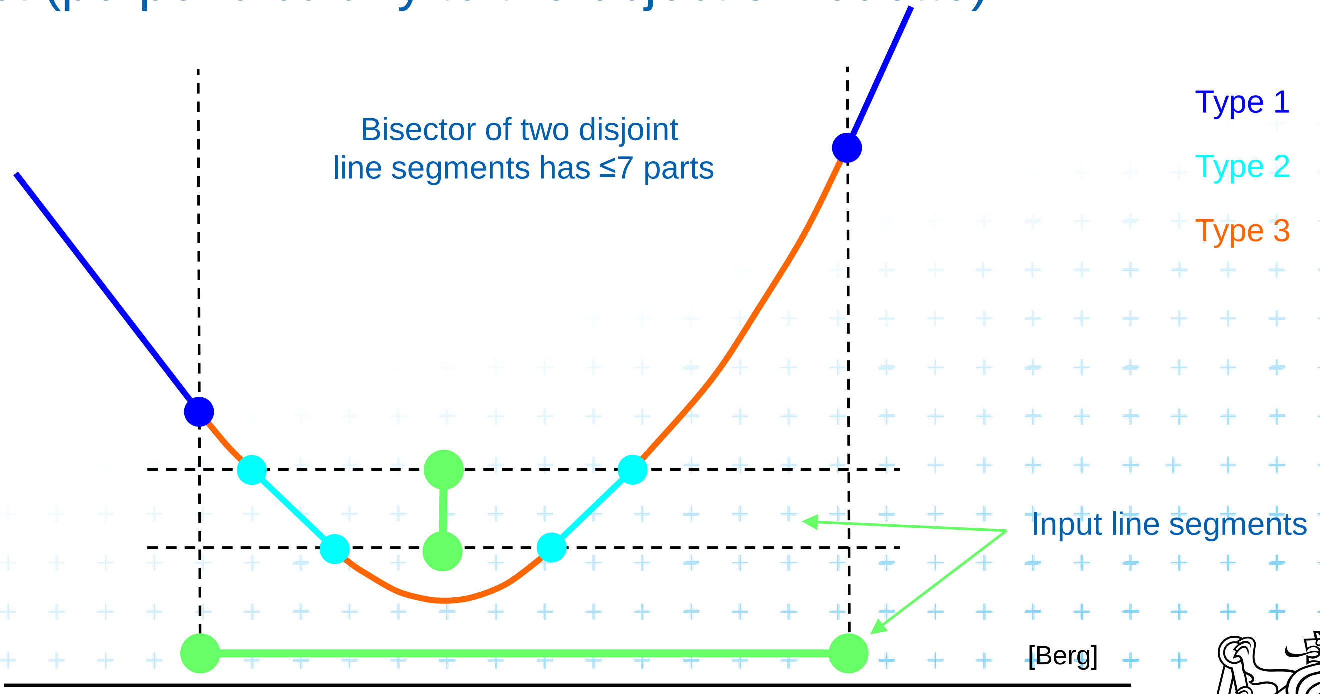

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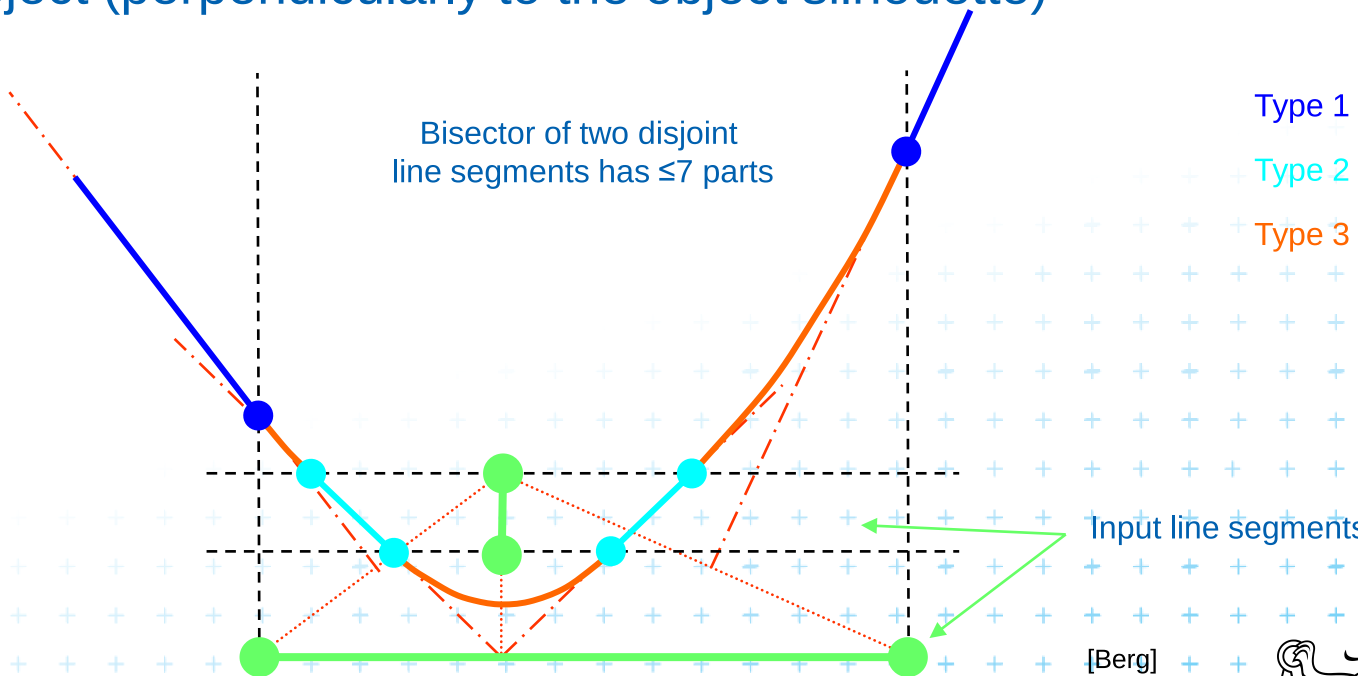

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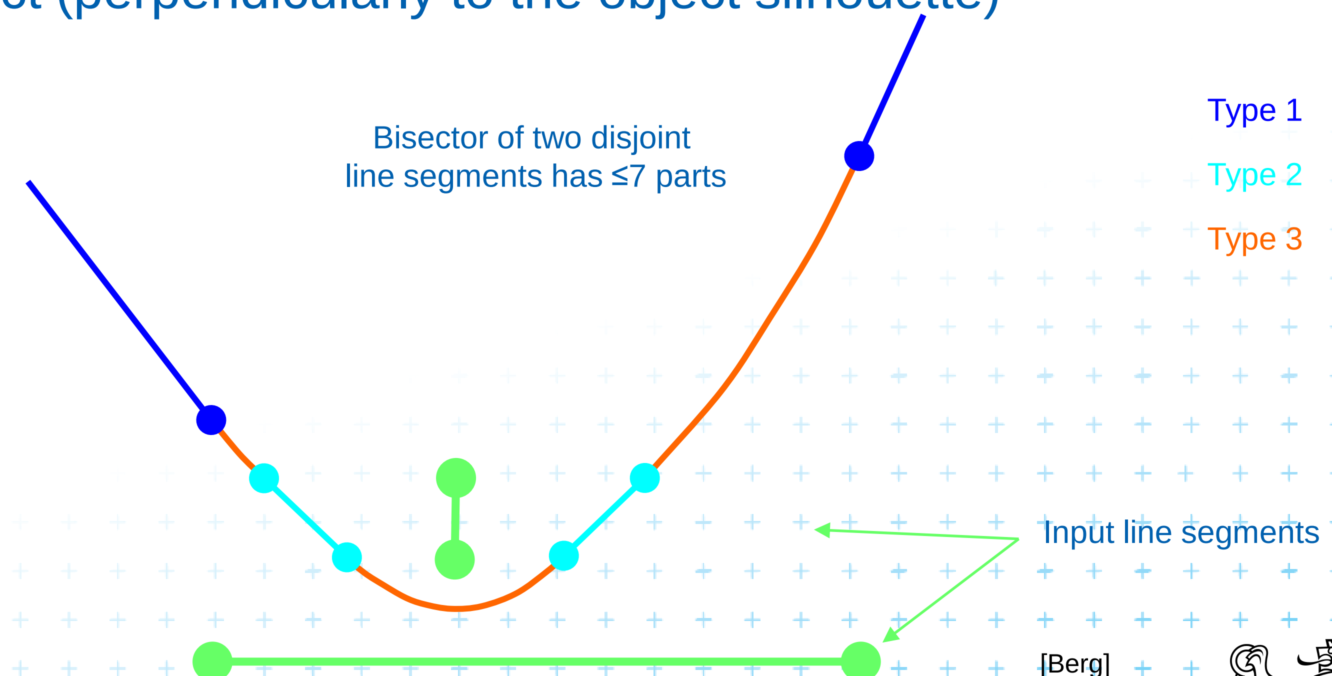
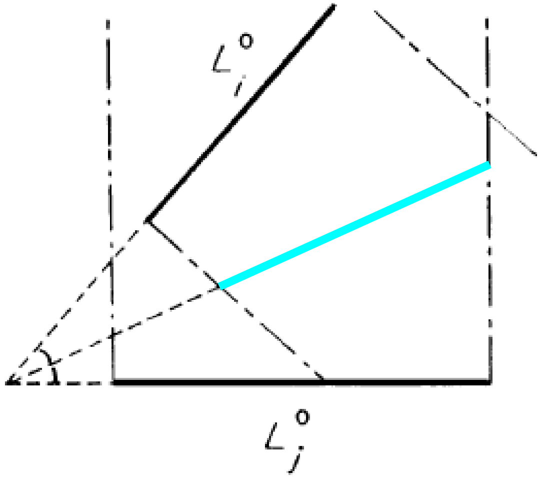


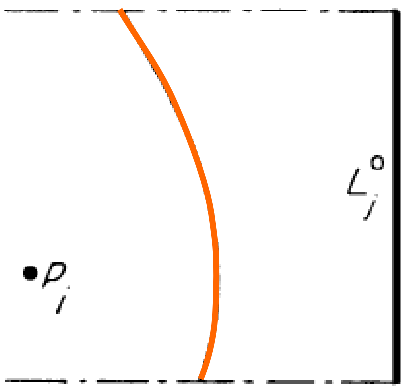
[Berg]

VD in greater details

Type 2



Type 3

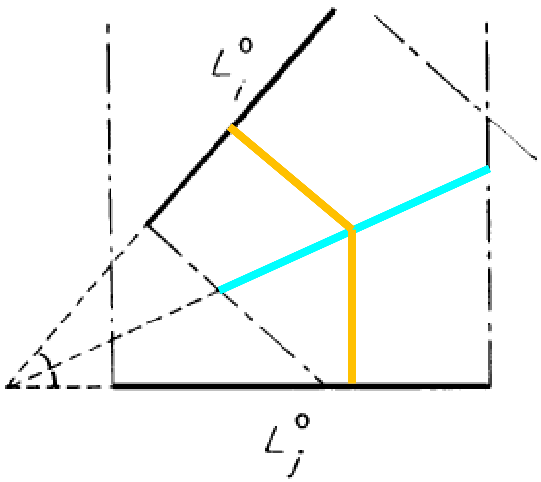


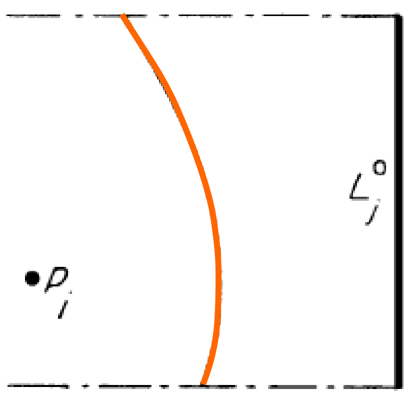
[Reiberg]

Bisector of two line segment interiors (in intersection of perpendicular slabs only)

Bisector of (end-)point and line segment interior

VD in greater details





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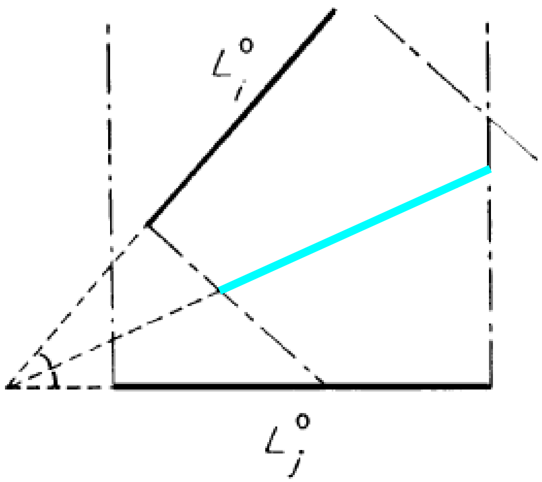

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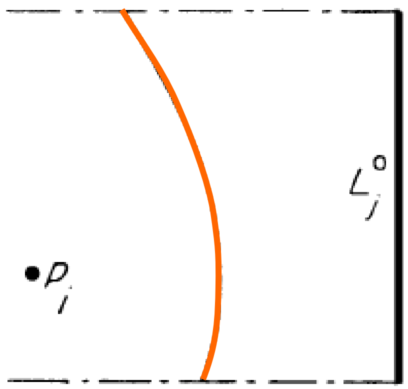

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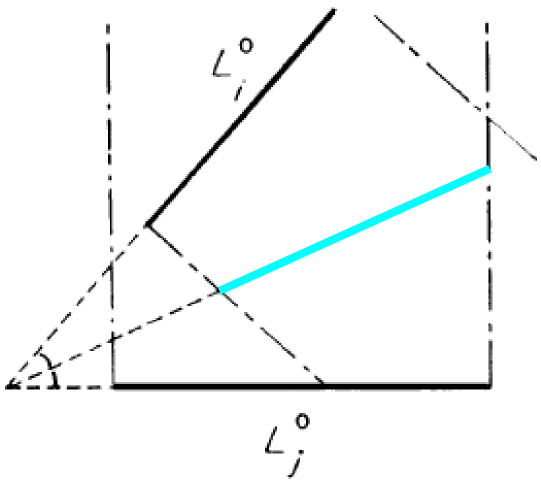

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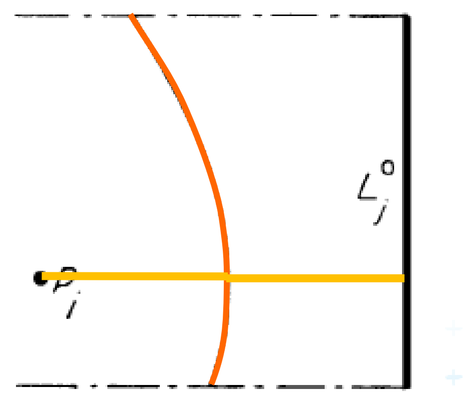

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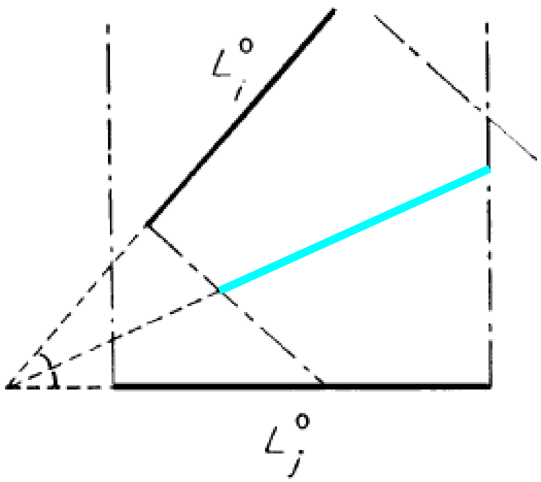

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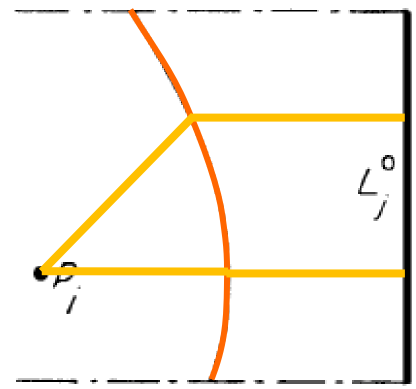

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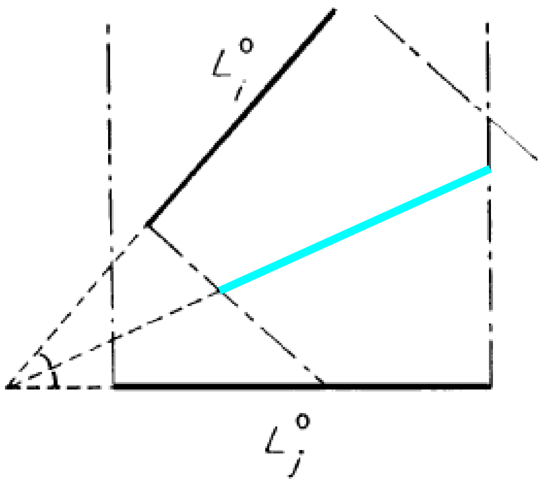

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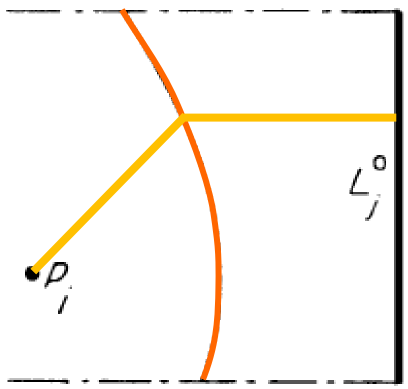

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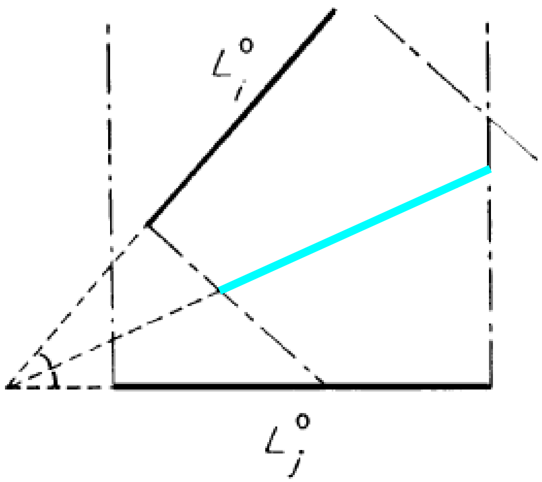

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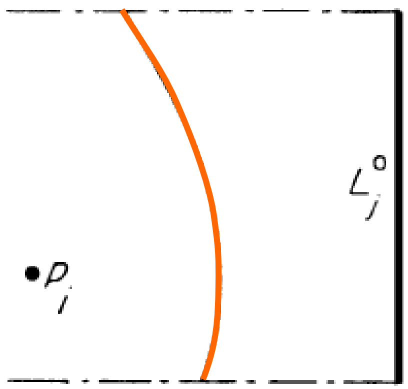


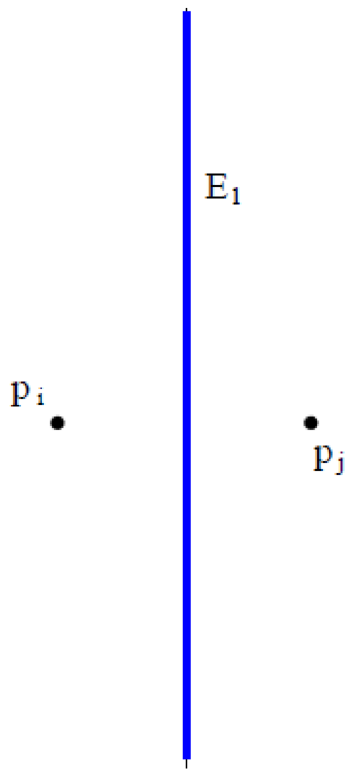
[Reiberg]

Bisector of two line segment interiors (in intersection of perpendicular slabs only)

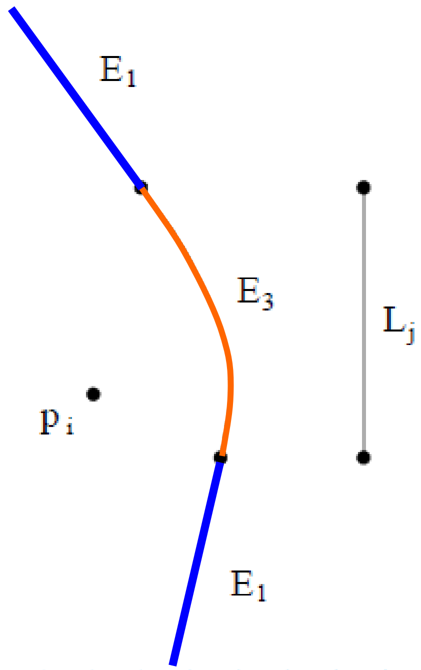
Bisector of (end-)point and line segment interior

VD of points and line segments examples

2 points | Point & segment | 2 line segments

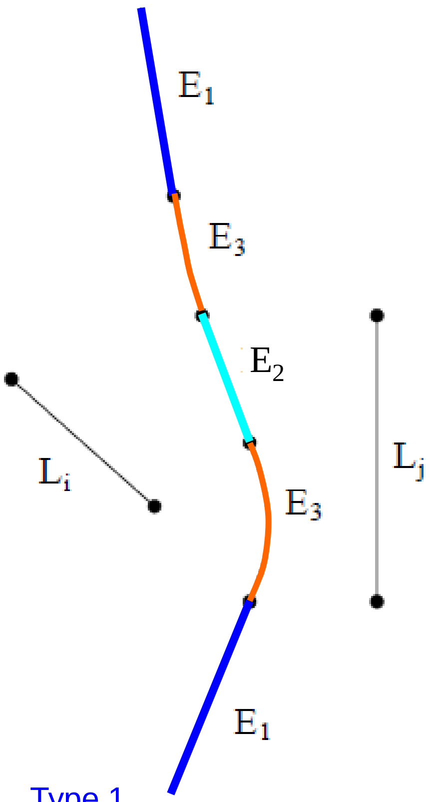


Type 1



Type 1

Type 3



Type 1

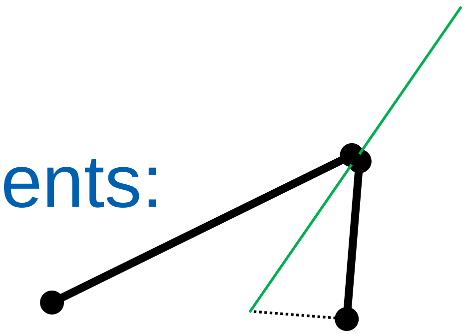
Type 2

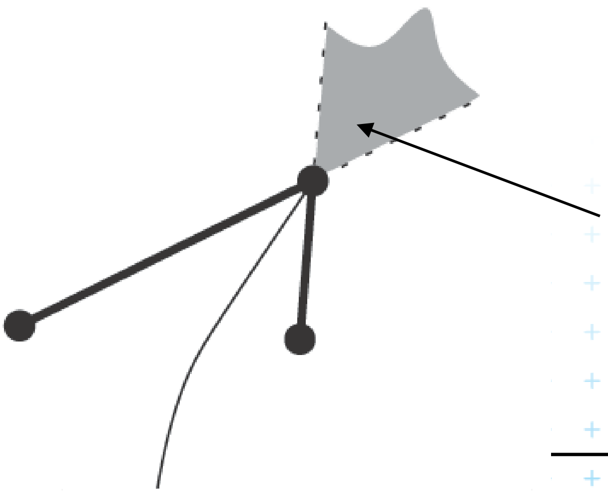
Type 3

[Reiberg]

Voronoi diagram of line segments

- Has more complex bisectors of line segments
  - VD contains line segments and parabolic arcs
- Still $O(n)$ combinatorial complexity
- Assumptions on the input line segments:
  - non-crossing
  - strictly disjoint end-points (slightly shorten the segm.)



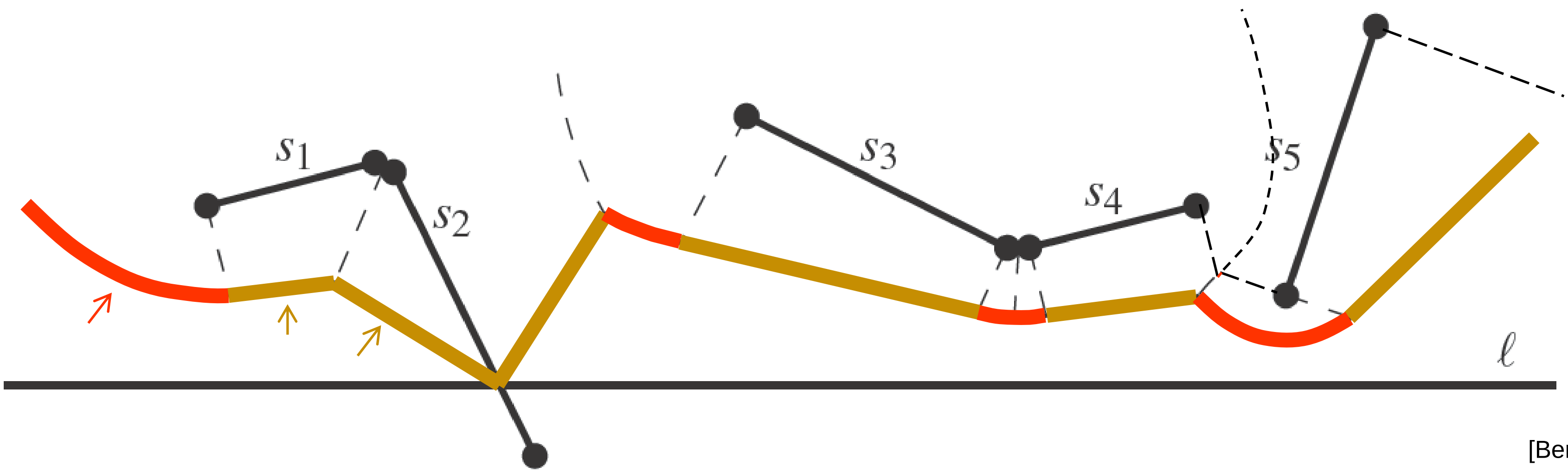


if(we allow touching segments)

Shared endpoints cause complication:
The whole region is equally close
to two line segments

Fortune's algorithm for line segments

Shape of beach line for line segments



[Berg]

Beach line = points with distance to the closest site above sweep line *l* equal to the distance to *l*

Beach line contains

- *parabolic arcs* when closest to a site end-point
- *straight line segments* when closest to a site interior
(or just the part of the site interior above *l* if the site *s* intersects *l*)

(This is the shape of the beach line)

Beach line **breakpoints types** site = line segment

Breakpoint *p* on the beach line is equidistant from l and equidistant and closest to:

points

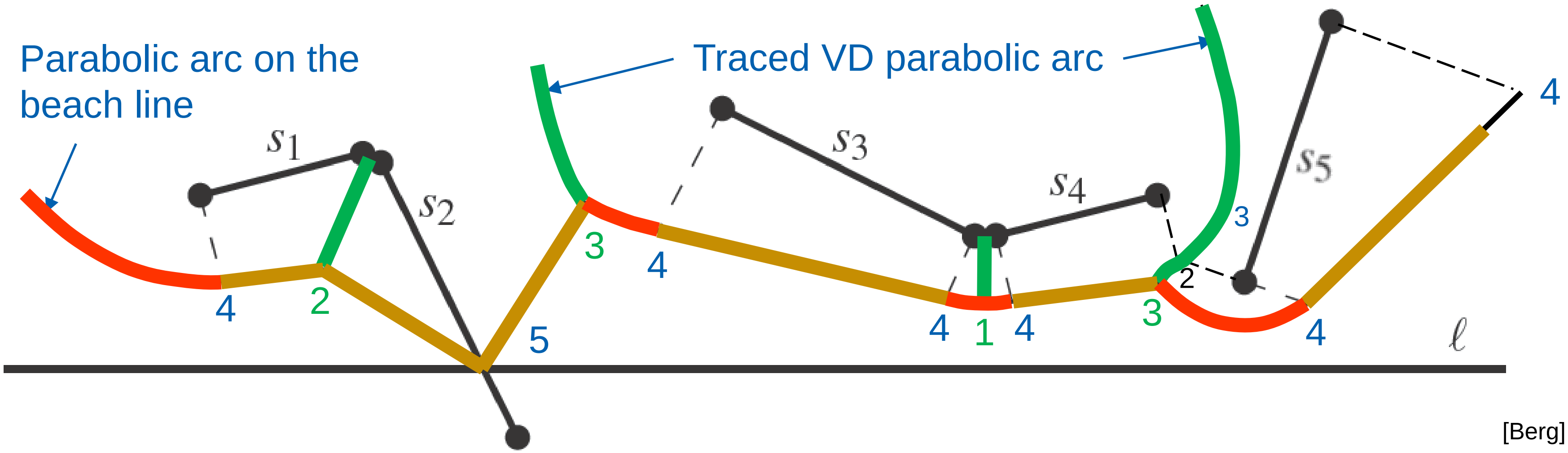
1. two site end-points => *p* traces a VD line segment

segments

2. two site interiors => *p* traces a VD line segment
3. end-point and interior => *p* traces a VD parabolic arc
4. one site end-point => *p* traces a line segment (border of the slab perpendicular to the site)
5. site interior intersects the scan line l => *p* = intersection, traces the input line segment

Cases 4 and 5 involve only one site and therefore do not form a Voronoi diagram edge (are used by alg.only)

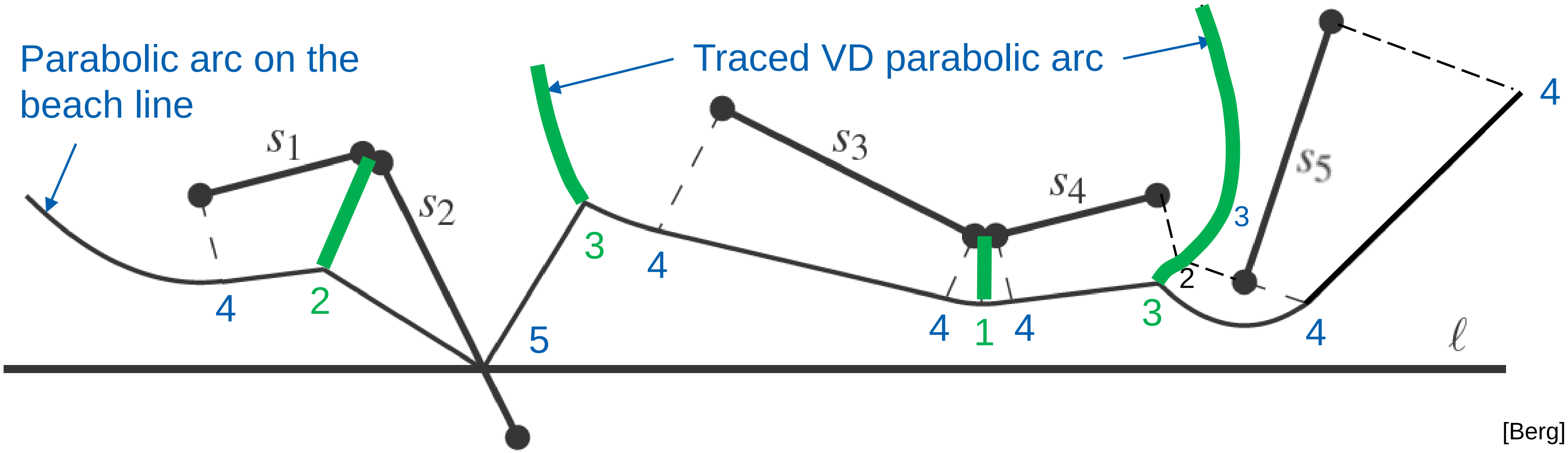
Breakpoints types - what they trace on VD



- 1,2 trace a Voronoi line segment (part of VD edge) DRAW
- 3 traces a Voronoi parabolic arc (part of VD edge) DRAW
- 4,5 trace a line segment (used only by the algorithm) MOVE
  - 4 limits the slab perpendicular to the line segment
  - 5 traces the intersection of input segment with a sweep line

(This is the shape of the traced VD arcs)

Breakpoints types - what they trace on VD



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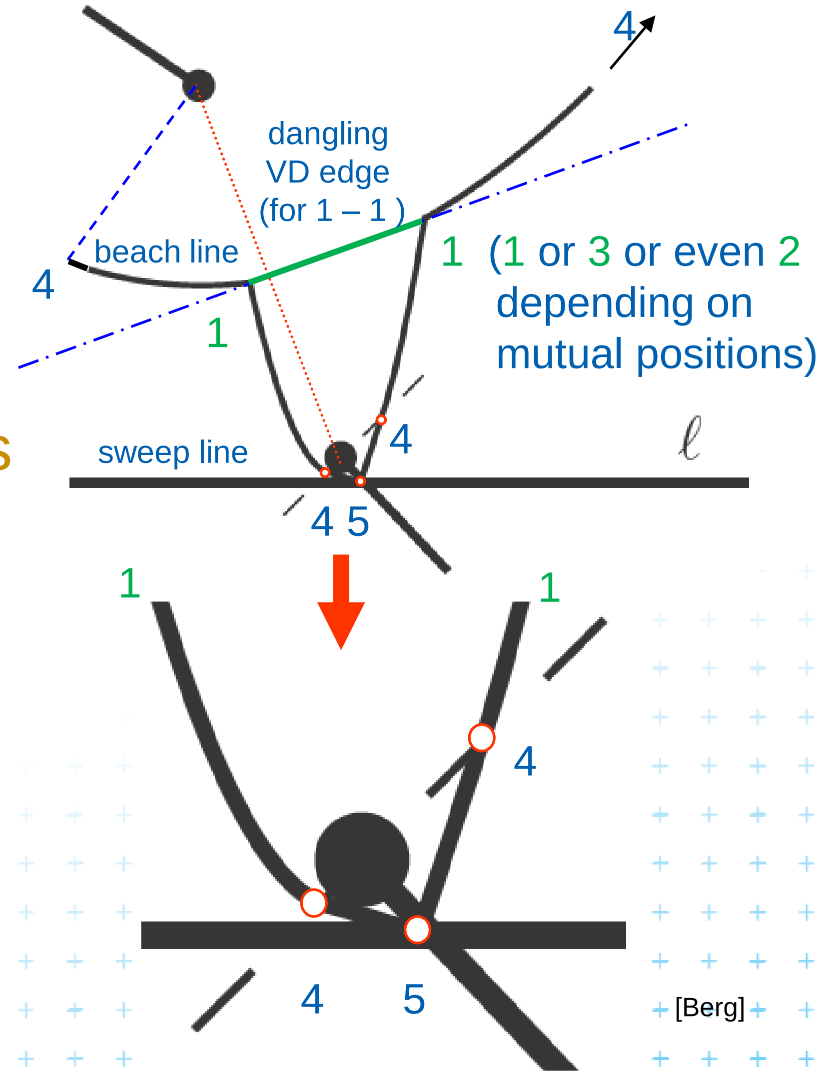
(This is the shape of the traced VD arcs)

Site event – sweep line reaches an endpoint

I. At upper endpoint of

- Arc above is split into two 1-1
- four new arcs are created (2 segments + 2 parabolas)
  4-5, 5-4 1-4, 4-1
- Breakpoints for two segments are of type 4-5-4
- Breakpoints for parabolas depend on the surrounding sites
  - Type 1 for two end-points
  - Type 3 for endpoint and interior
  - etc…

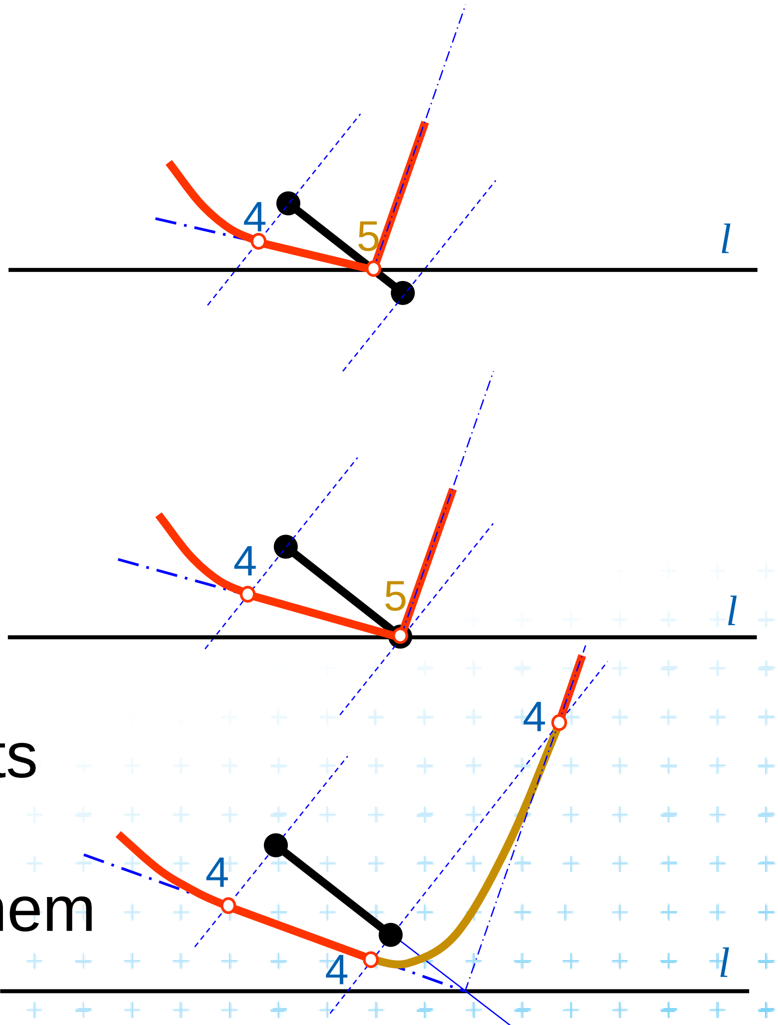


Site event – sweep line reaches an endpoint

II. At lower endpoint of ●—●

- Intersection with interior (breakpoint of type 5)

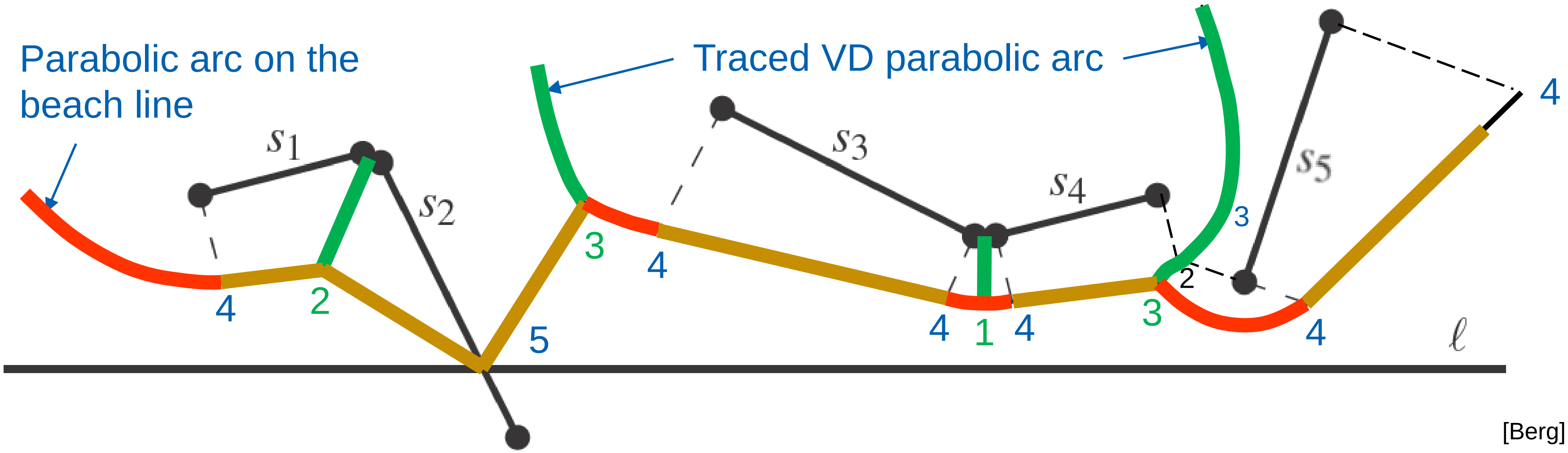
- is replaced by two breakpoints (of type 4) with parabolic arc between them



Circle event – lower point of circle of 3 sites

- Two breakpoints meet (on the beach-line)
- Solution depends on their type
  - Any of first three types (1,2,or 3) meet (circle event)
    - 3 sites involved – Voronoi vertex created
  - Type 4 (*segment interiors*) with something else
    - two sites involved – breakpoint changes its type
    - Voronoi vertex not created (Voronoi edge may change its shape)
  - Type 5 (*on segment*) with something else
    - never happens for disjoint segments (meet with type 4 happens before)

Breakpoints types - what they trace on VD



[Berg]

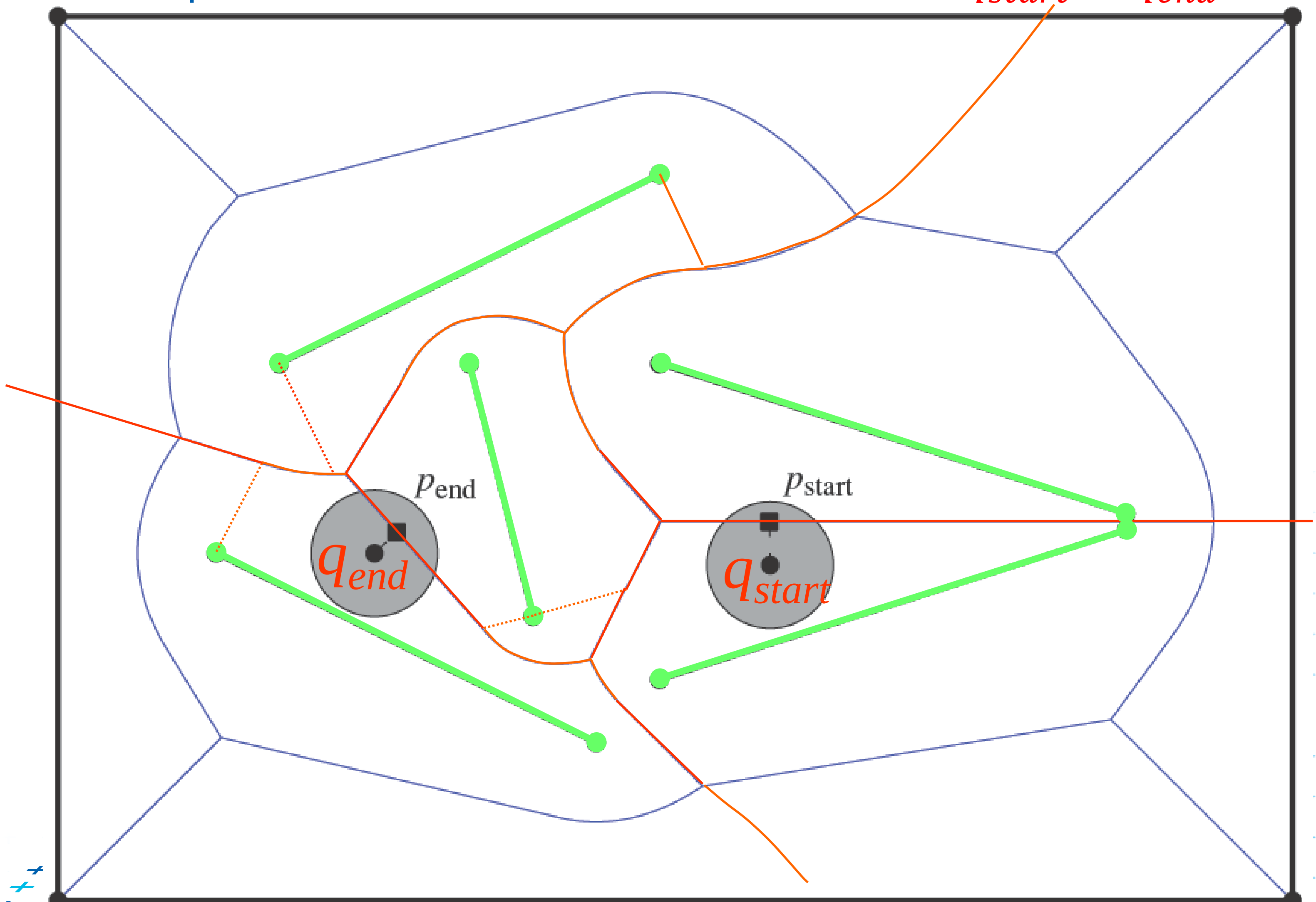
- 1,2 trace a Voronoi line segment (part of VD edge) DRAW
- 3 traces a Voronoi parabolic arc (part of VD edge) DRAW
- 4,5 trace a line segment (used only by the algorithm) MOVE
  - 4 limits the slab perpendicular to the line segment
  - 5 traces the intersection of input segment with a sweep line

(This is the shape of the traced VD arcs)

Motion planning example

Motion planning example - retraction

Find path for a circular robot of radius *r* from q_{start} to q_{end}



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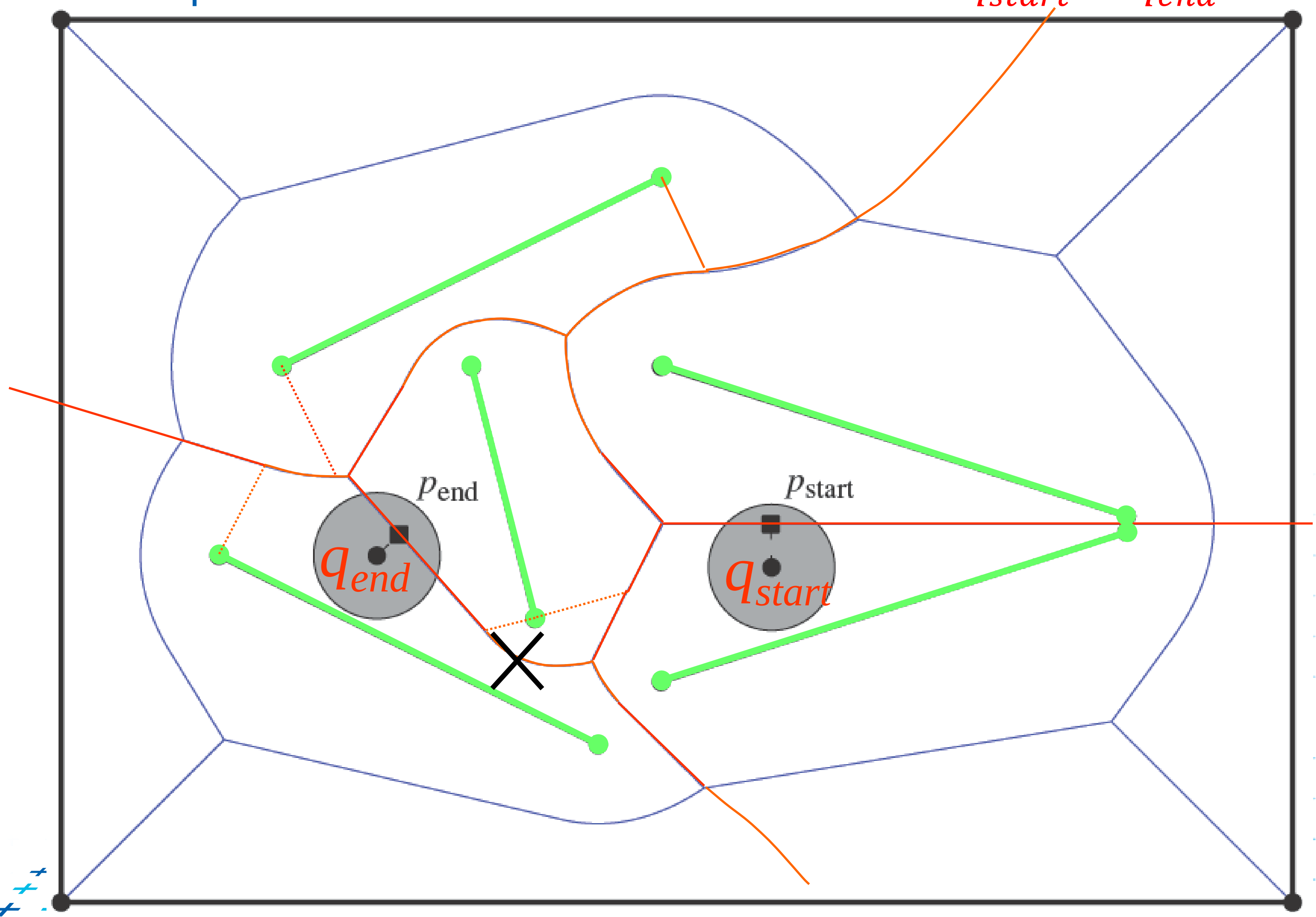

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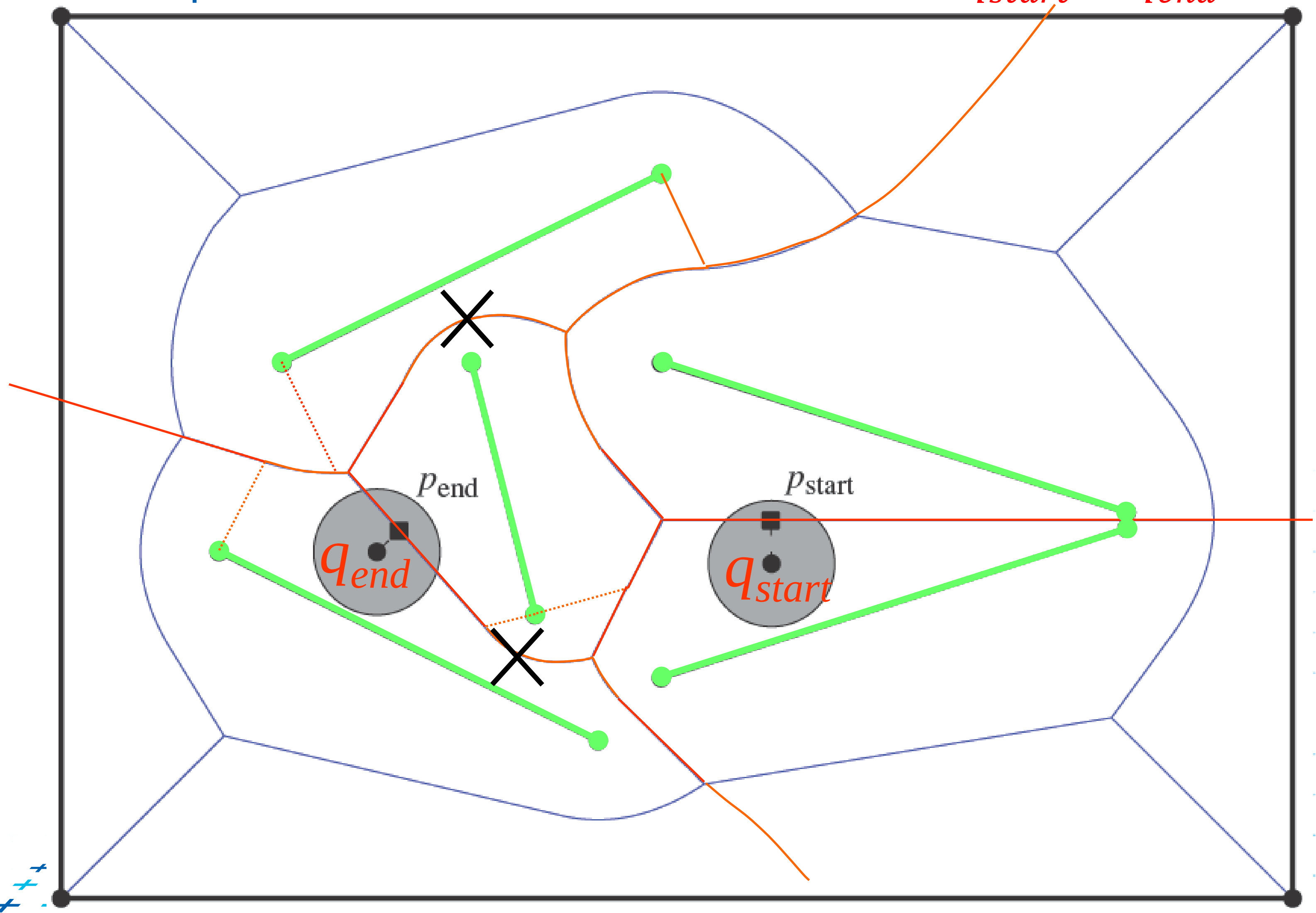
Motion planning example - retraction **Rušení hran**

Find path for a circular robot of radius *r* from q_{start} to q_{end}



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Motion planning example - retraction Rušení hran

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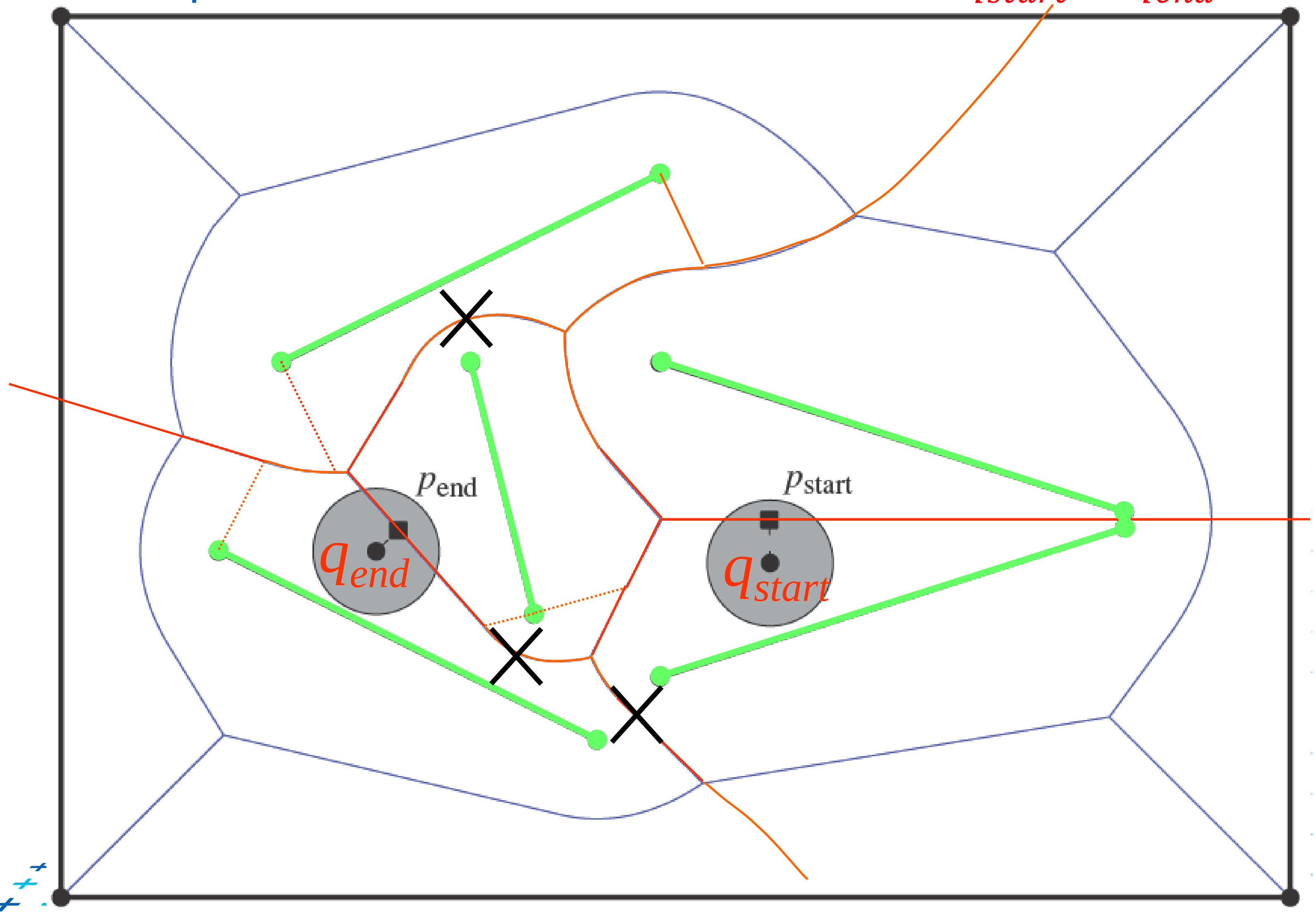

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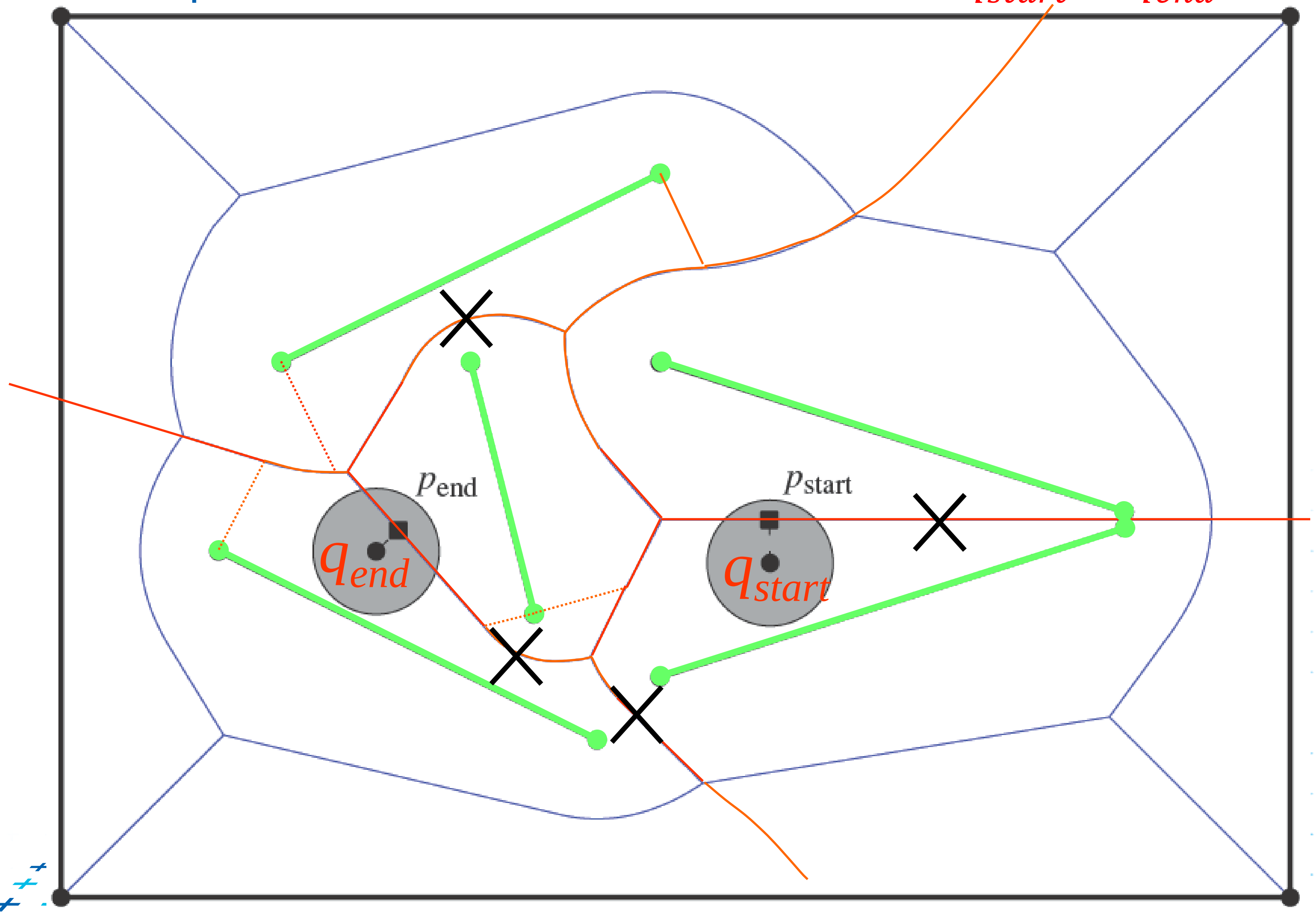

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Motion planning example - retraction

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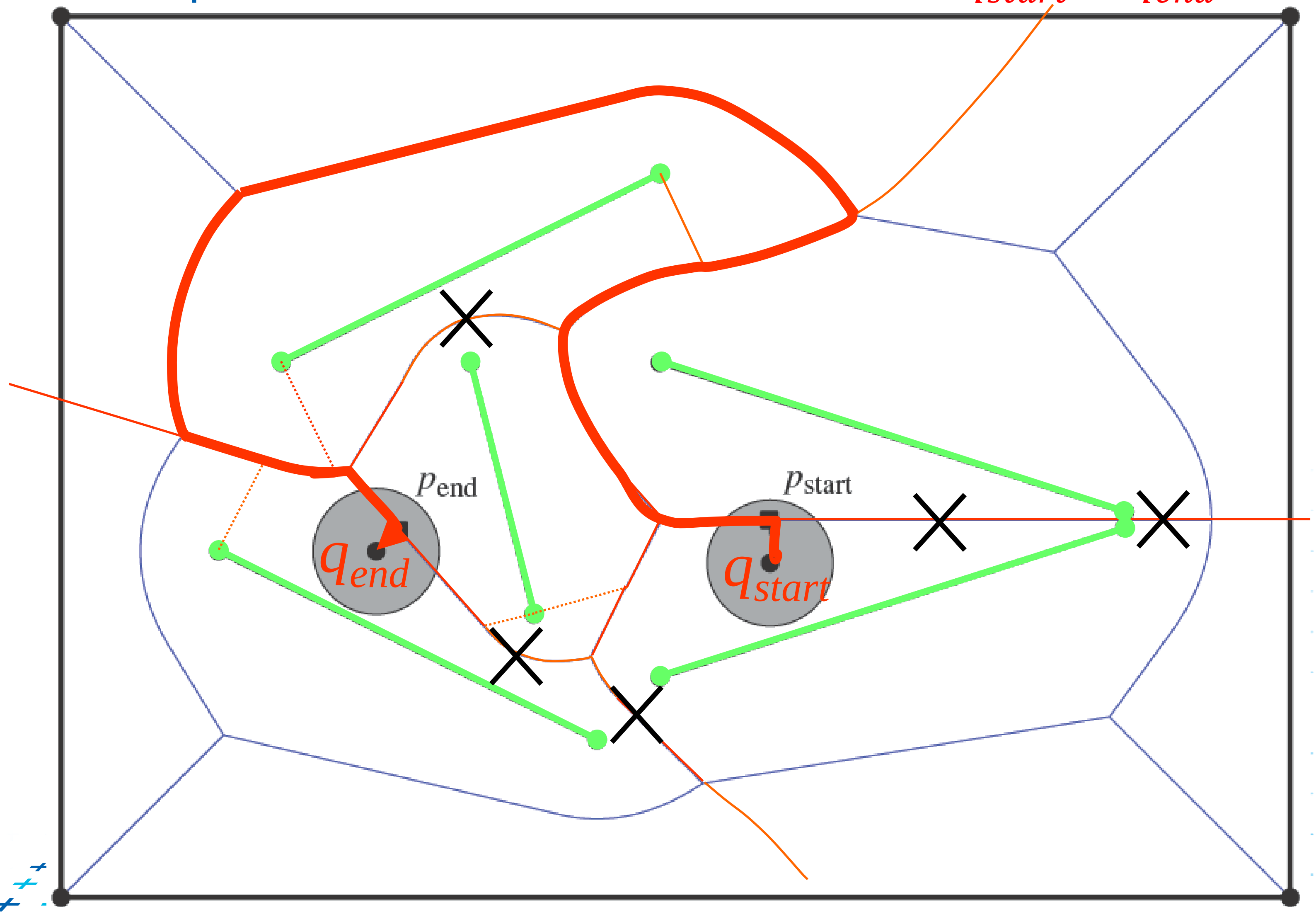


[Berg]

Motion planning example - retraction

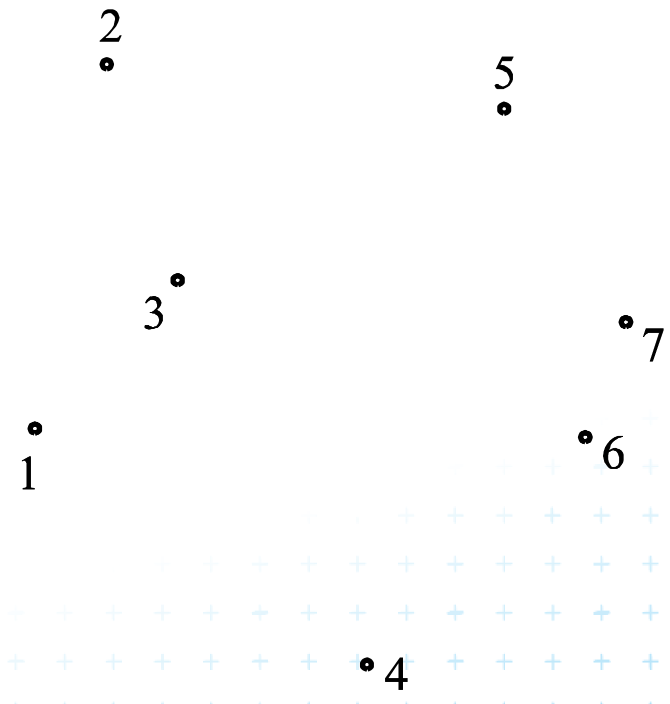
Find path for a circular robot of radius *r* from q_{start} to q_{end}

- Create Voronoi diagram of line segments, take it as a graph
- Project q_{start} *and* q_{end} to P_{start} and P_{end}*on the VD*
- Remove segments with distance to sites smaller than radius *r* of a robot
- Depth first search if path from P_{start} to P_{end} *exists*
- Report path $q_{start}\ P_{start} \ldots$ *path* $\ldots\ P_{end}\ q_{end}$

- *O*(*n* log *n*) time using *O*(*n*) storage

Higher order VD

Order-2 Voronoi diagram (nearest to two sites)



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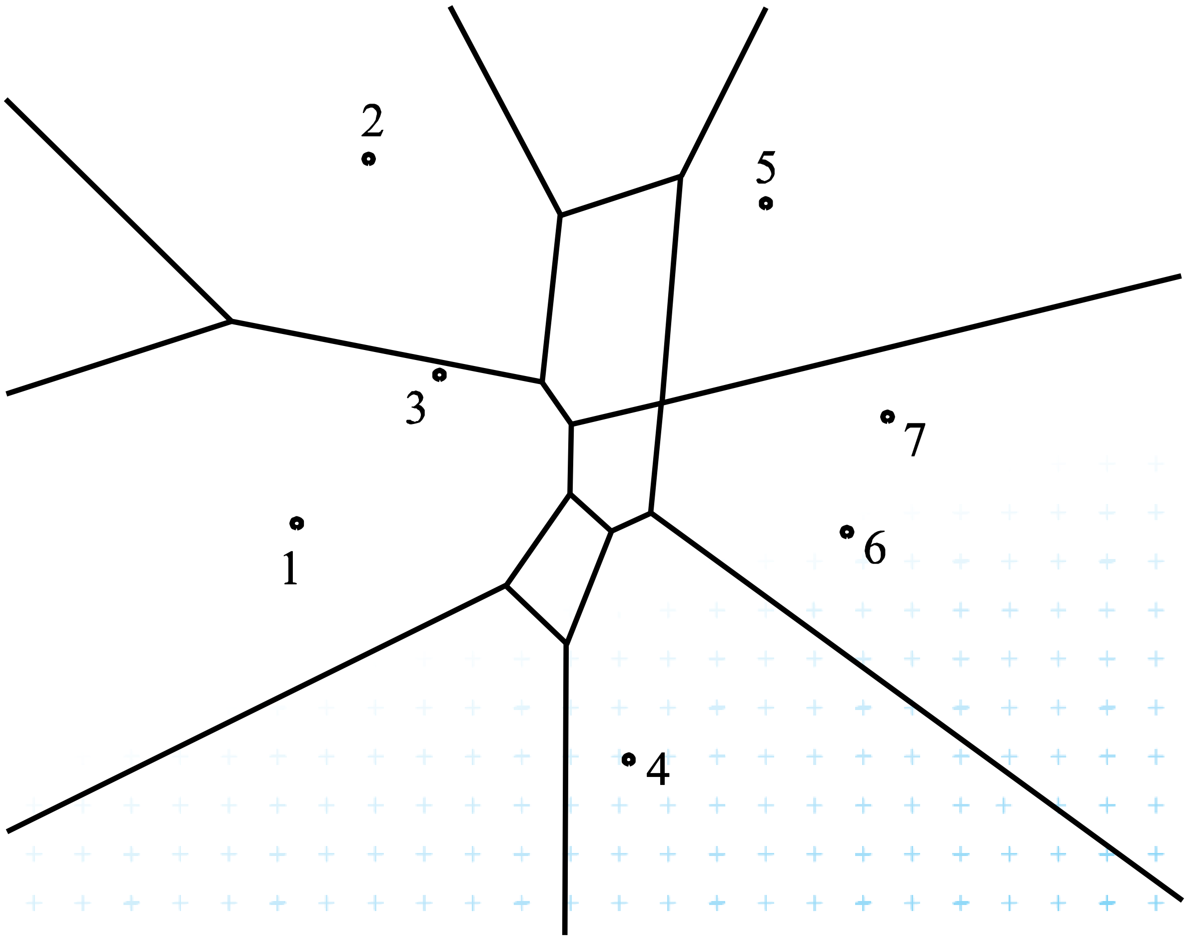

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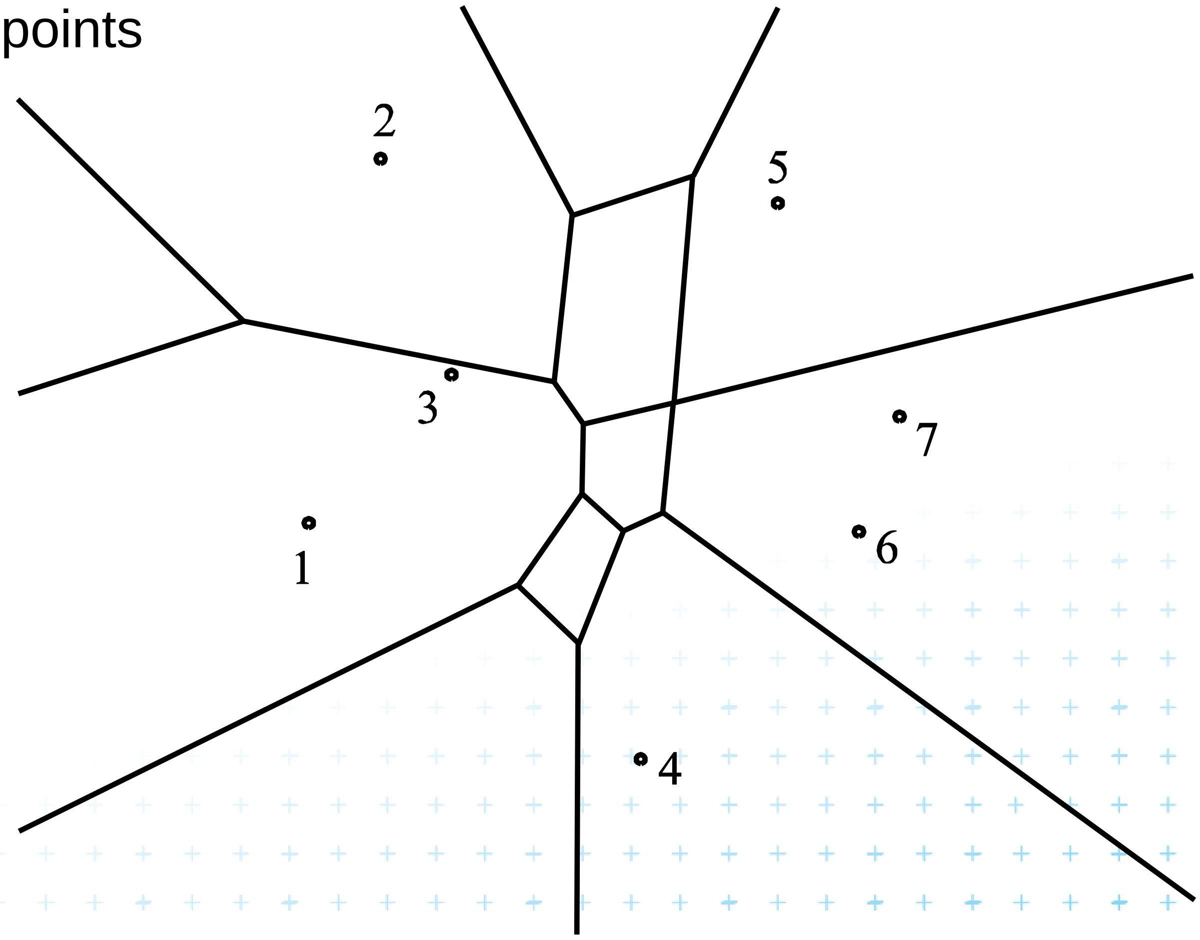
Order-2 Voronoi diagram (nearest to two sites)

Cell $V(p_i, p_j)$: the set of points of the plane closer to each of p_i and p_j than to any other site



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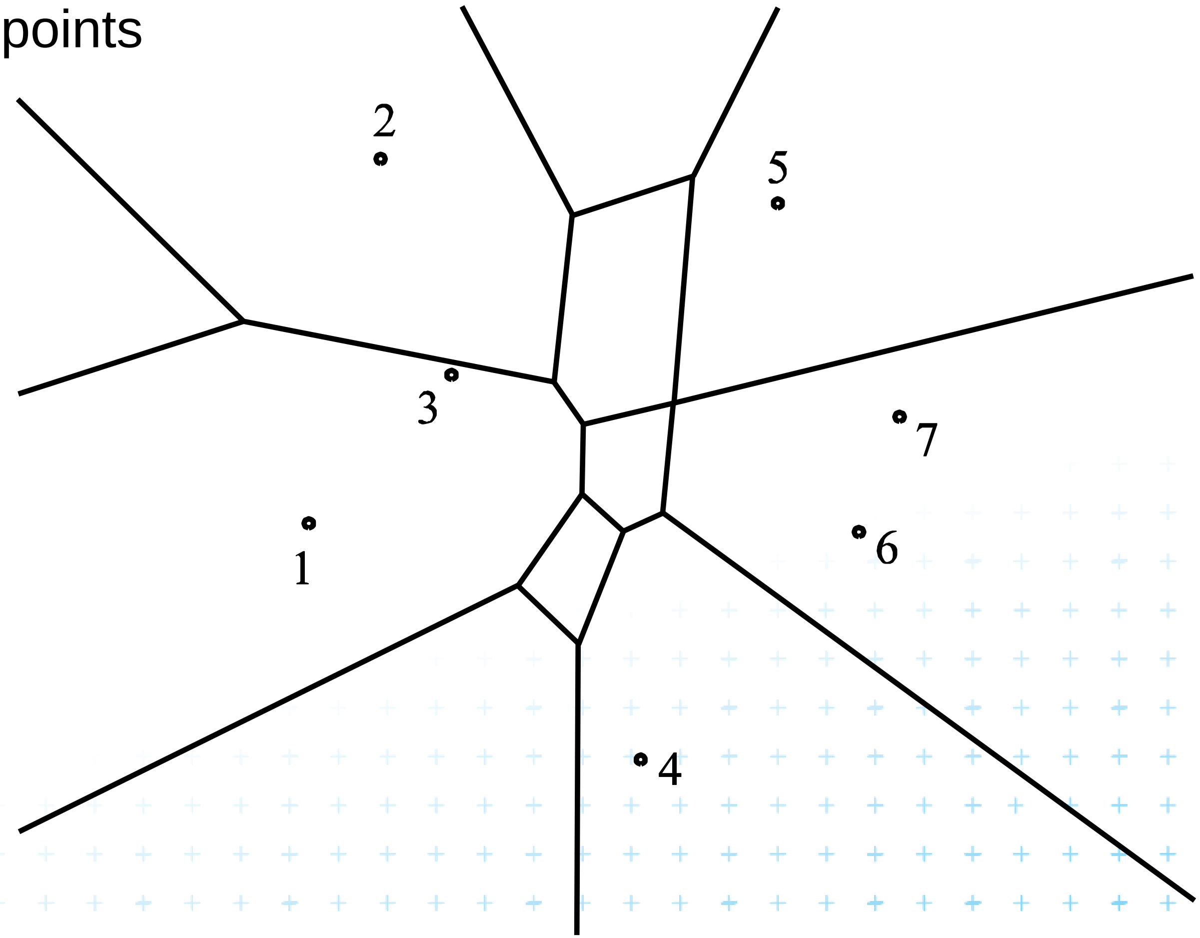

Property

The order-2 Voronoi regions are convex

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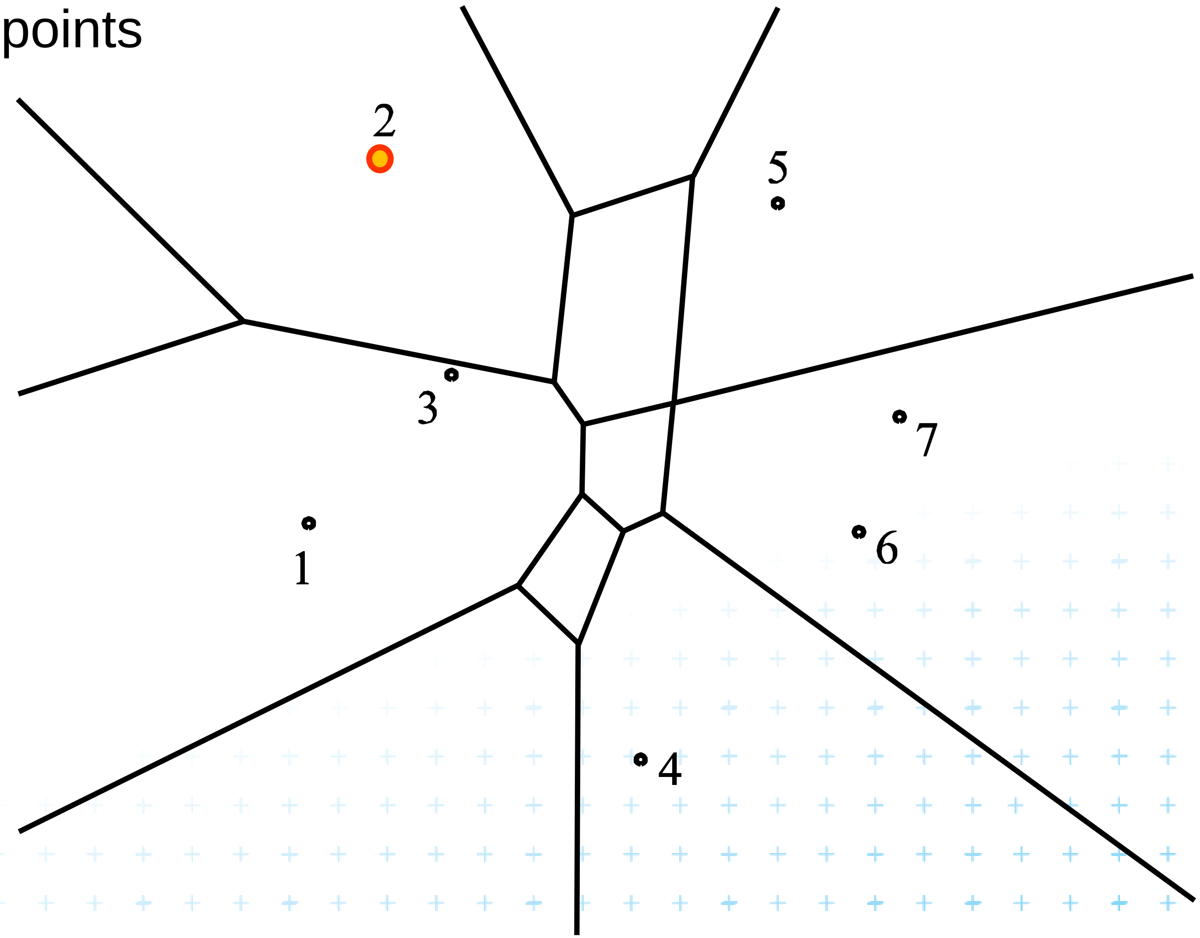

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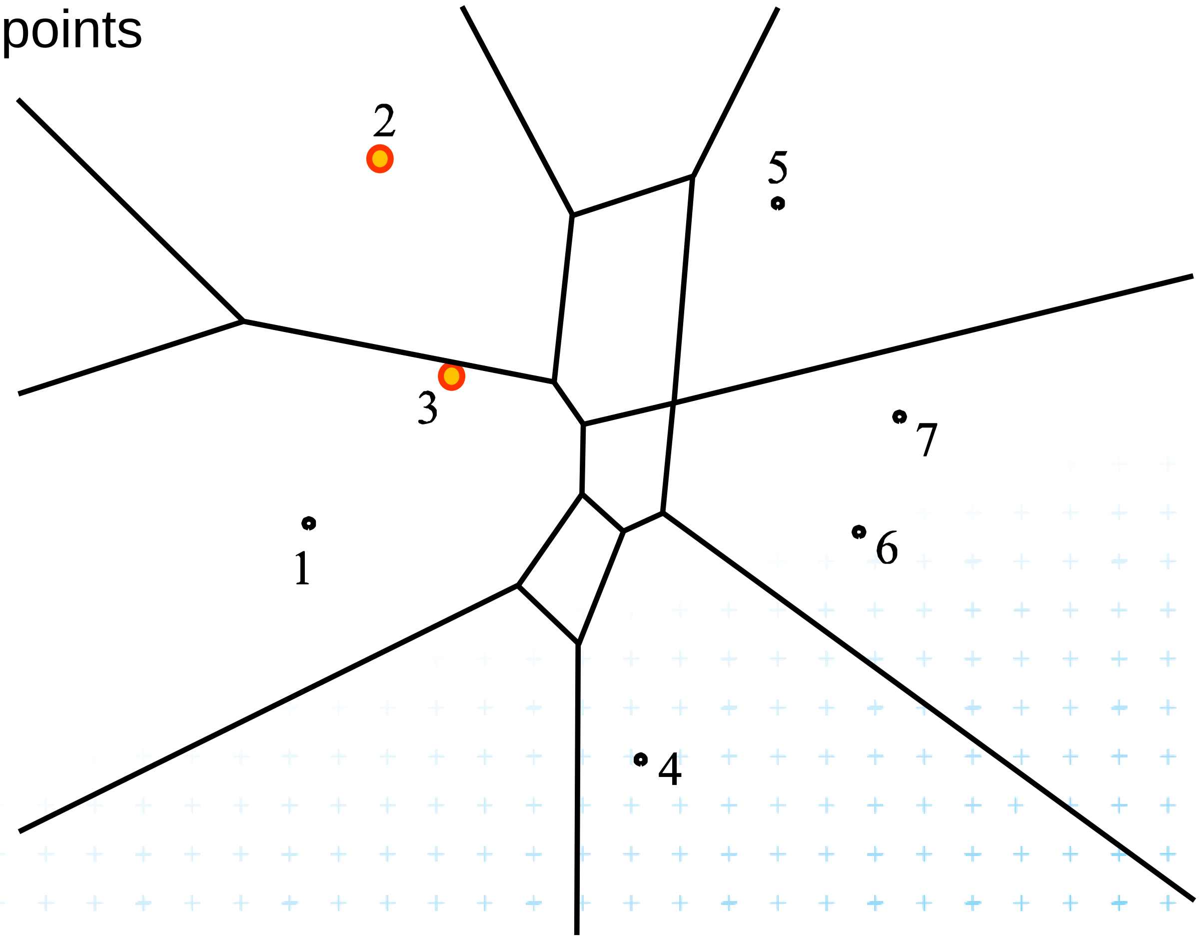

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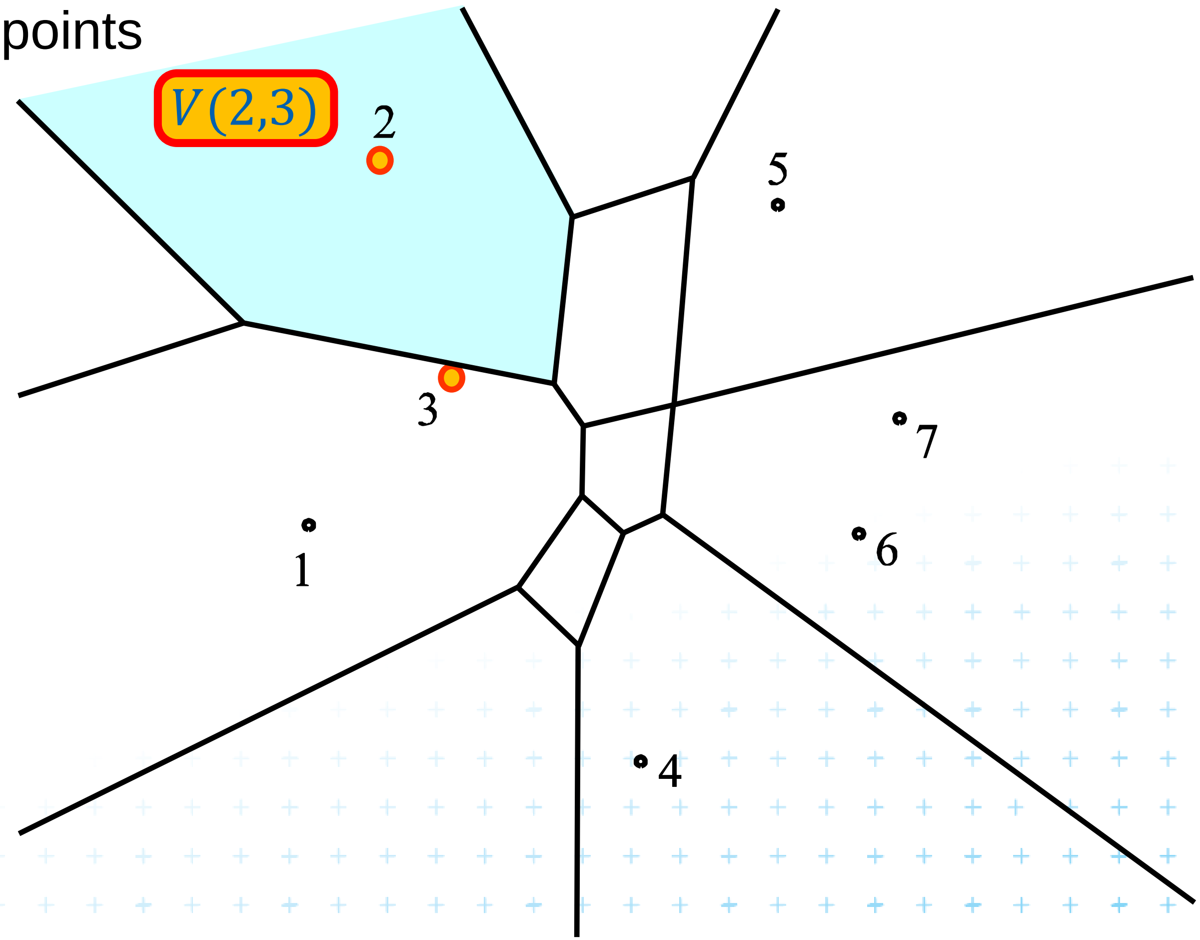

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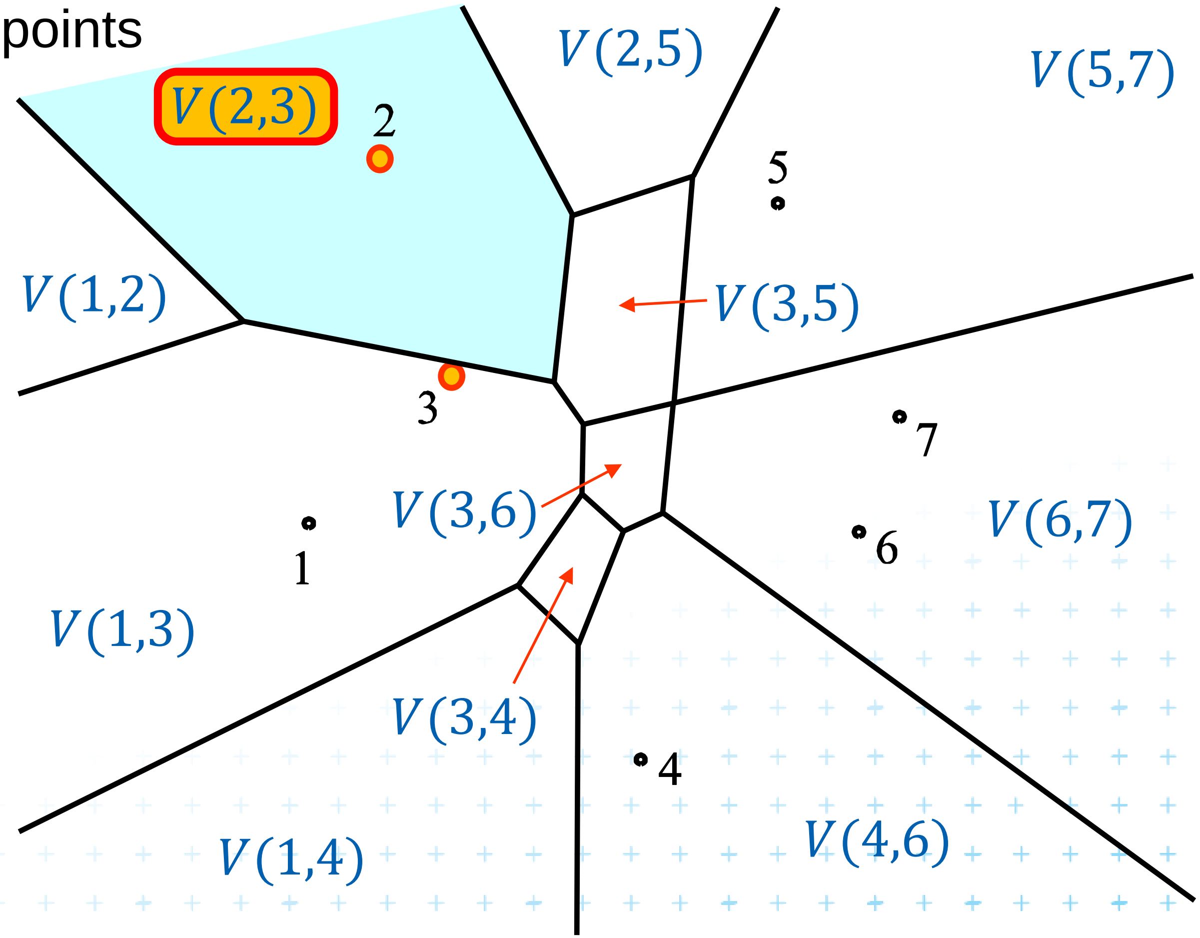
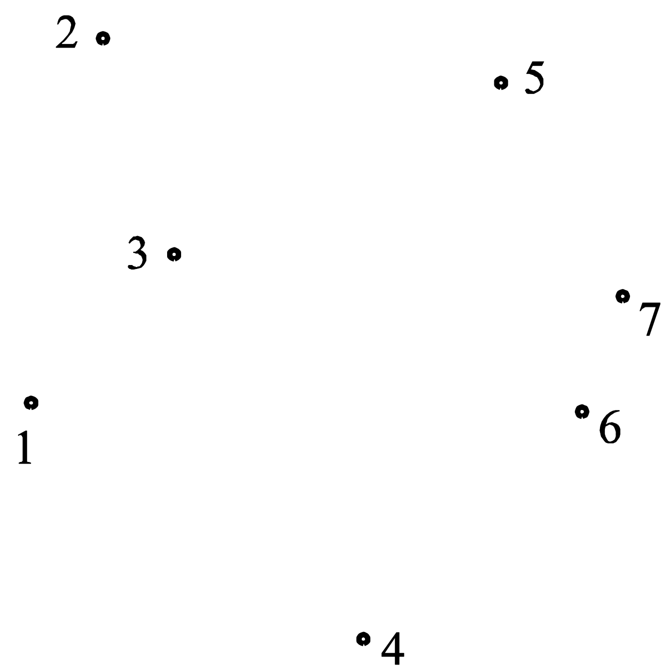


[Nandy]

Construction of V(3,5) = V(5,3)



[Nandy]

Intersection of all halfplanes except $h(3{,}5)$ and $h(5{,}3)$:

$$V(3{,}5) = \bigcap_{x \neq 5} h(3, x) \cap \bigcap_{x \neq 3} h(5, x)$$

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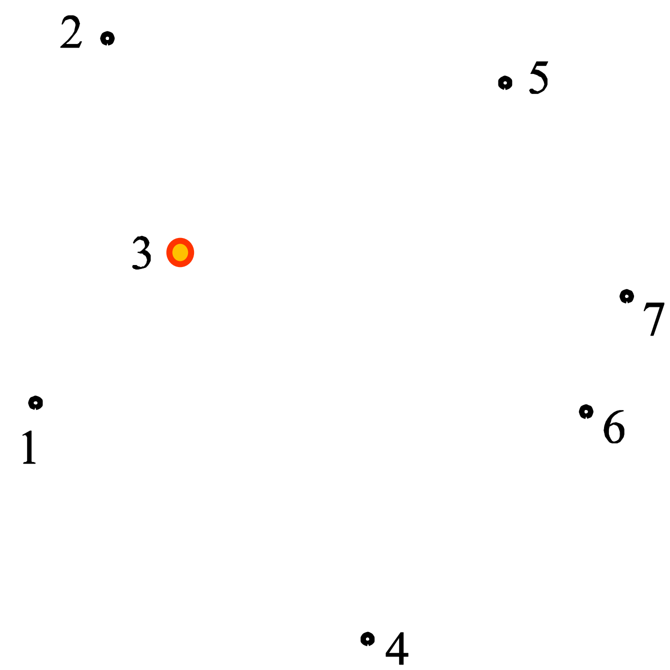

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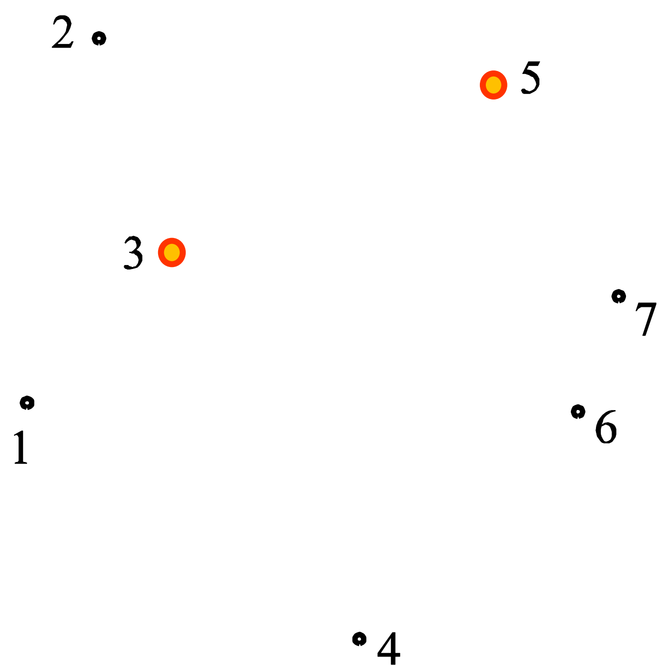

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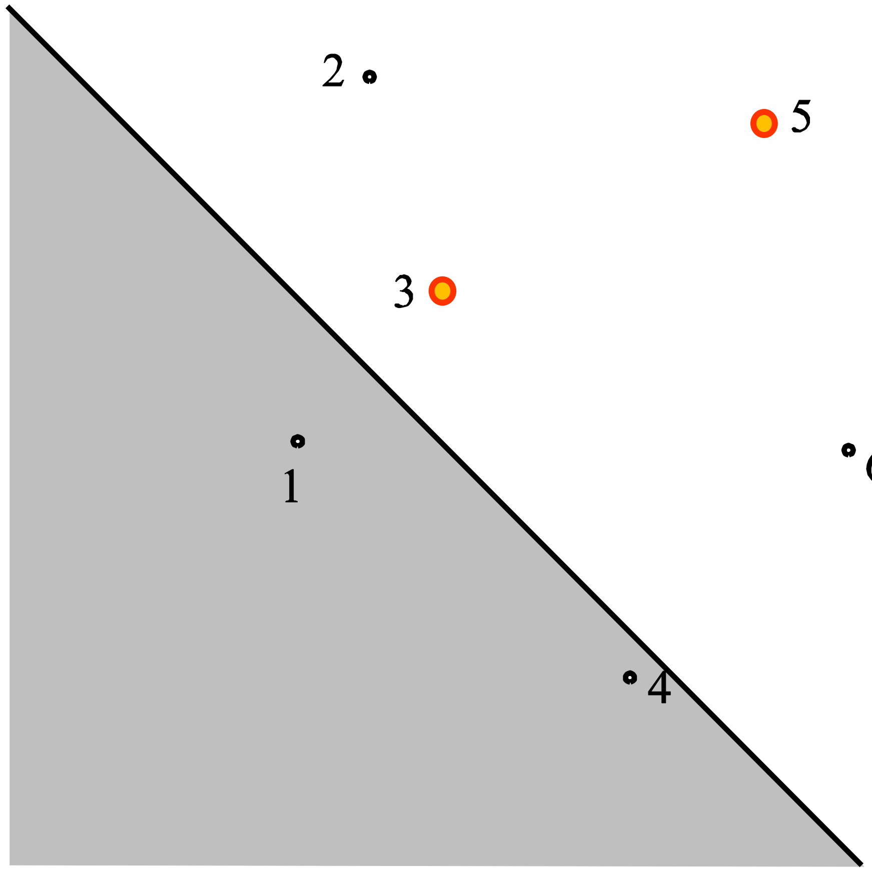

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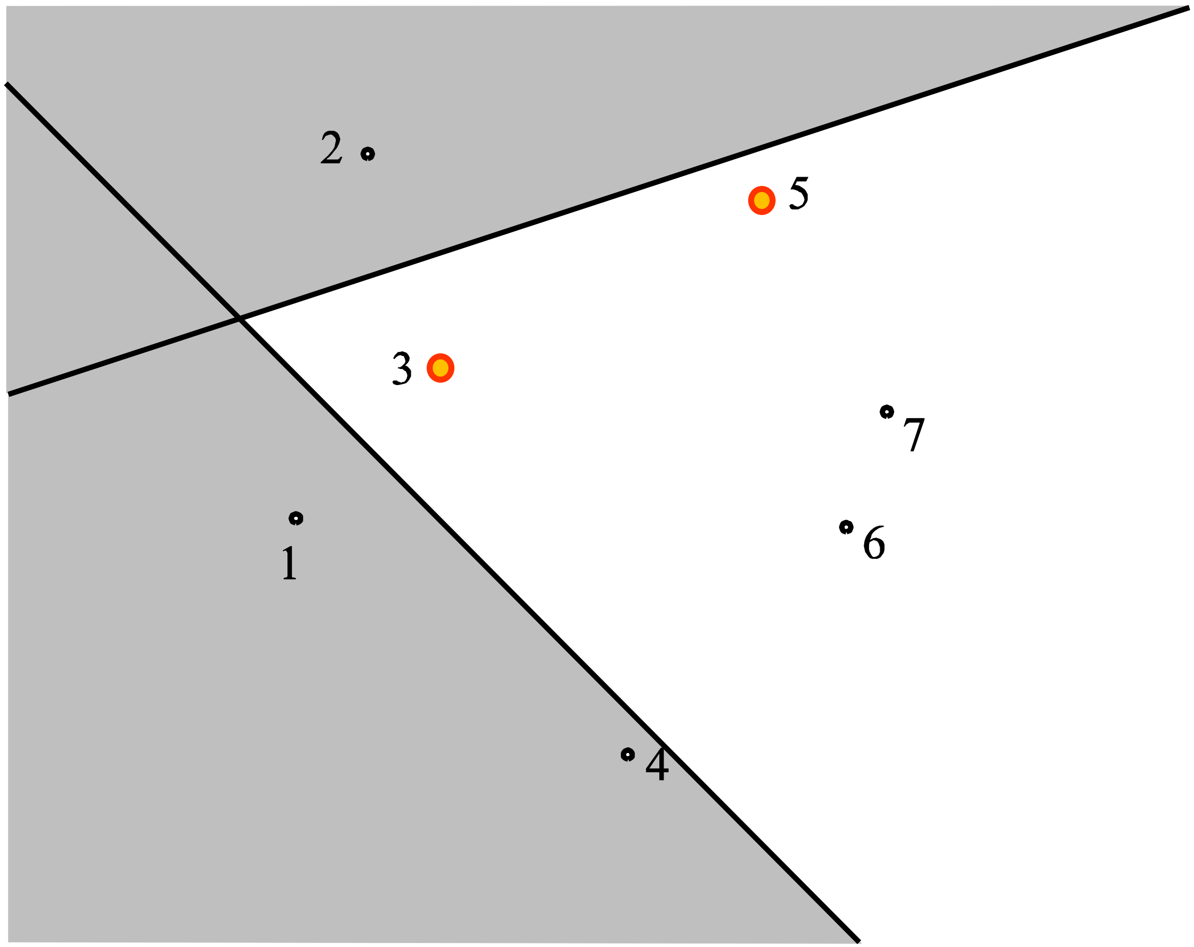

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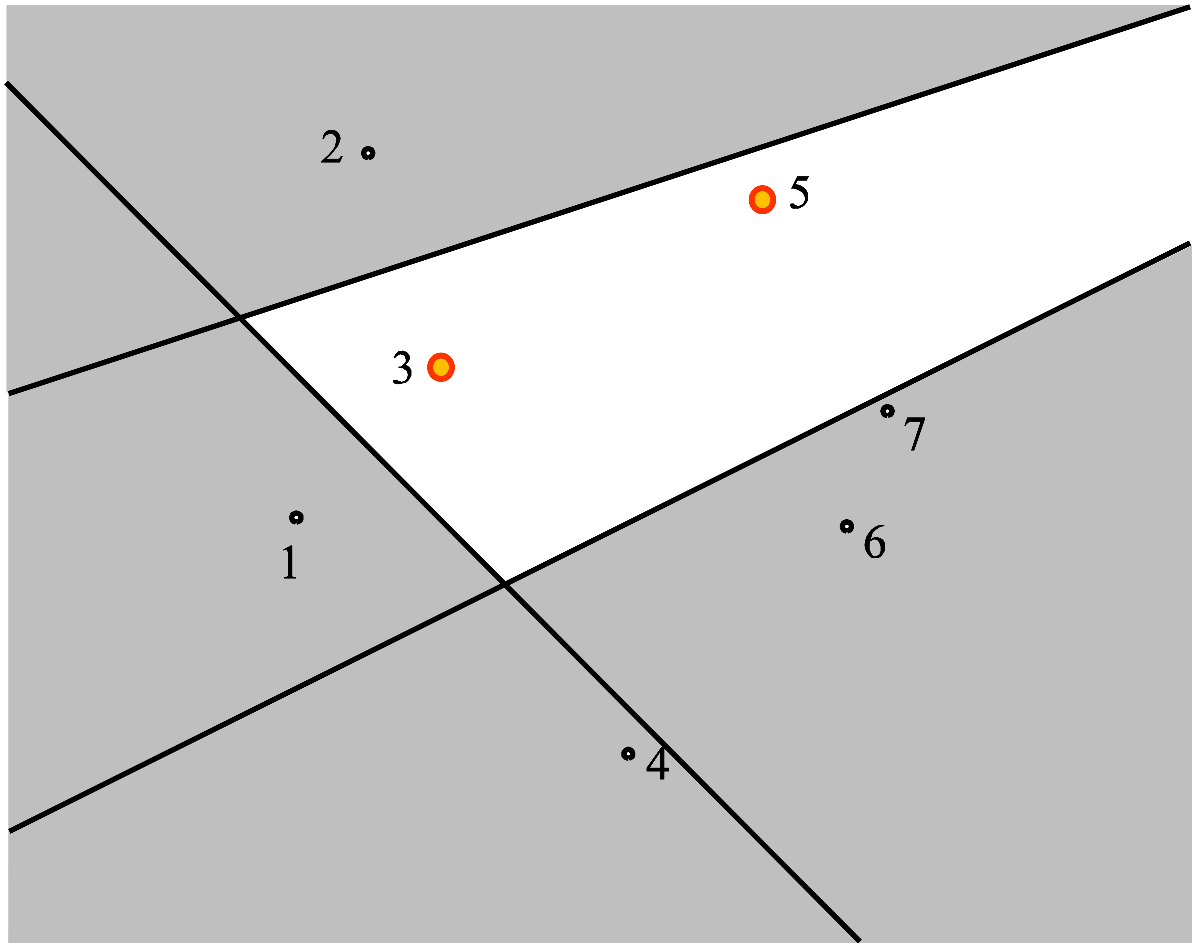

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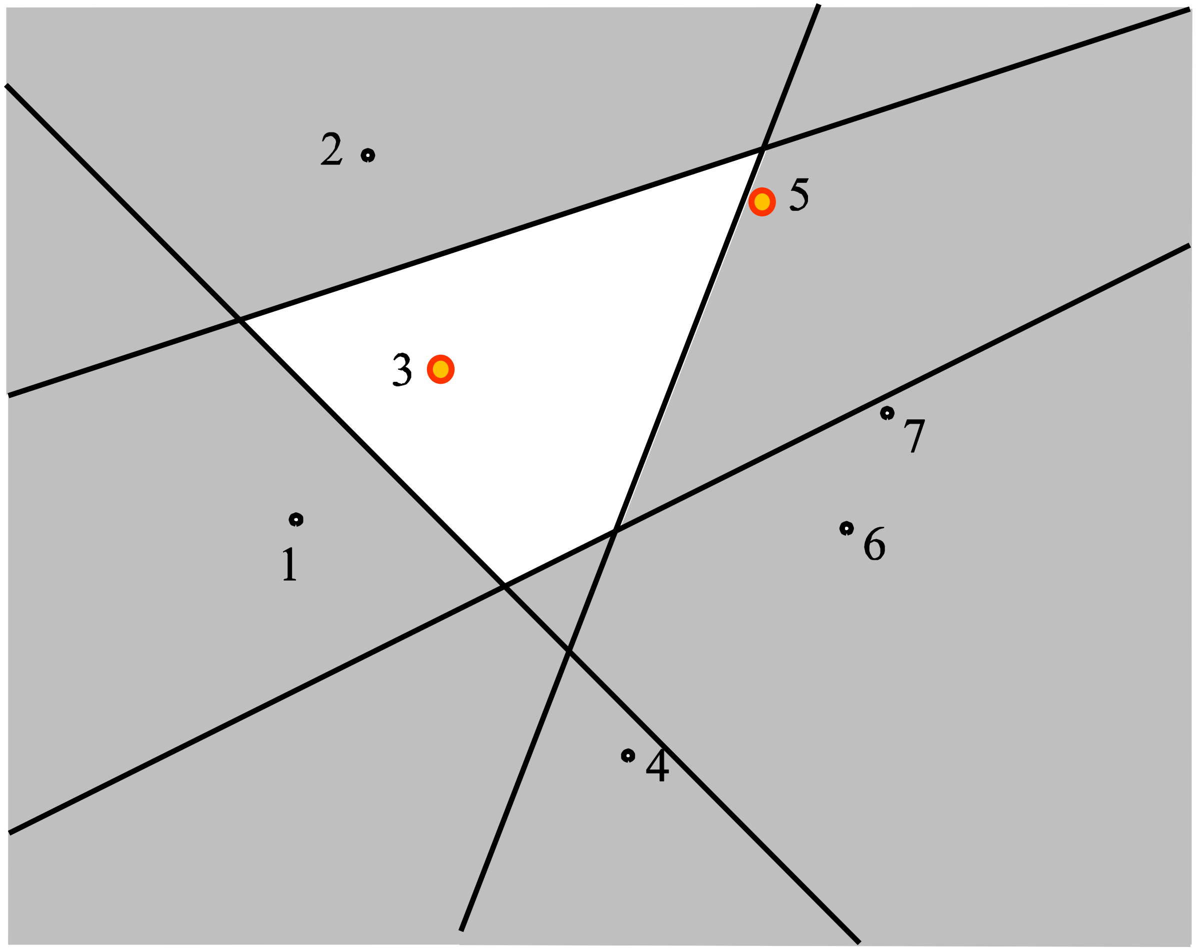

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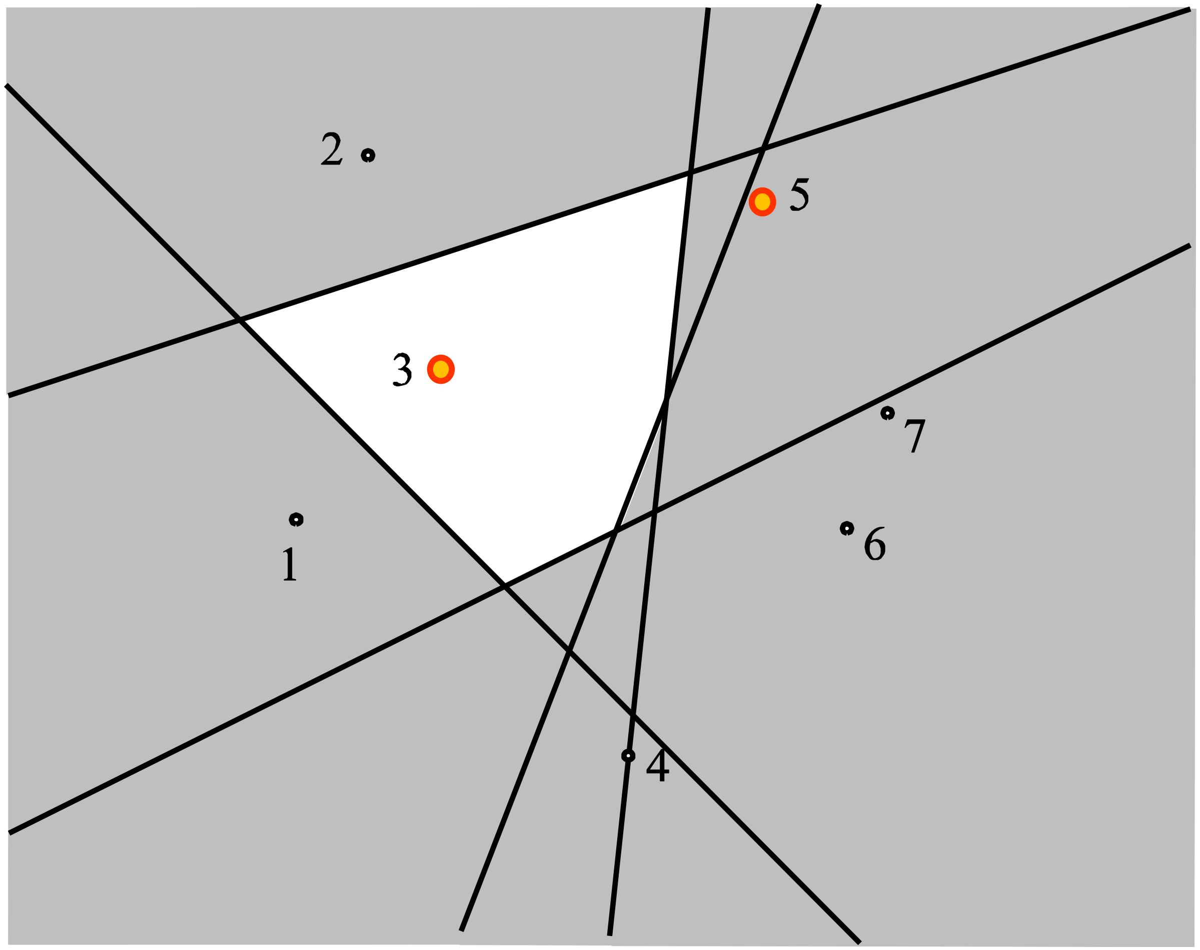

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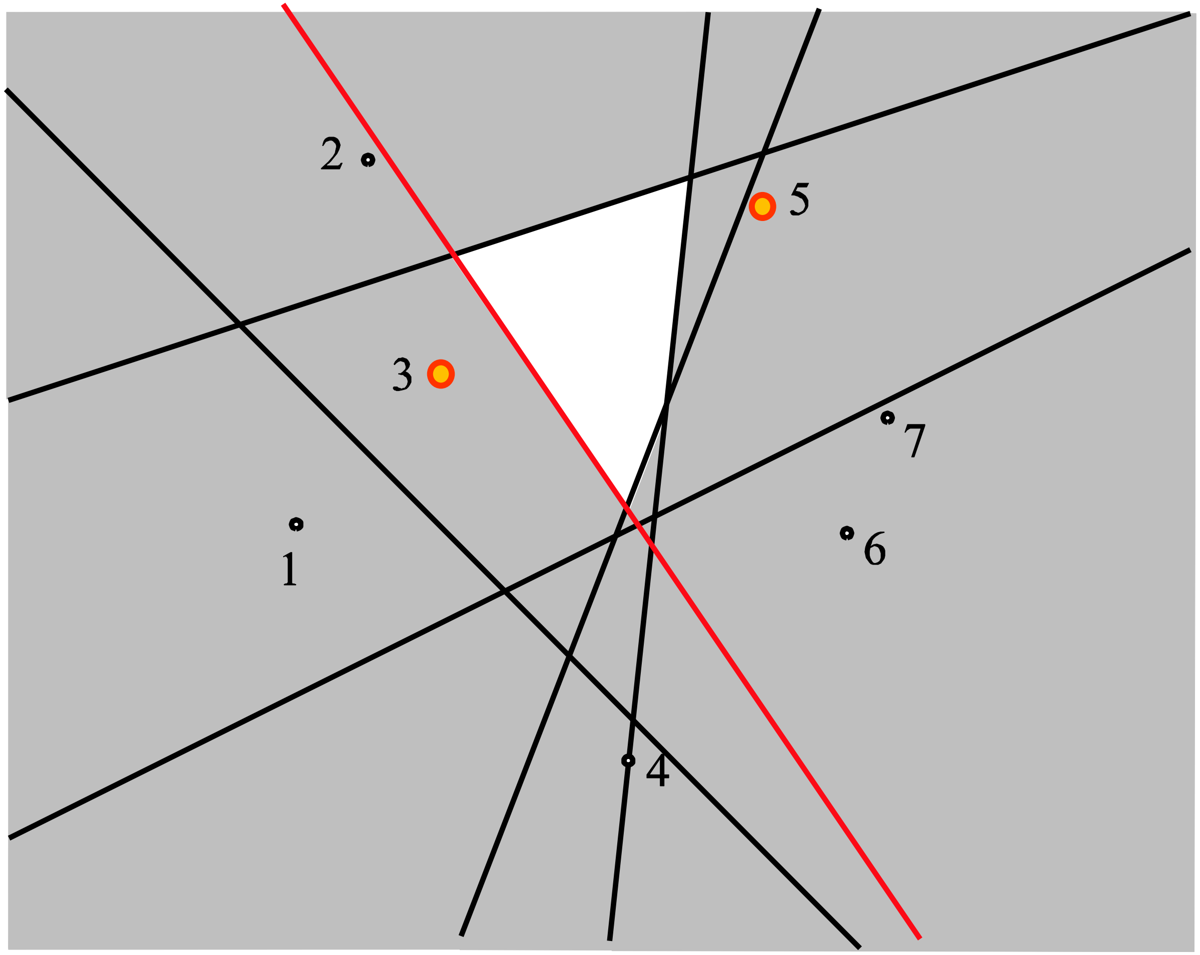

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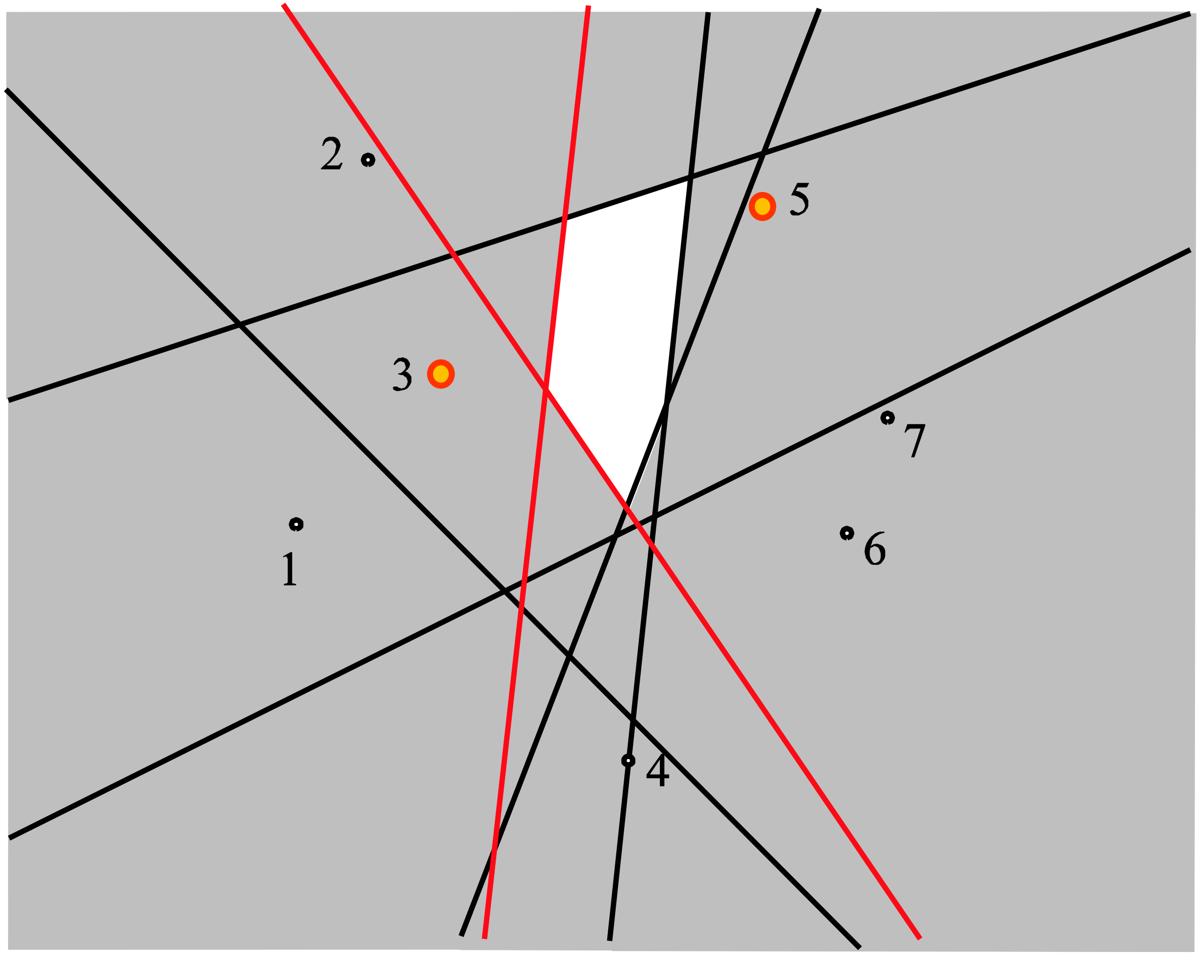

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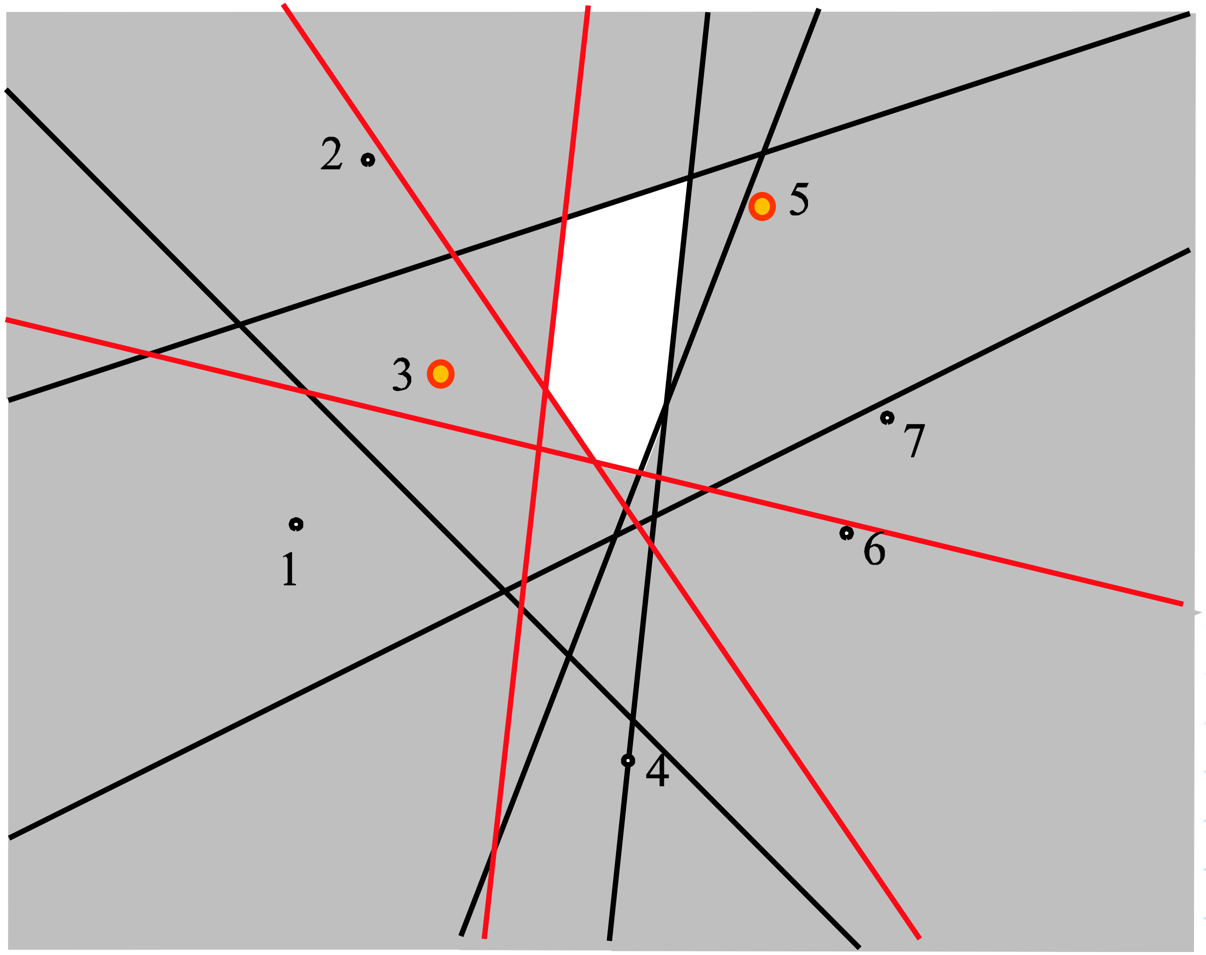

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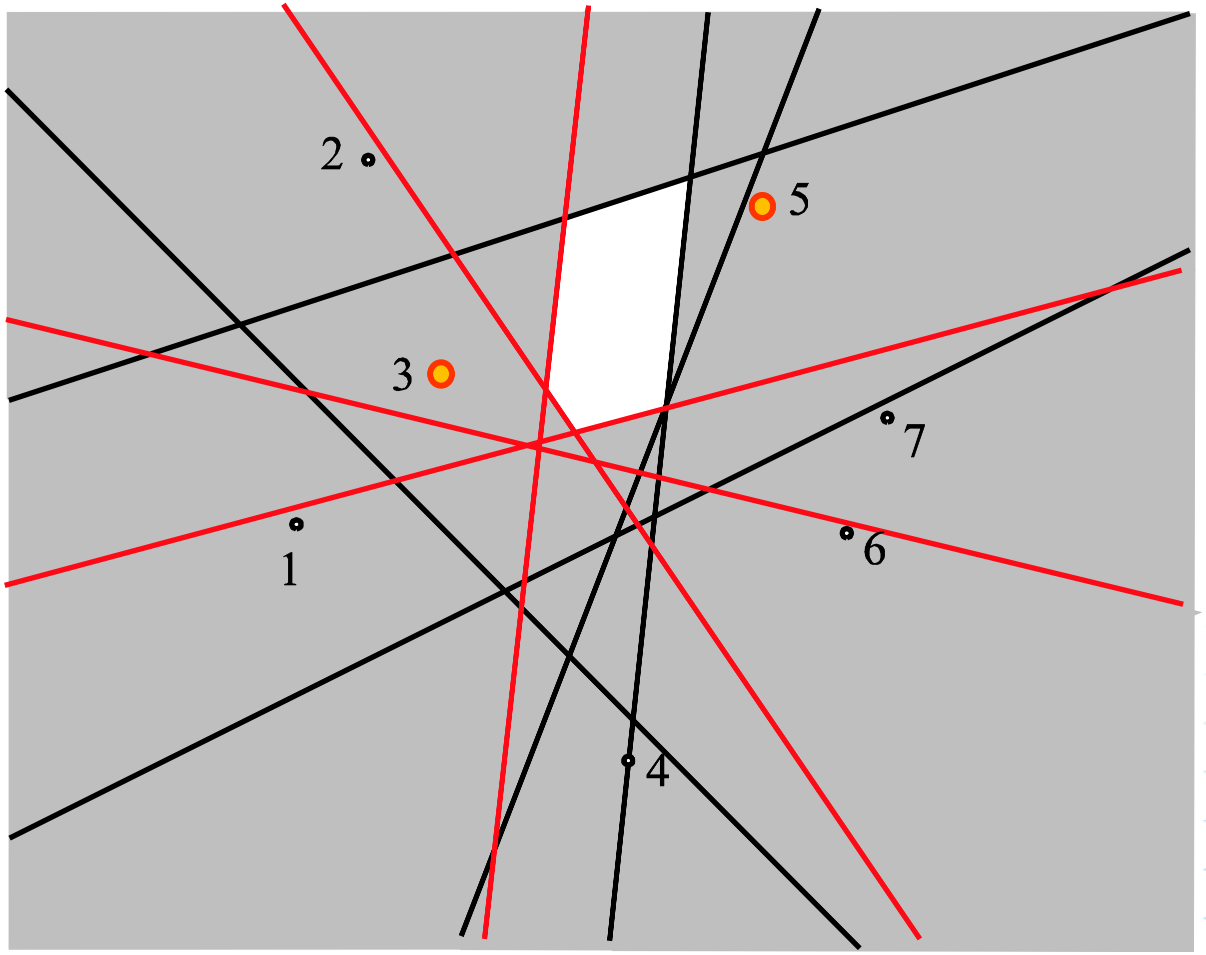

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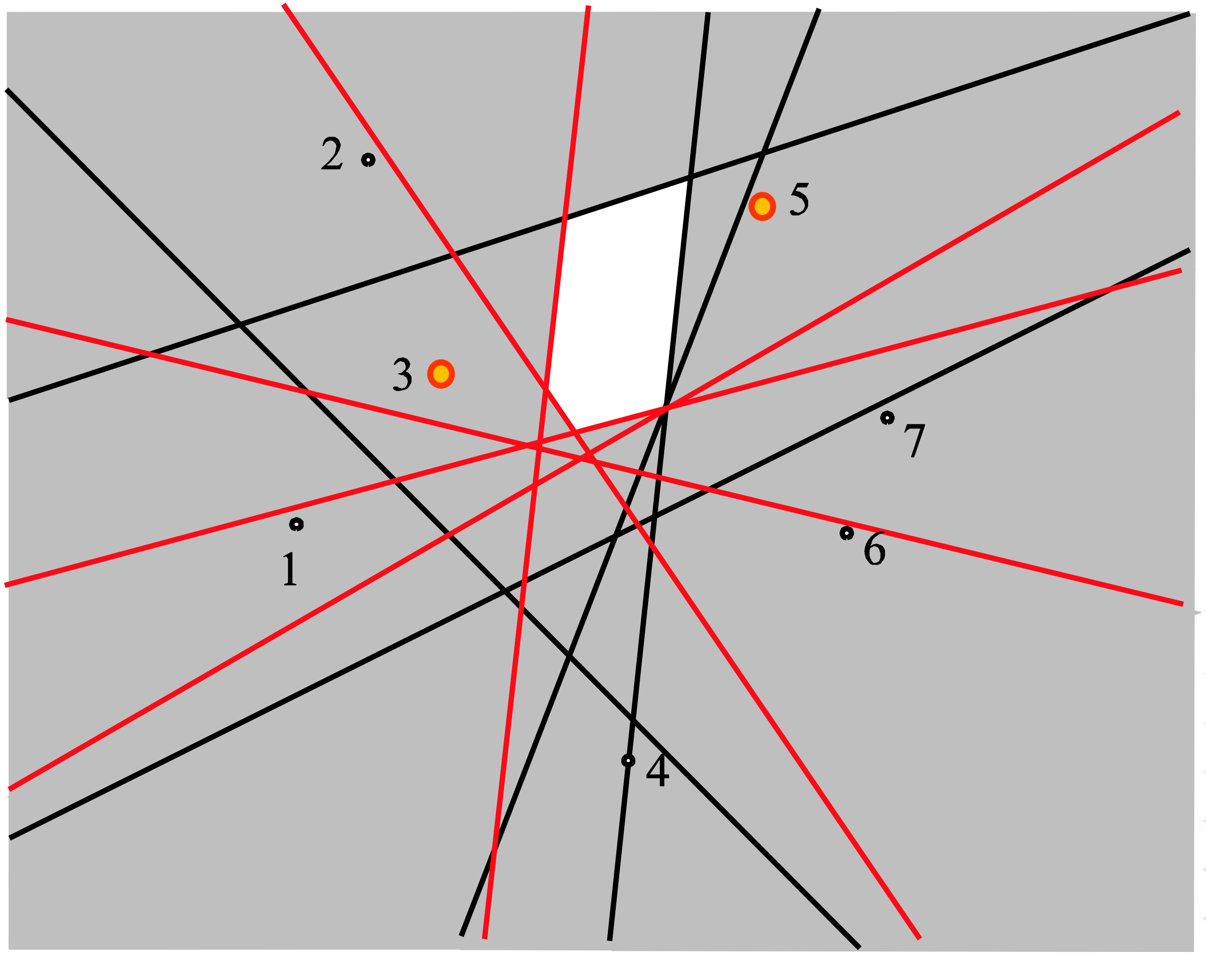

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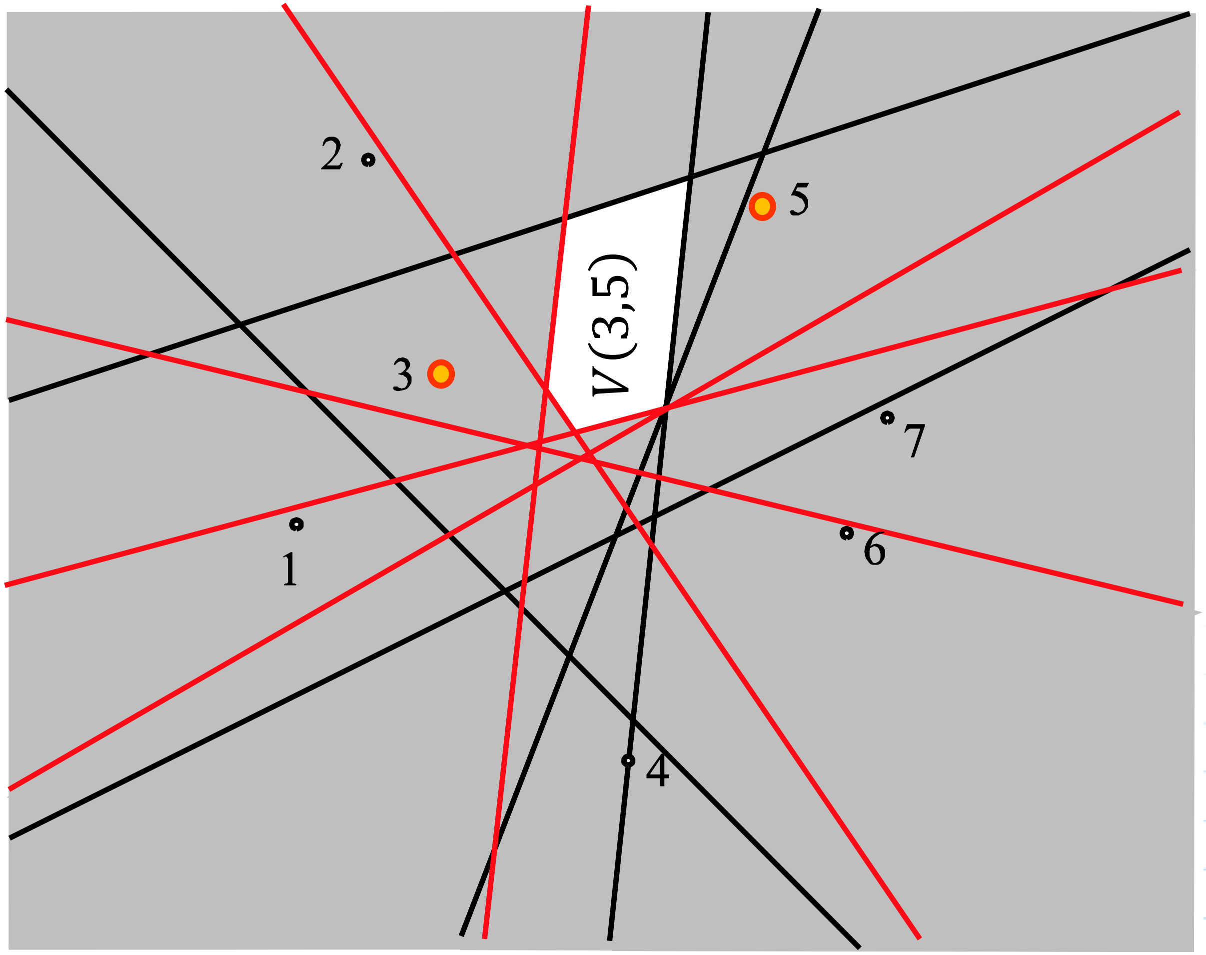

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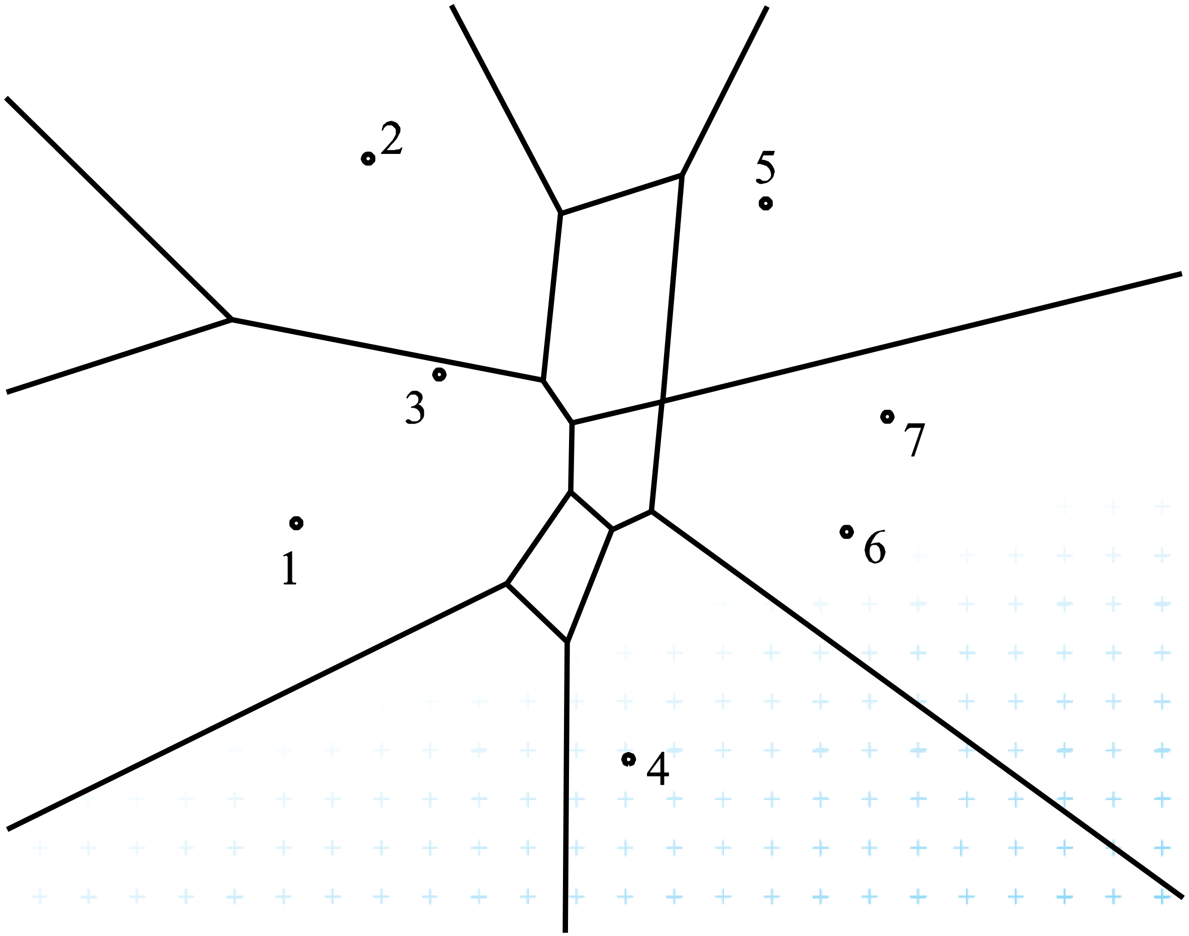
Order-2 Voronoi edges



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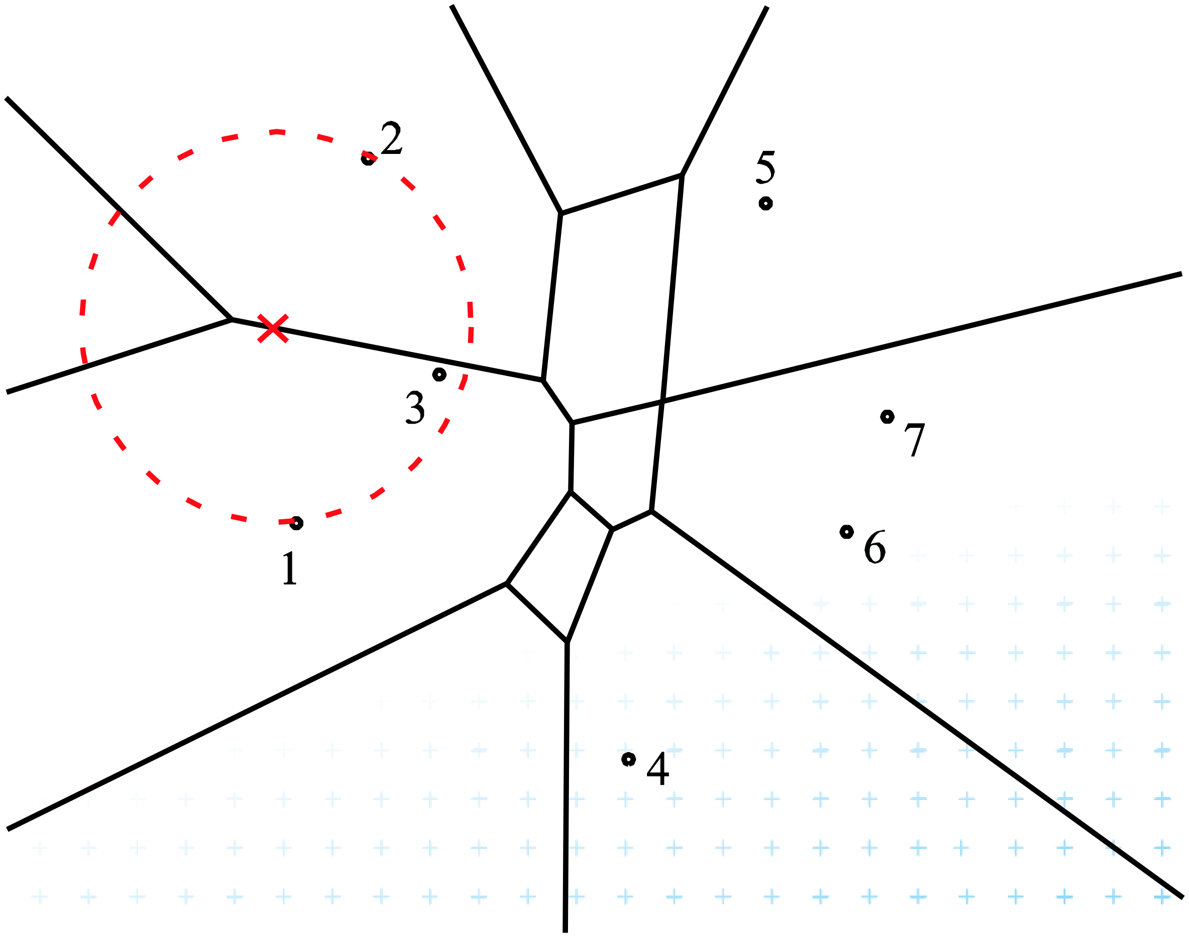
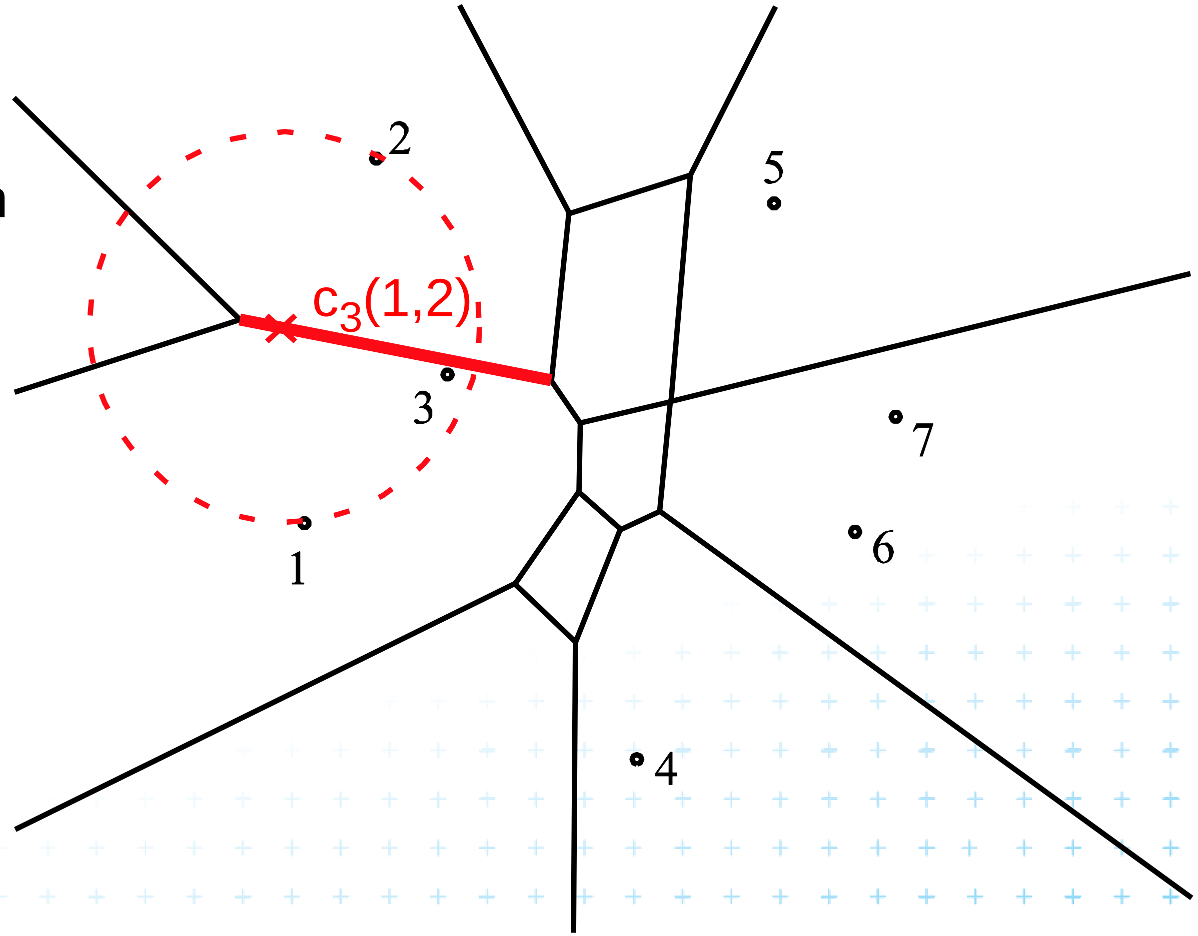


[Nandy]

Order-2 Voronoi edges

edge : set of centers of circles passing through 2 sites *s* and *t* and containing site *p*

=> $c_p(s,t)$



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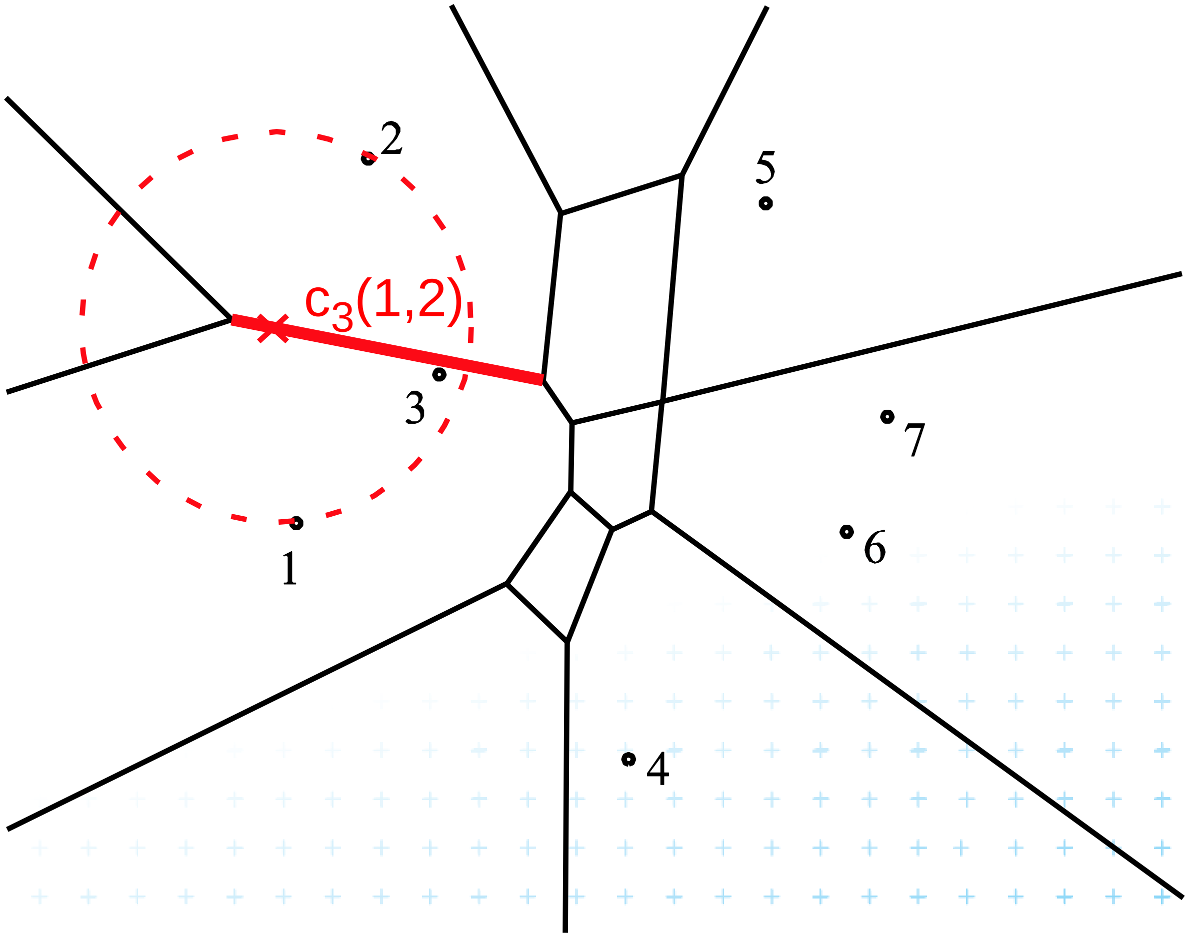

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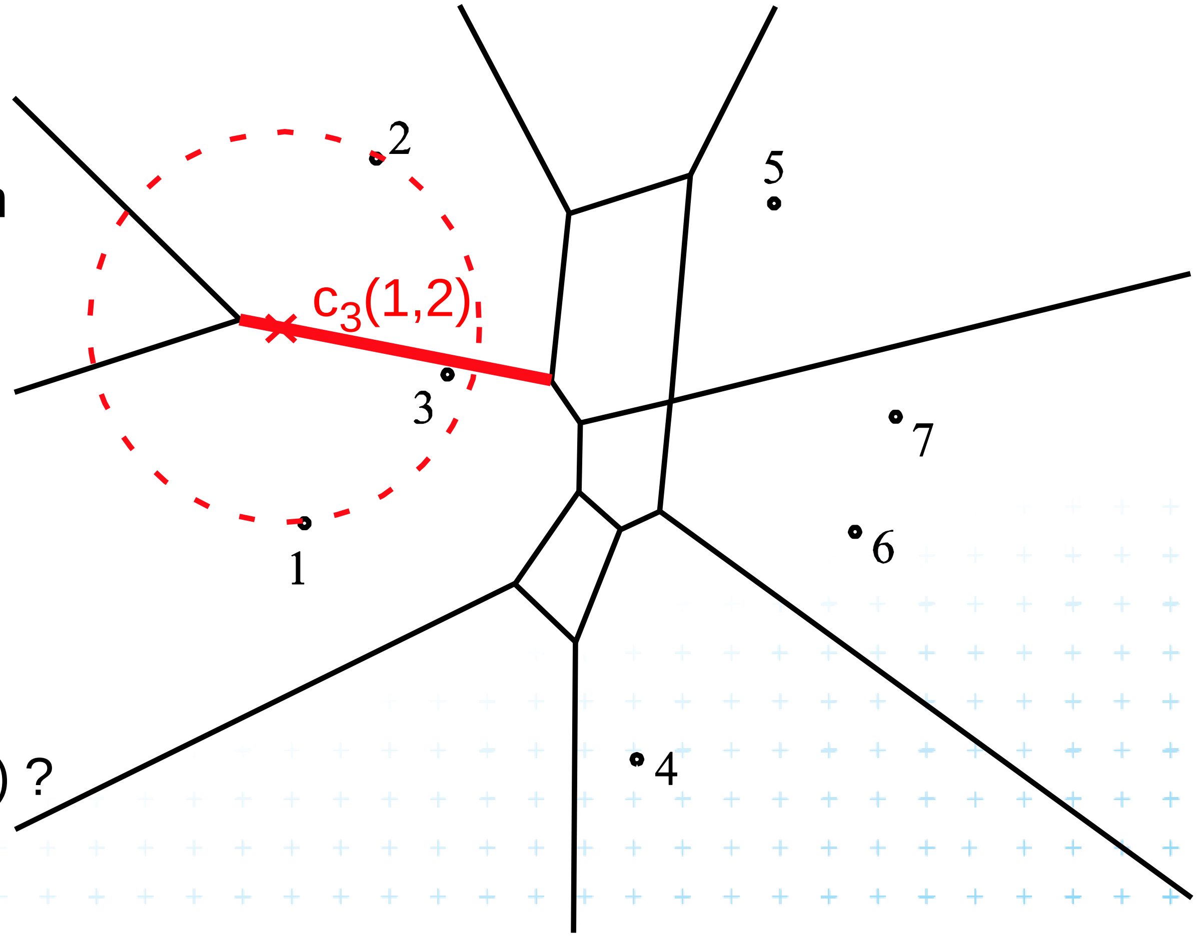

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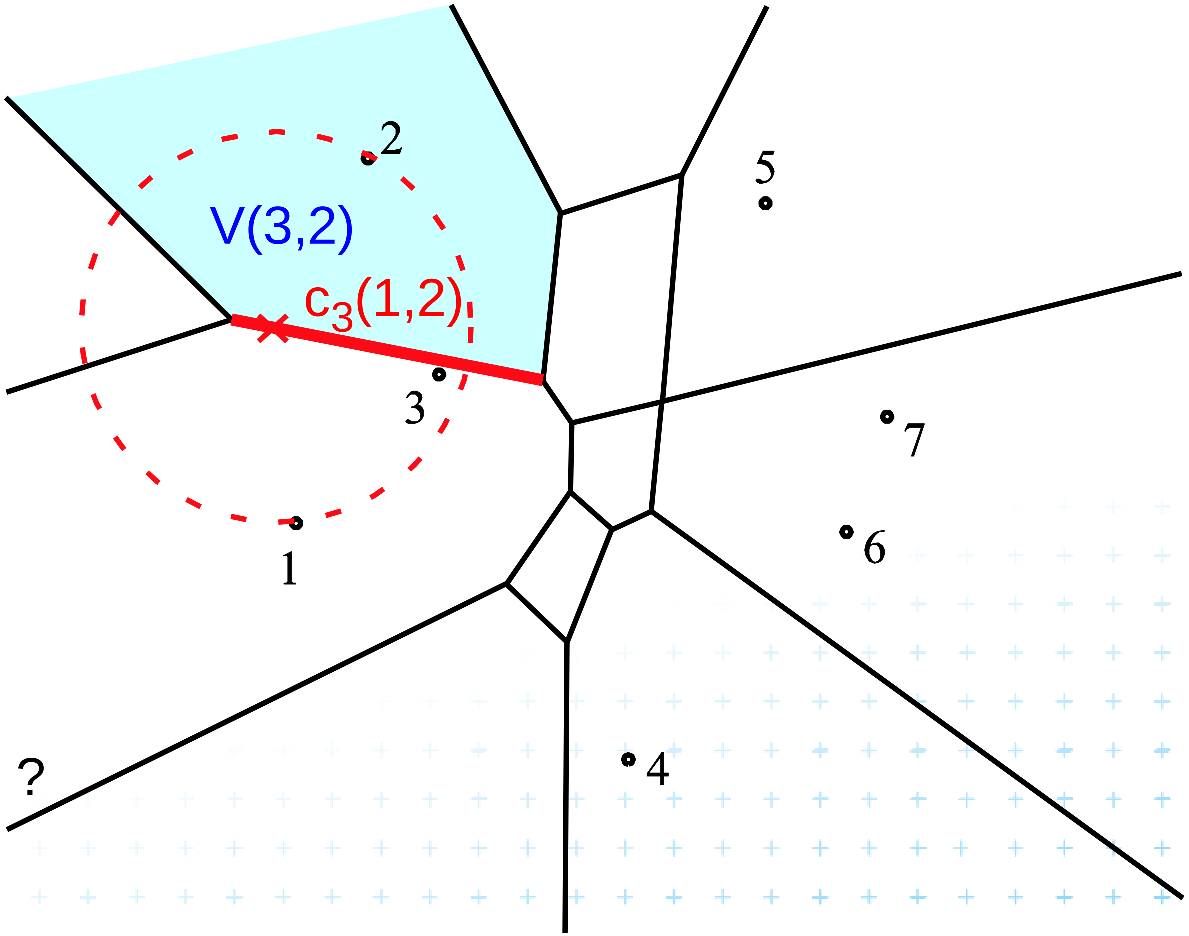

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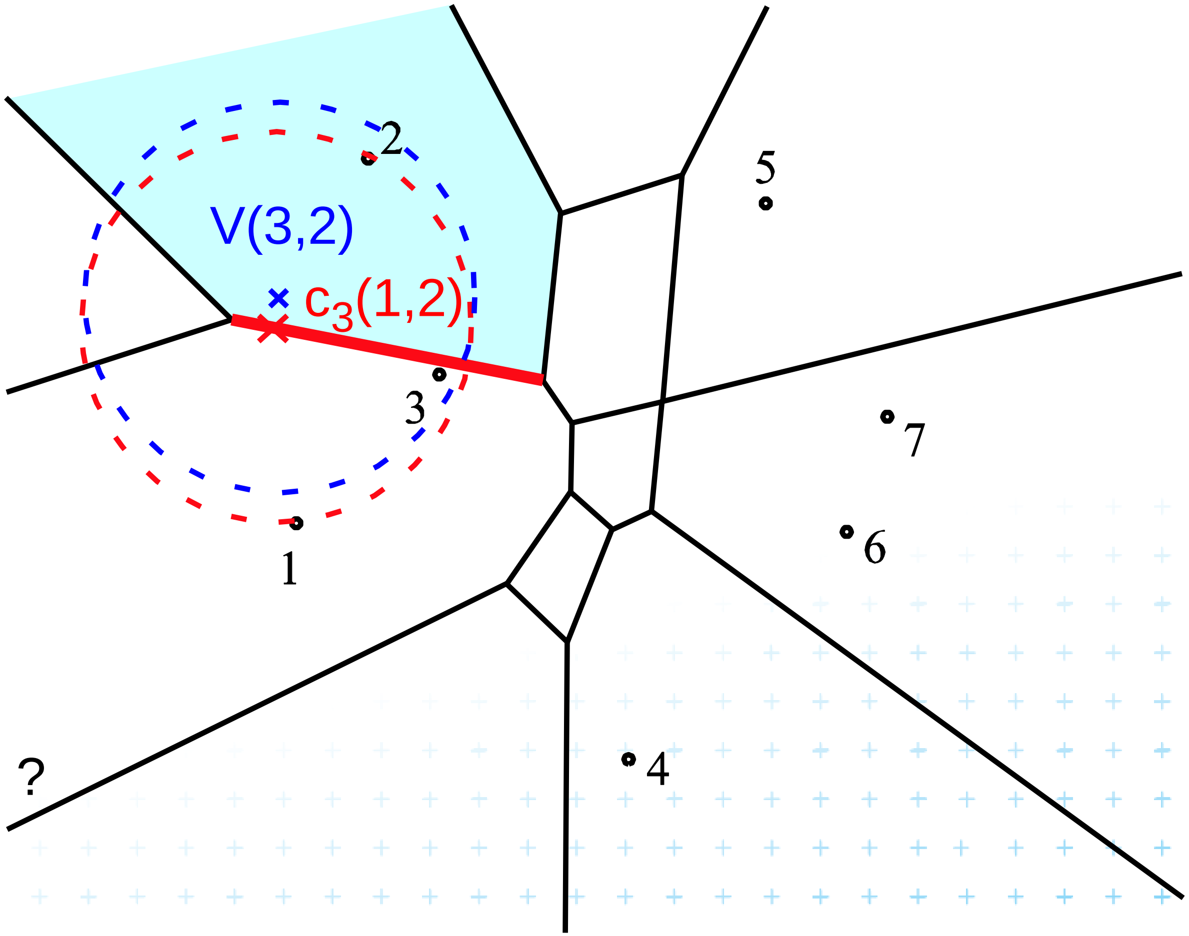

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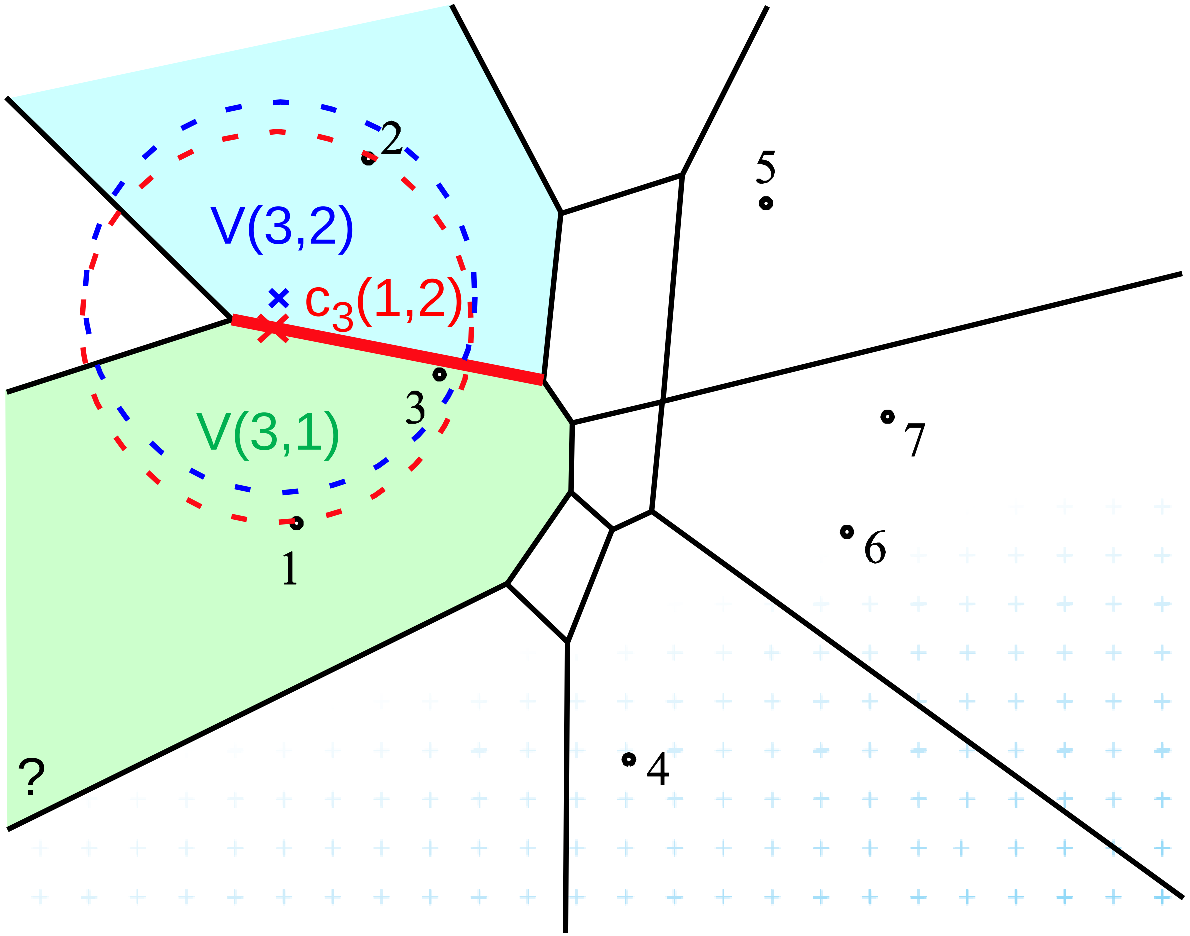

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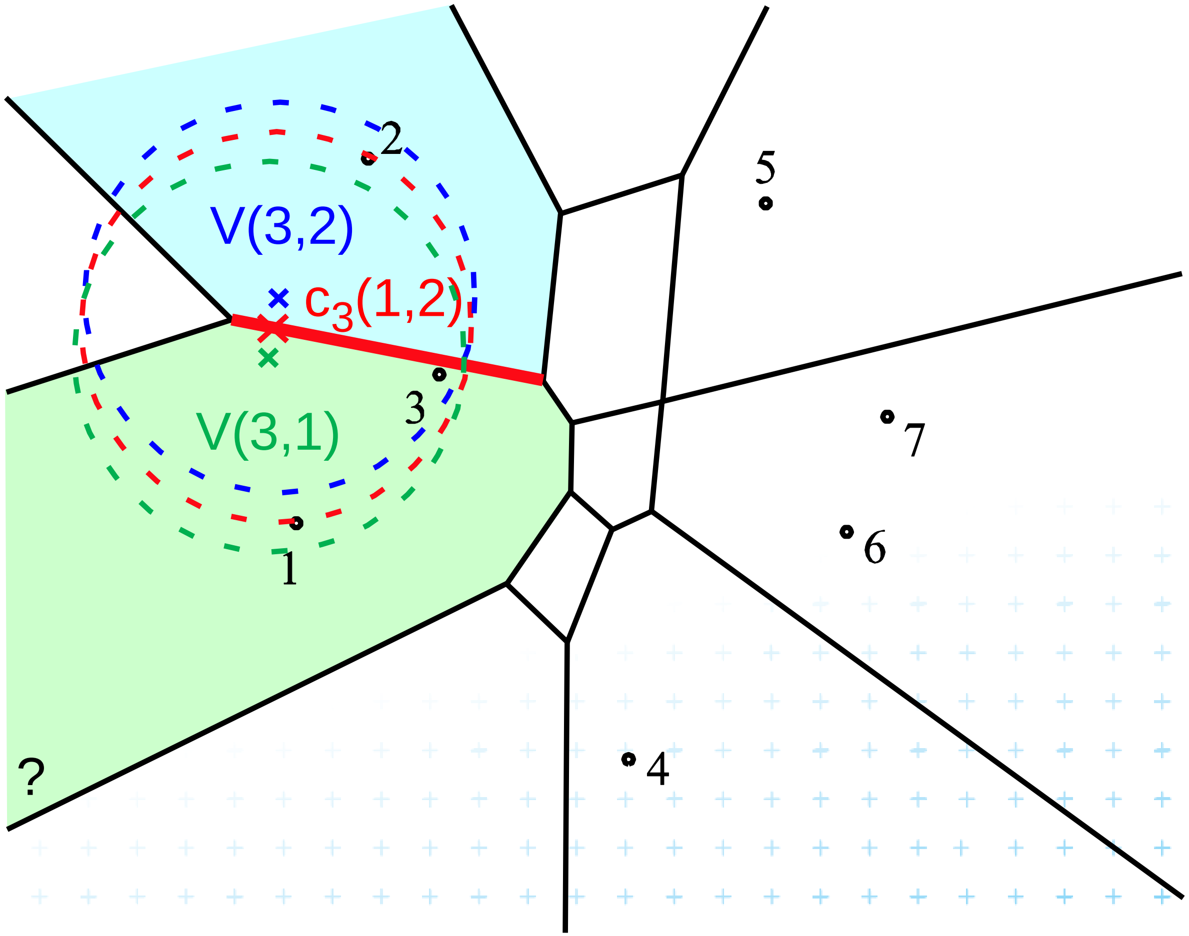

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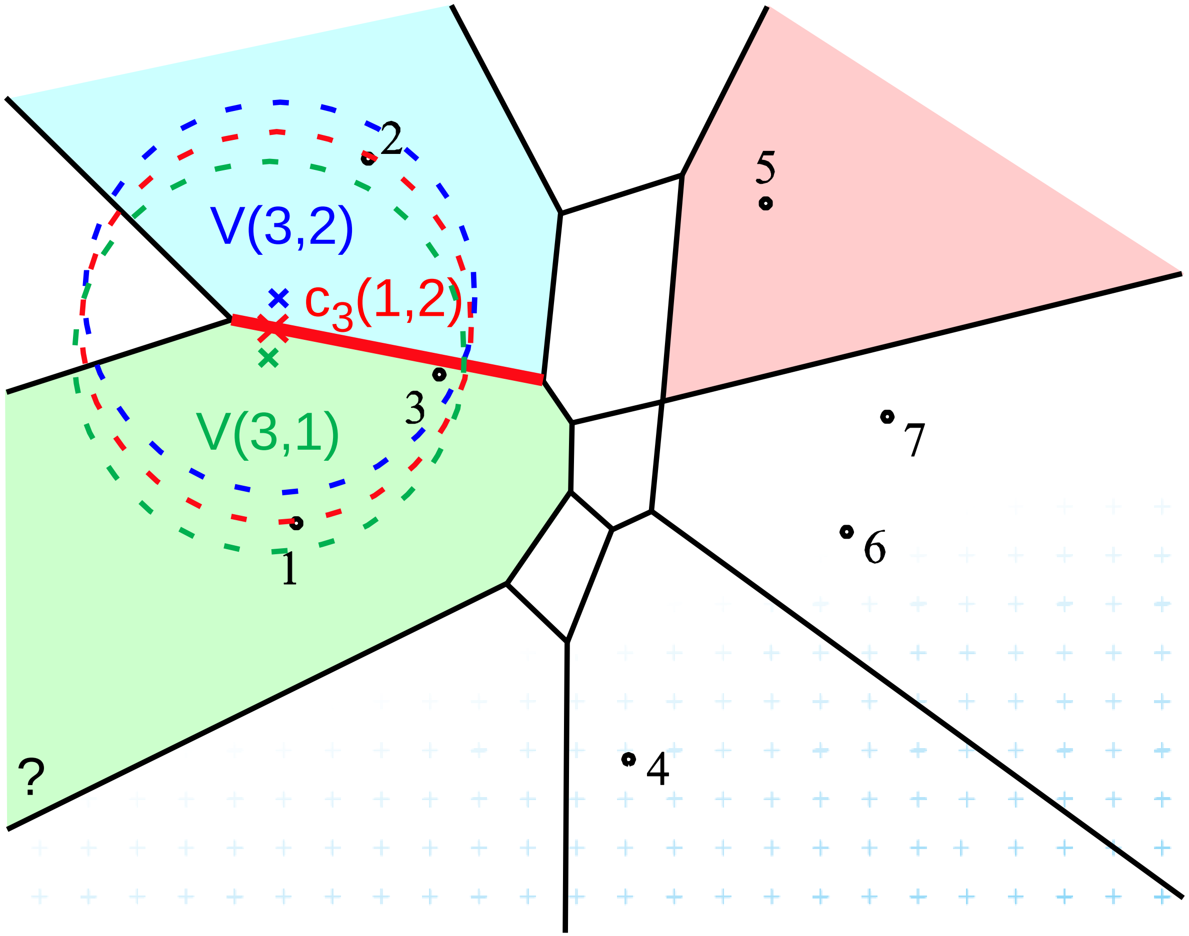

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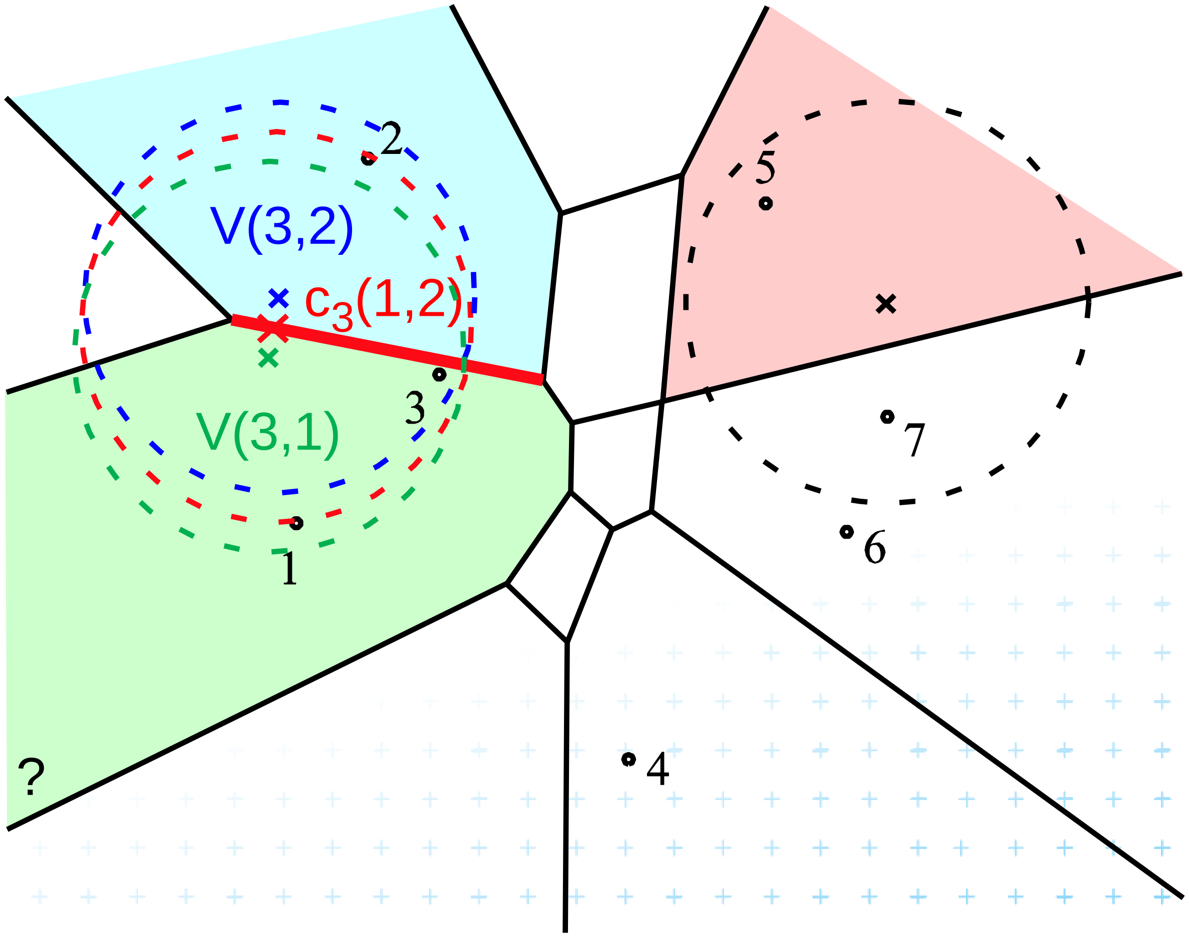

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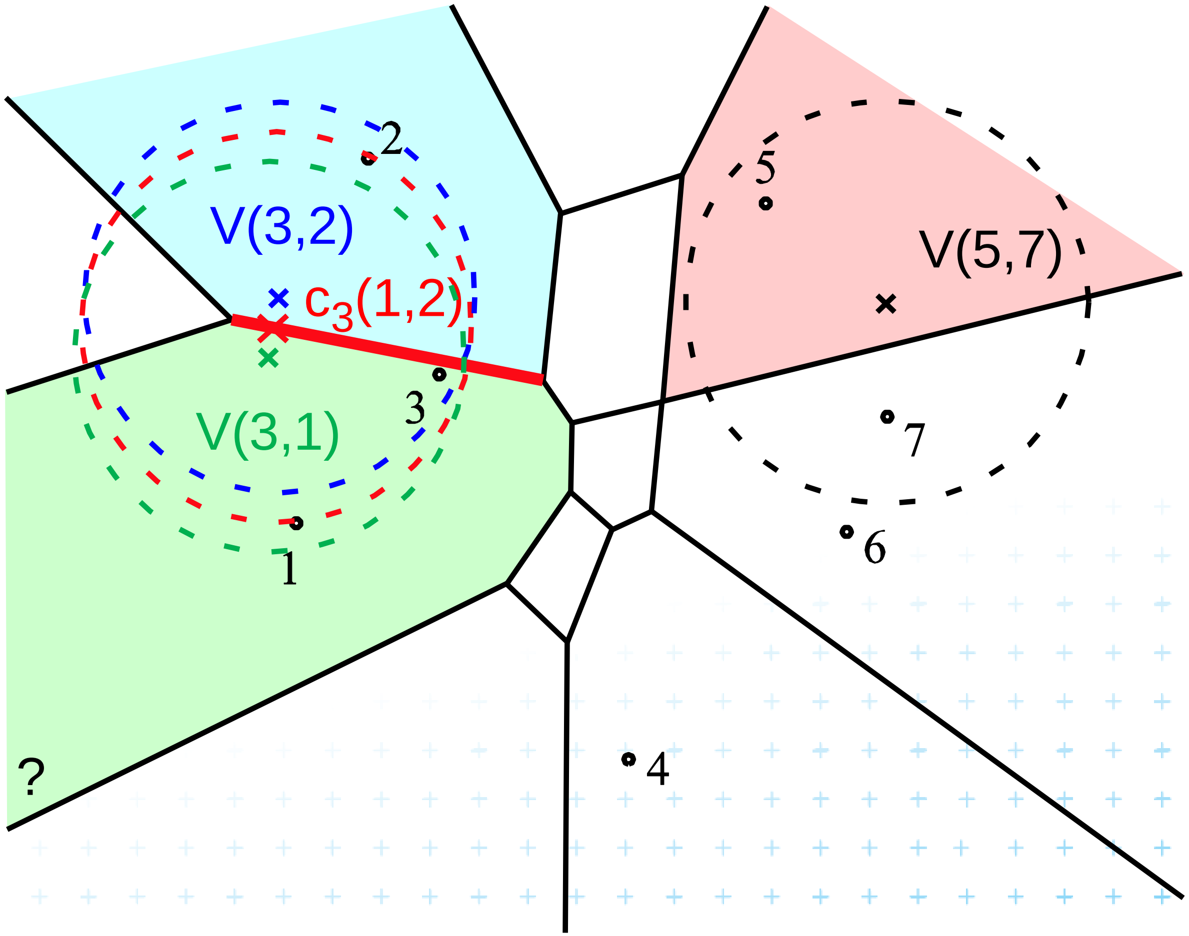

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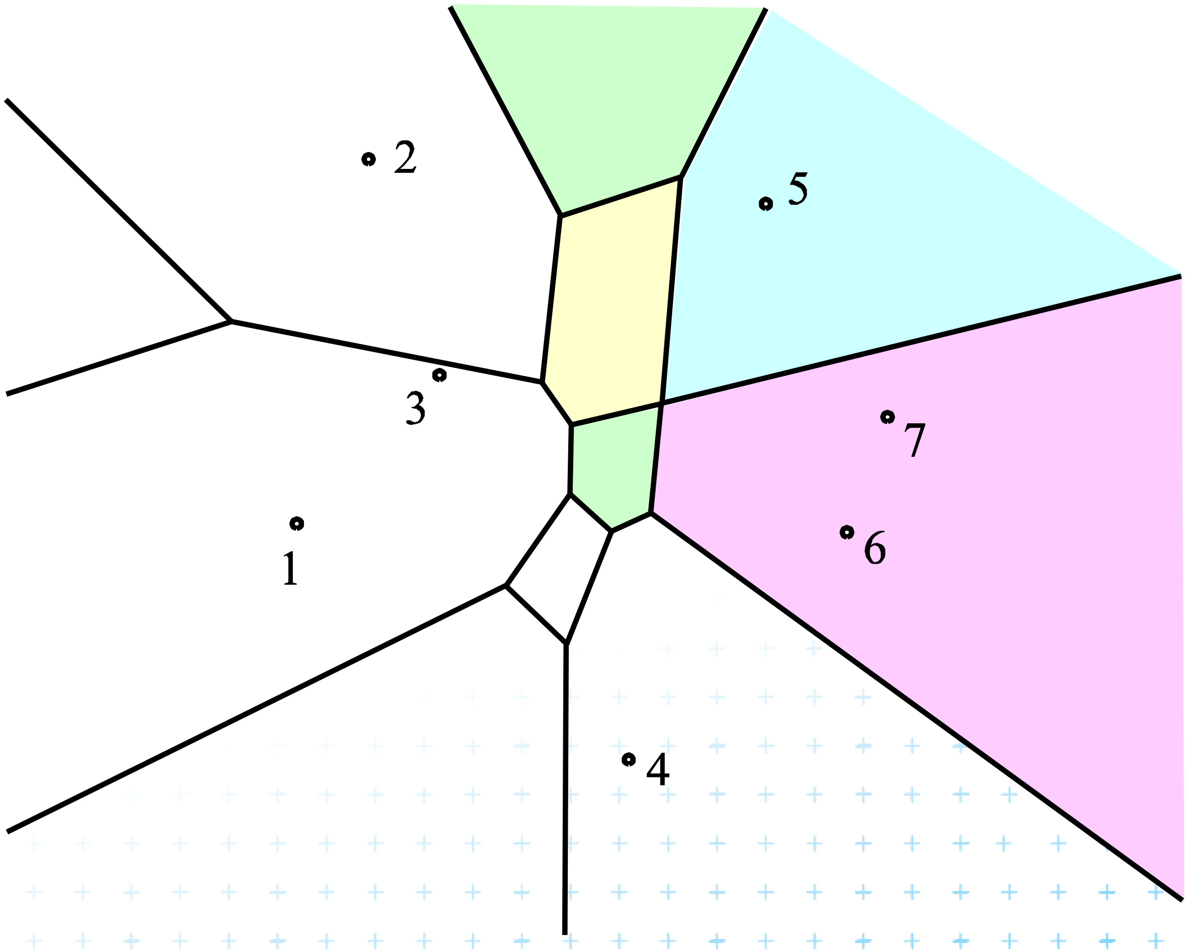
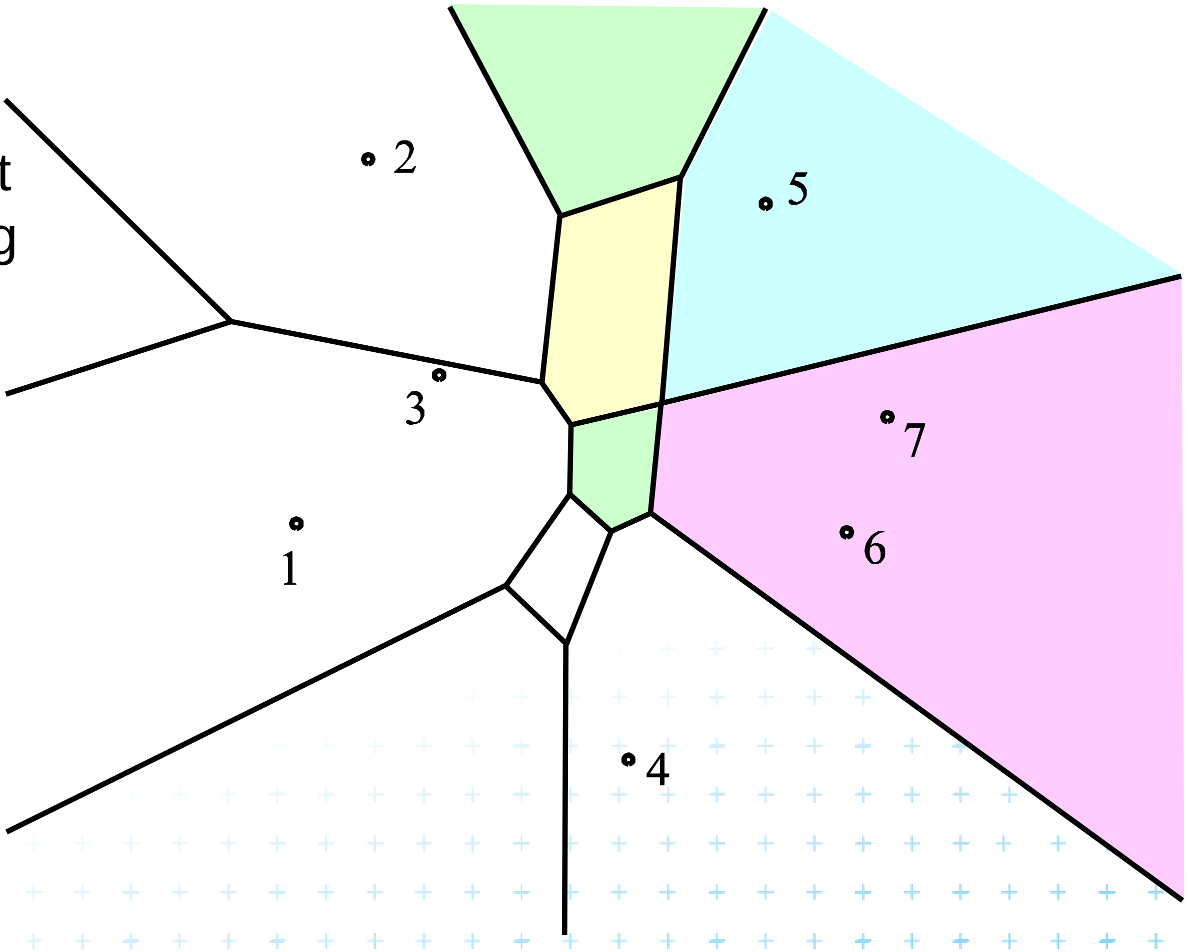


Order-2 Voronoi vertices

vertex : center of a circle passing through at least 3 sites Q and containing either site p or nothing



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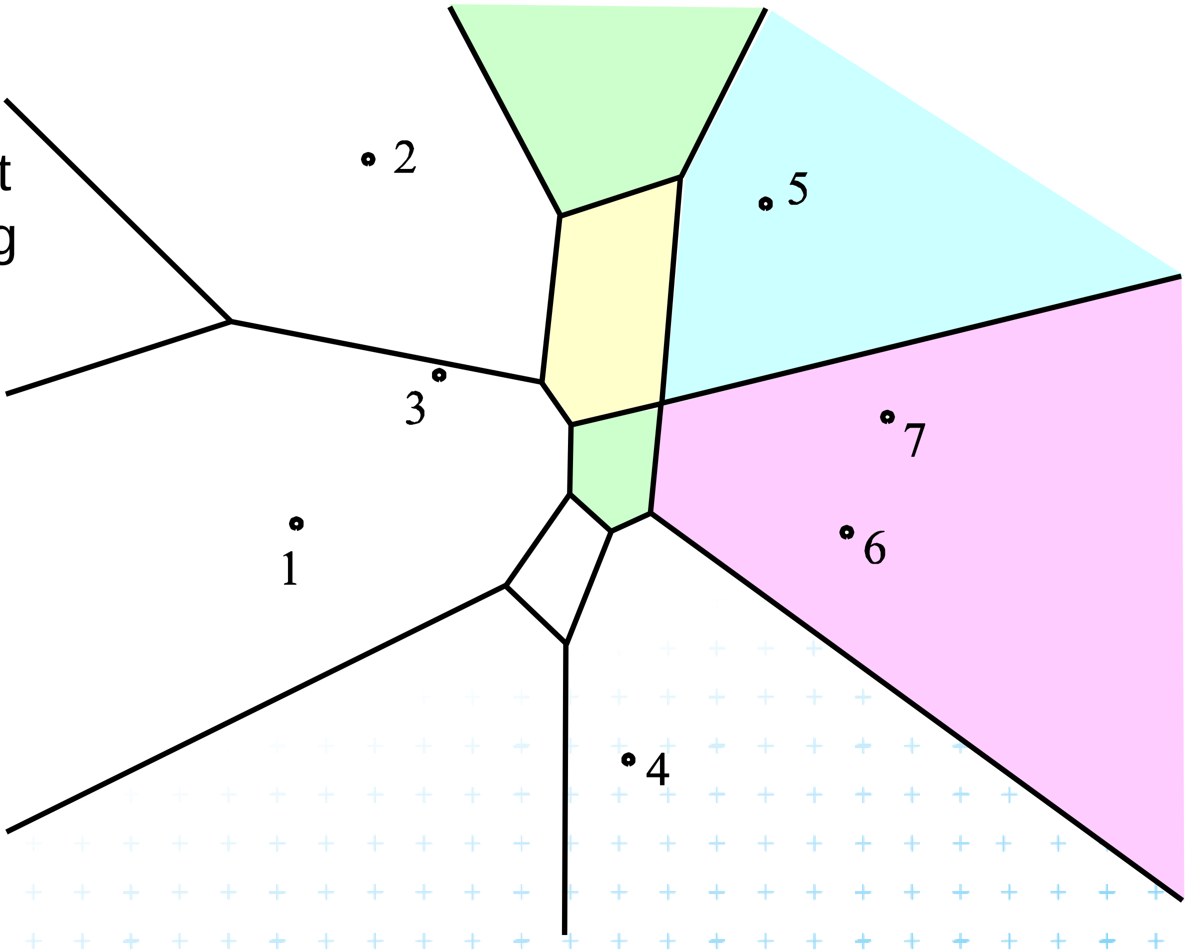

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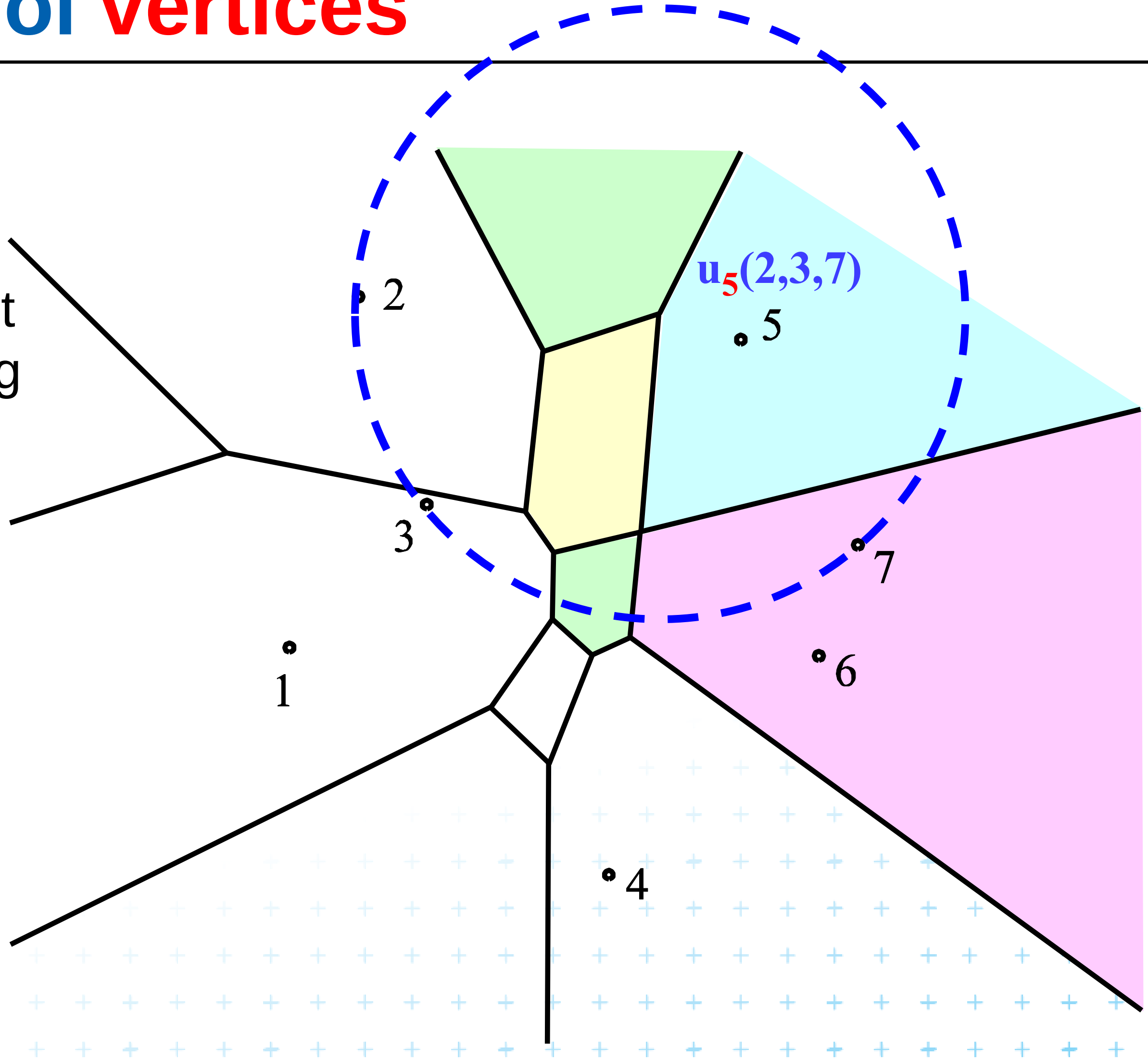
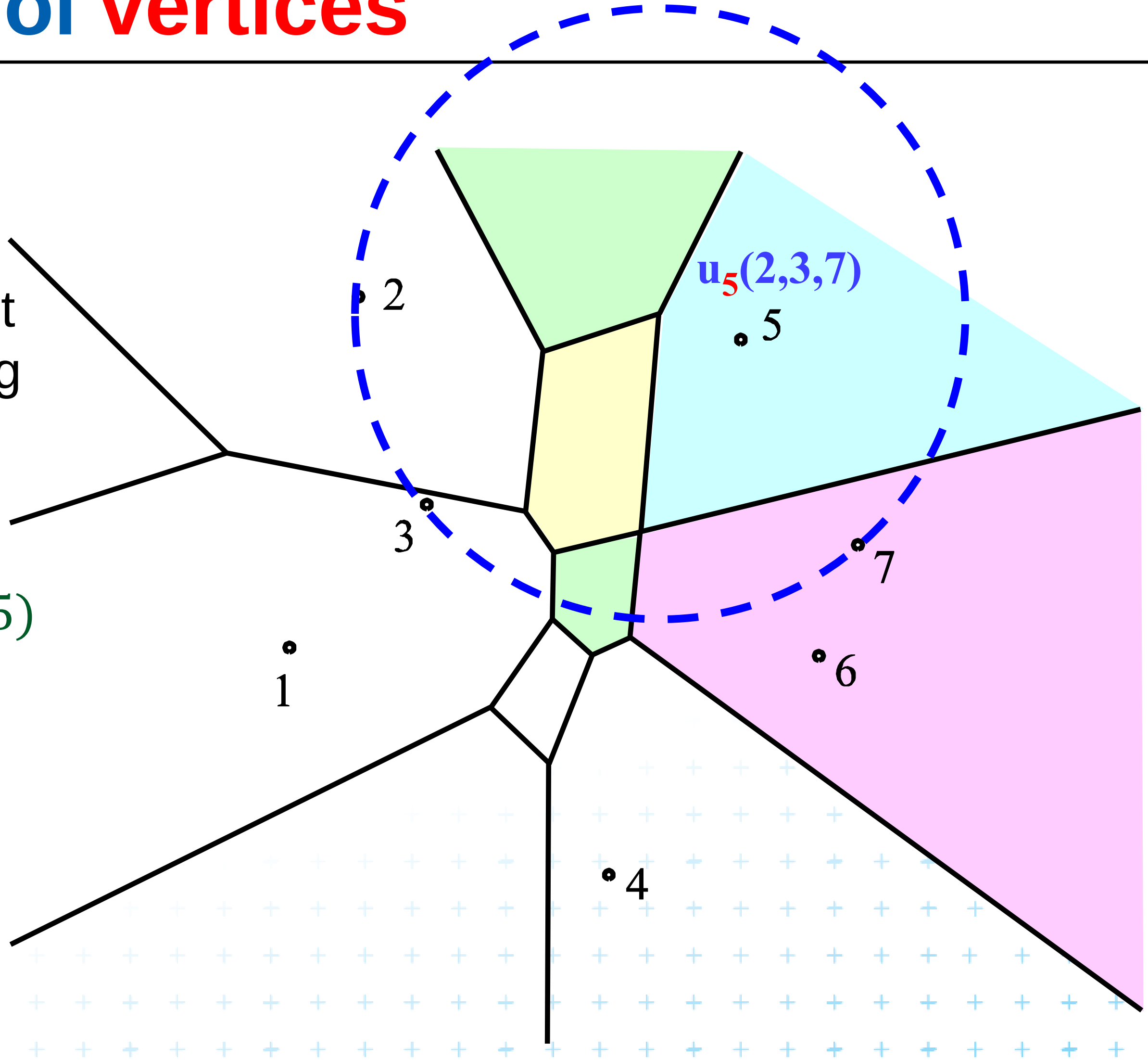


Order-2 Voronoi vertices

vertex : center of a circle passing through at least 3 sites Q and containing either site p or nothing

$\Rightarrow u_p(Q)$ or $u_\varnothing(Q \cup p)$
$u_5(2,3,7)$, $u_\varnothing(3,6,7,5)$



Order-2 Voronoi vertices

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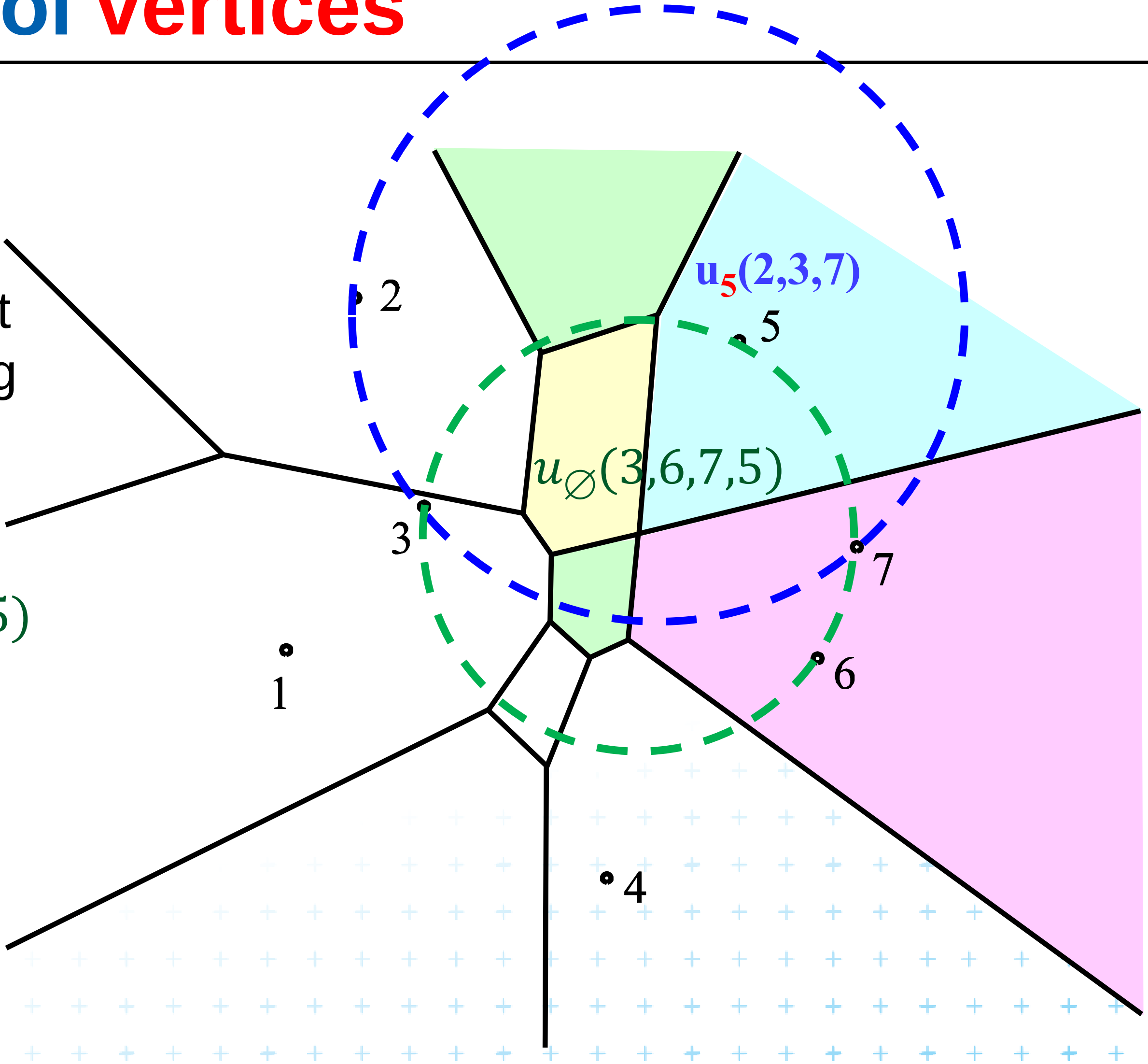
$u_5(2,3,7)$, $u_\varnothing(3,6,7,5)$

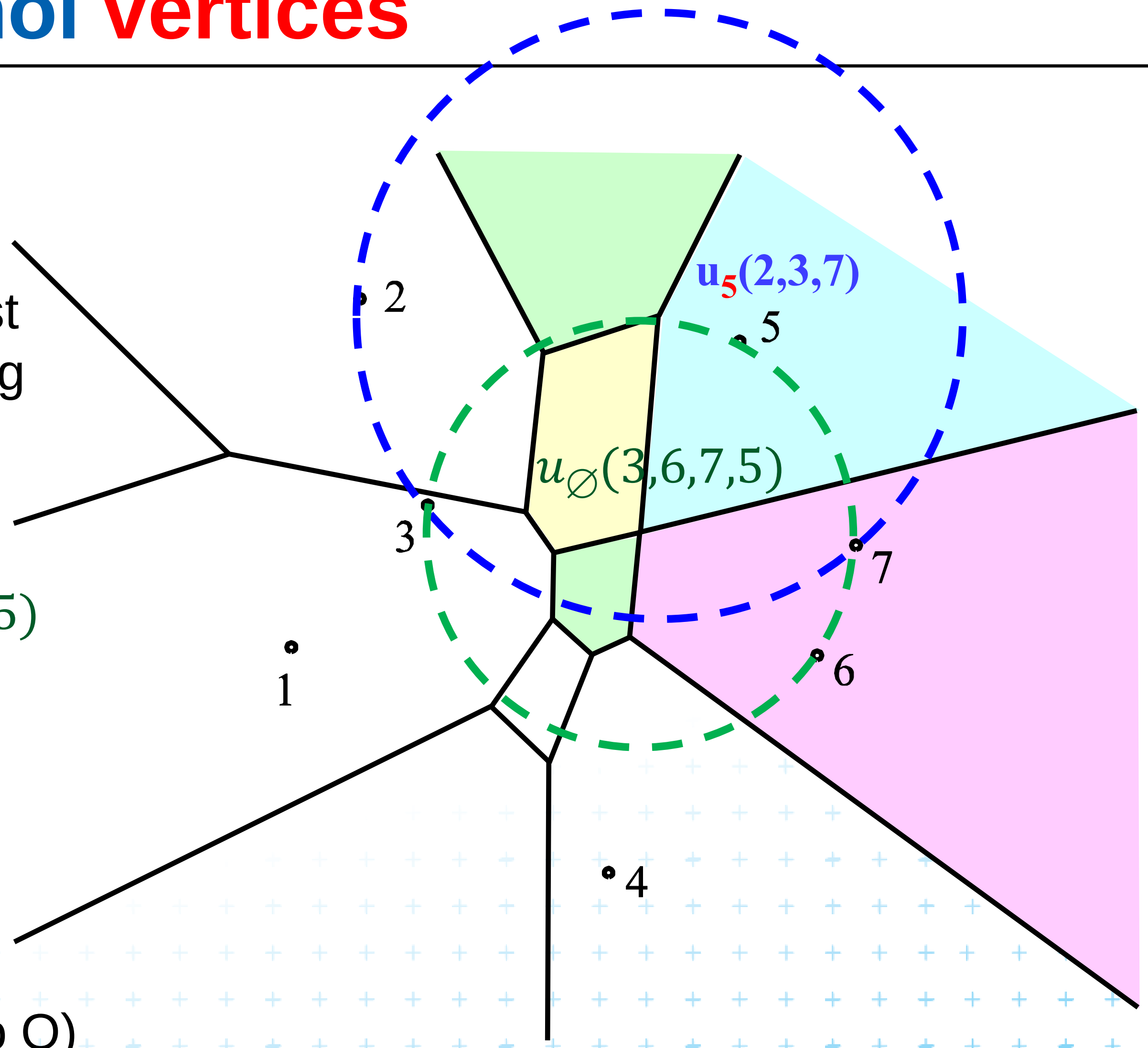


Order-2 Voronoi vertices

vertex : center of a circle passing through at least 3 sites Q and containing either site p or nothing

$\Rightarrow u_p(Q)$ or $u_\emptyset(Q \cup p)$

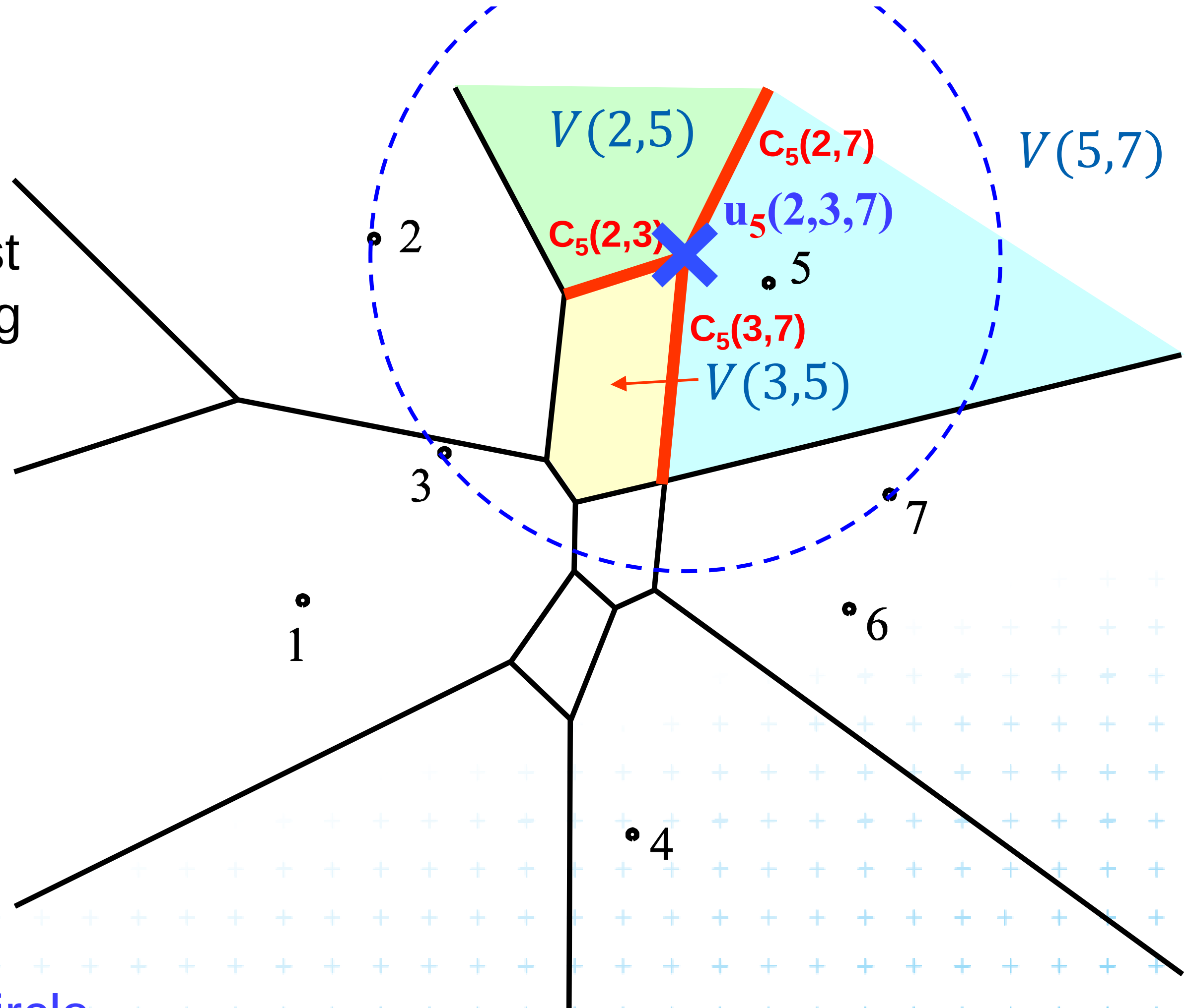
$u_5(2,3,7)$, $u_\emptyset(3,6,7,5)$



(circle circumscribed to Q)

Order-2 Voronoi vertex $u_p(Q)$

vertex : center of a circle passing through at least 3 sites Q and containing either site p or nothing



Case $u_p(Q)$
$u_5(2,3,7)$

Cell 5 is inside for all incident edges:

$C_5(2,3)$
$C_5(2,7)$
$C_5(3,7)$

=> 5 is inside for the circle with center in Voronoi vertex

Order-2 Voronoi **vertex** $u_{\varnothing}(Q \cup p)$

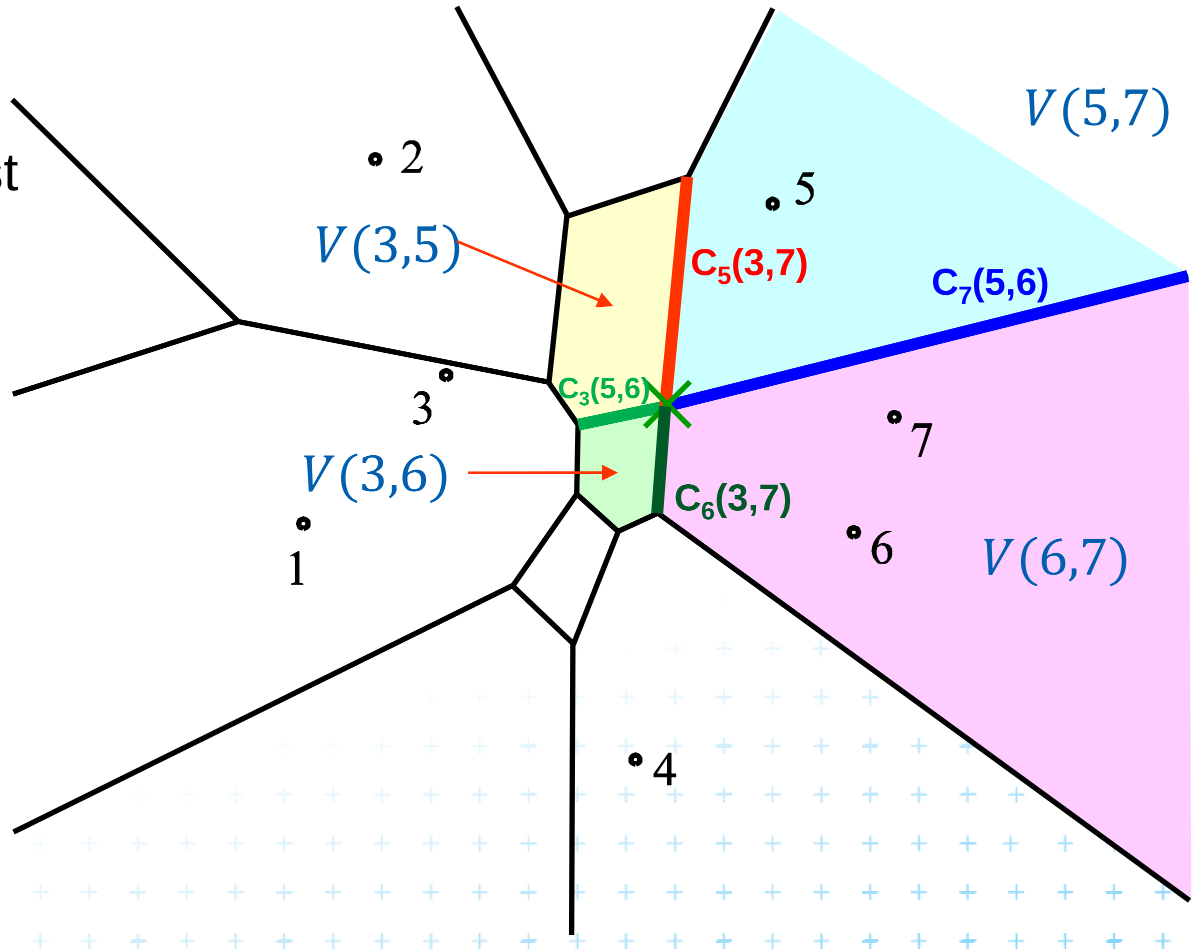
vertex : center of a circle passing through at least 3 sites and containing either site p or nothing

Case $u_{\varnothing}(Q \cup p)$
$u_{\varnothing}(3,5,6,7)$

Cell 5 is not inside for all incident edges:

$C_5(3,7)$
$C_6(3,7)$
$C_3(5,6)$
$C_7(5,6)$

=> 5 is on circle with center in Voronoi vertex

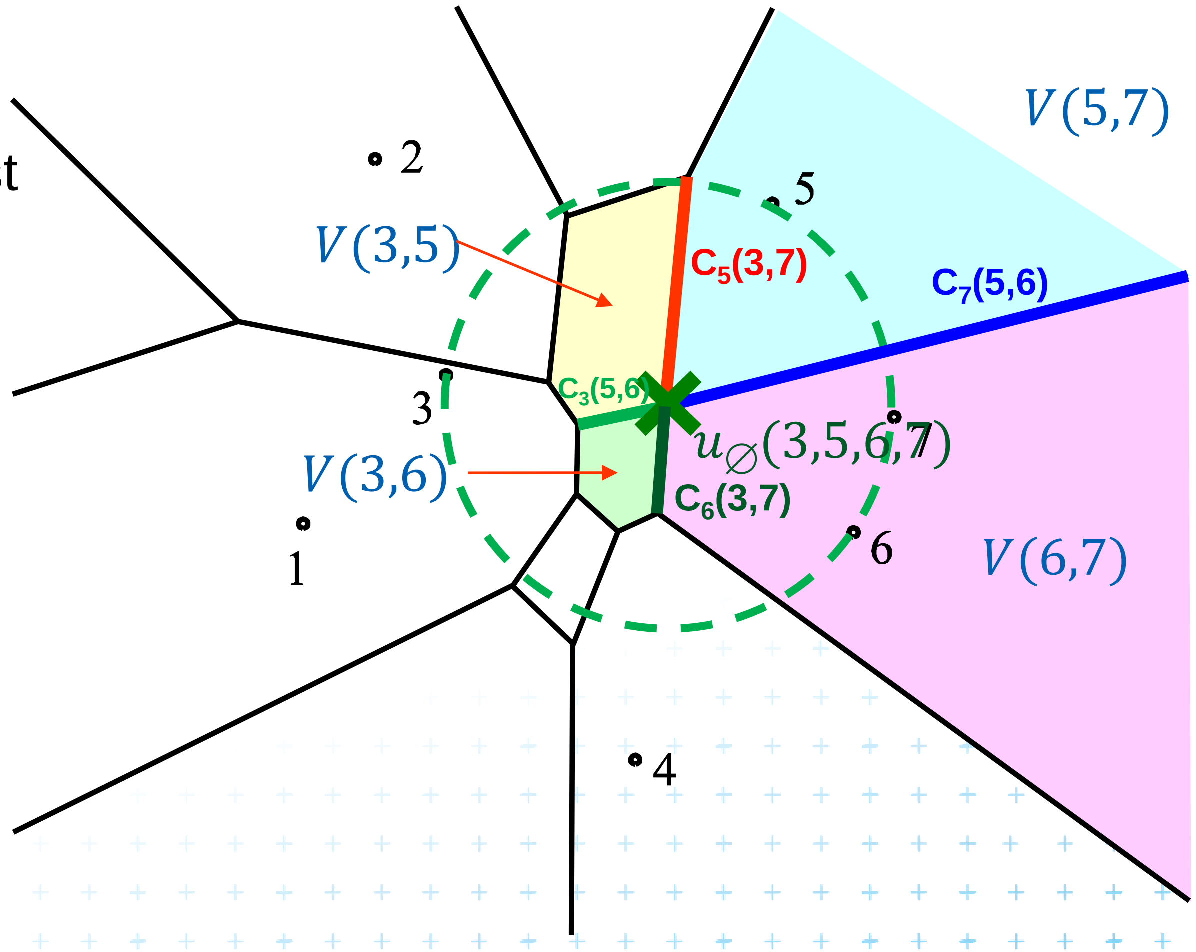


Order-2 Voronoi **vertex** $u_{\emptyset}(Q \cup p)$

vertex : center of a circle passing through at least 3 sites and containing either site p or nothing

Case $u_{\emptyset}(Q \cup p)$

$u_{\emptyset}(3,5,6,7)$



Cell 5 is not inside for all incident edges:

$C_5(3,7)$

$C_6(3,7)$

$C_3(5,6)$

$C_7(5,6)$

=> 5 is on circle with center in Voronoi vertex

Order-k Voronoi Diagram

Single step $V_k \rightarrow V_{k+1}$

The order-k diagram can be constructed from the order-$(k-1)$ diagram in $O(kn\log n)$ time

Globally

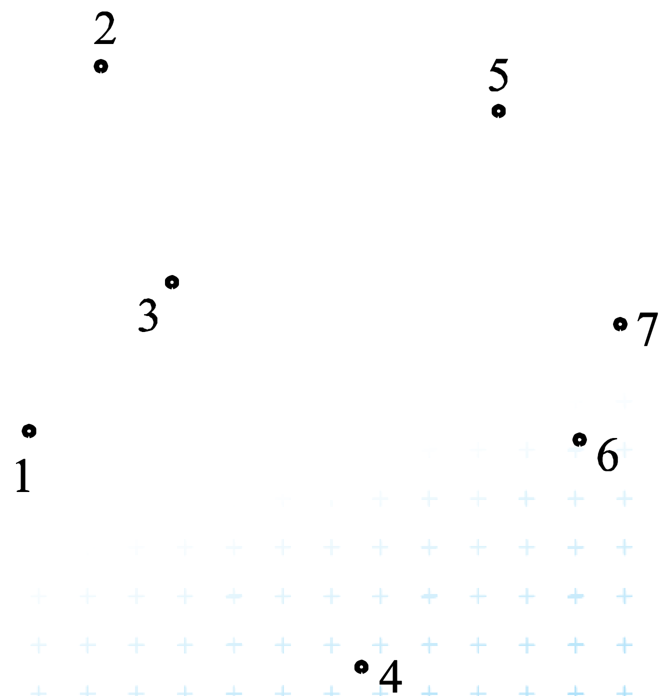
$$\sum_{i=1}^{k-1} O(in\log n) = O(k^2 n\log n)$$

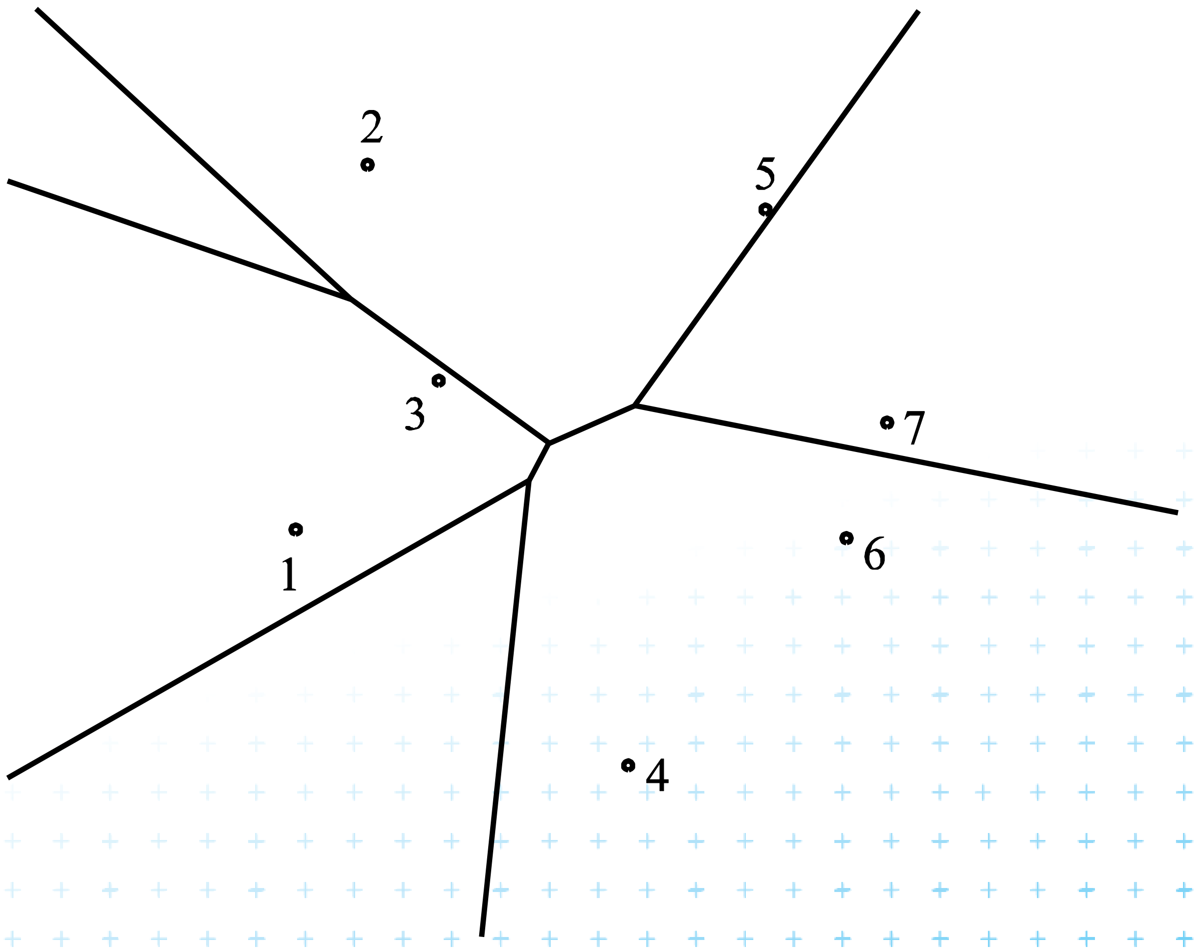
From $V_1 \rightarrow V_k$

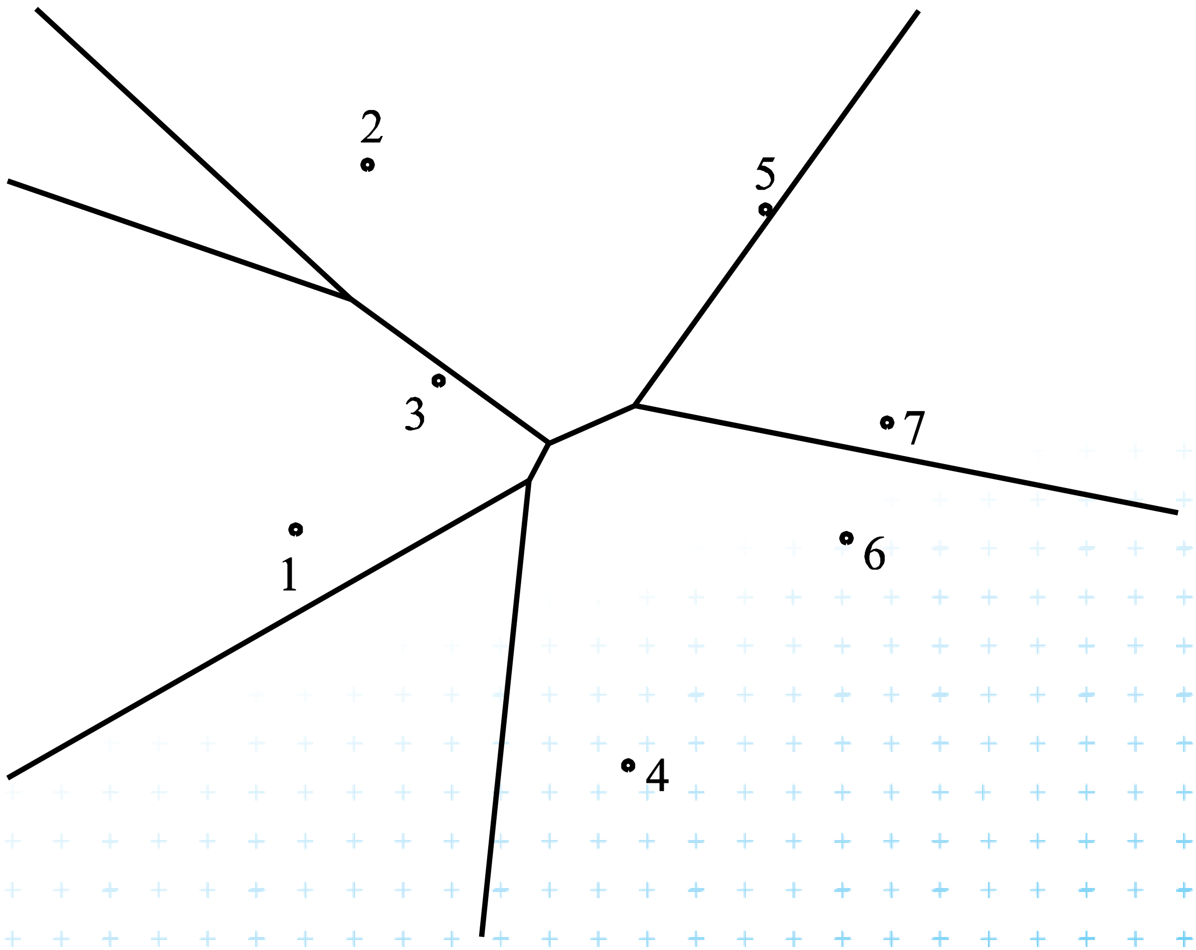
The order-k diagram can be iteratively constructed in $O(k^2 n\log n)$ time from the pointset of size n

[Preparata]

Order n-1 VD
(Farthest-point Voronoi diagram)


2
5
3
7
1
6
4


2
5
3
7
1
6
4


2
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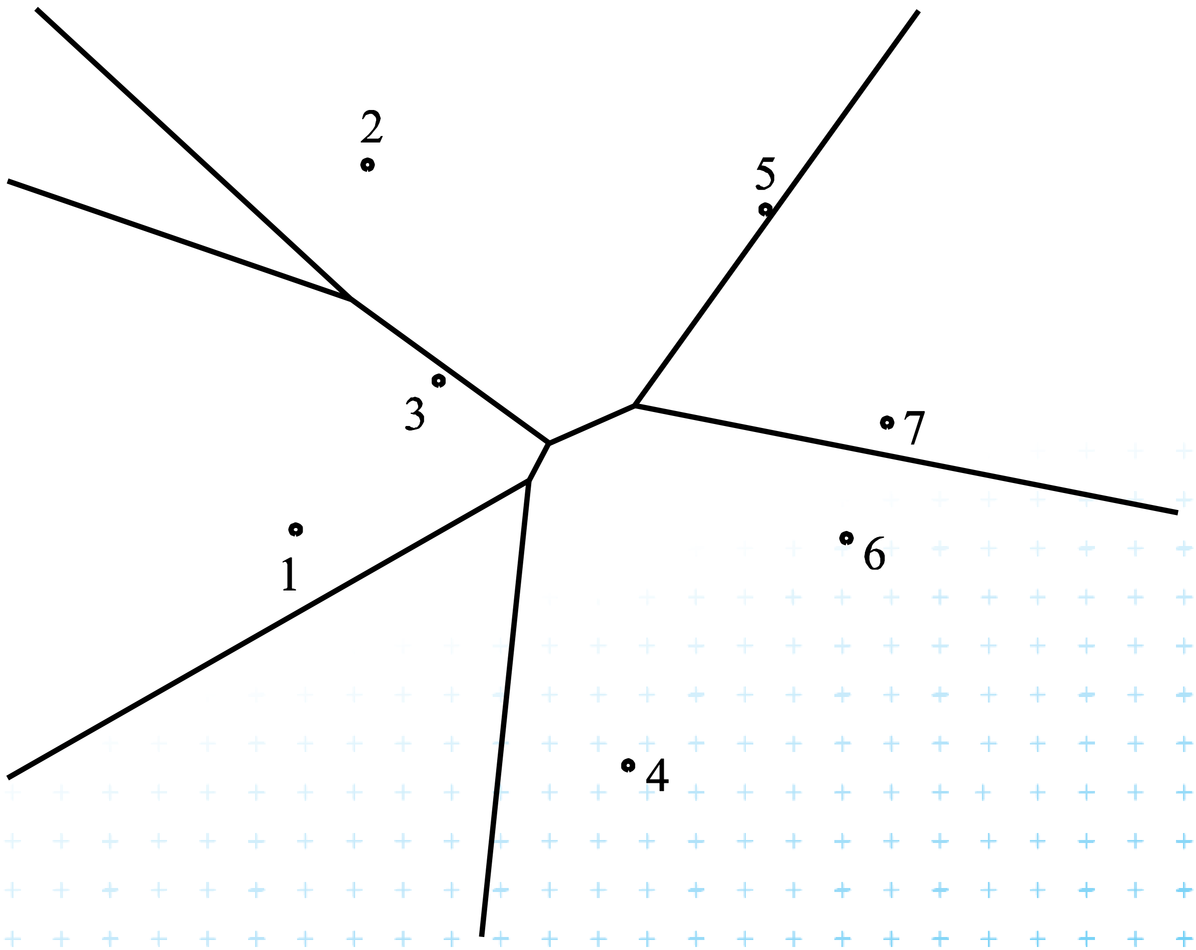
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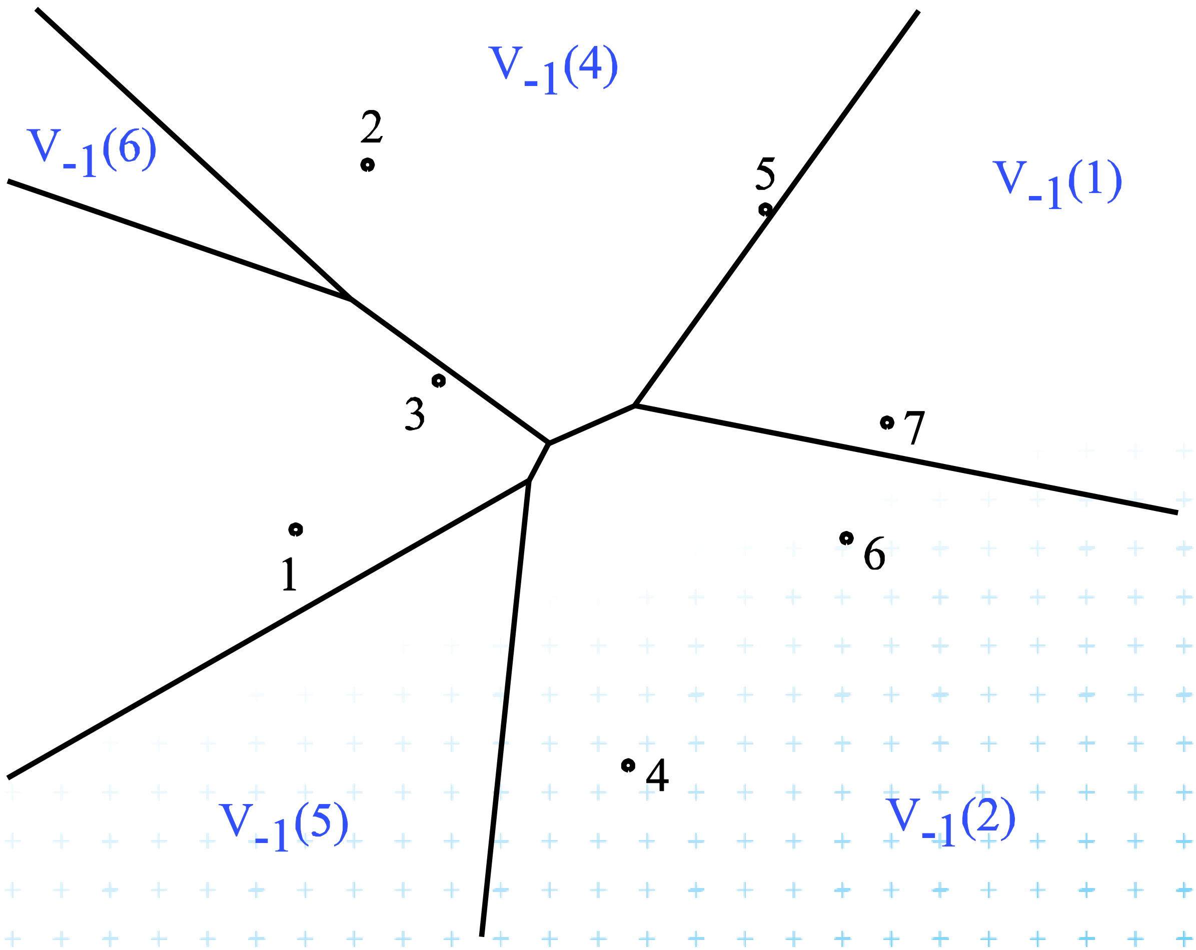
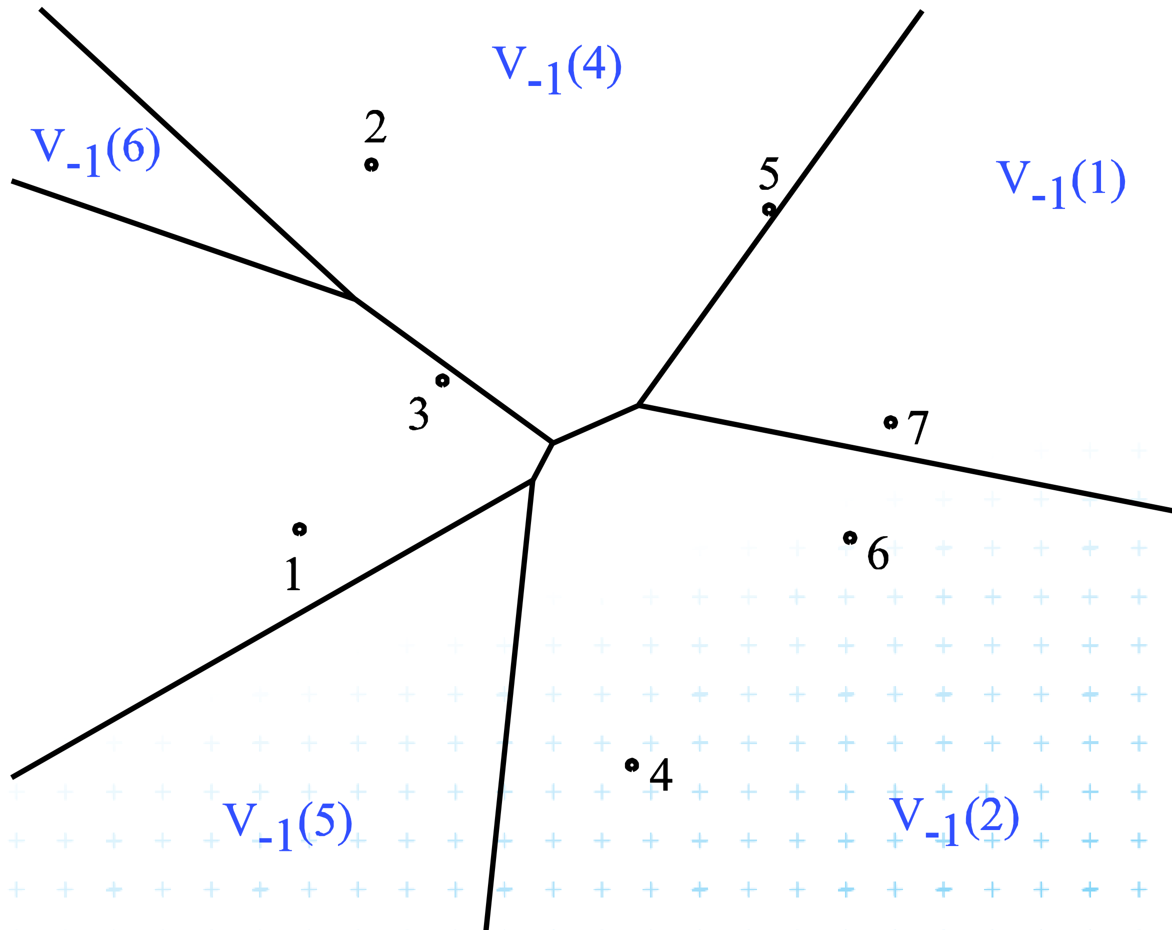


V-1(4)
V-1(6)
V-1(1)
V-1(5)
V-1(2)
2
5
3
7
1
6
4

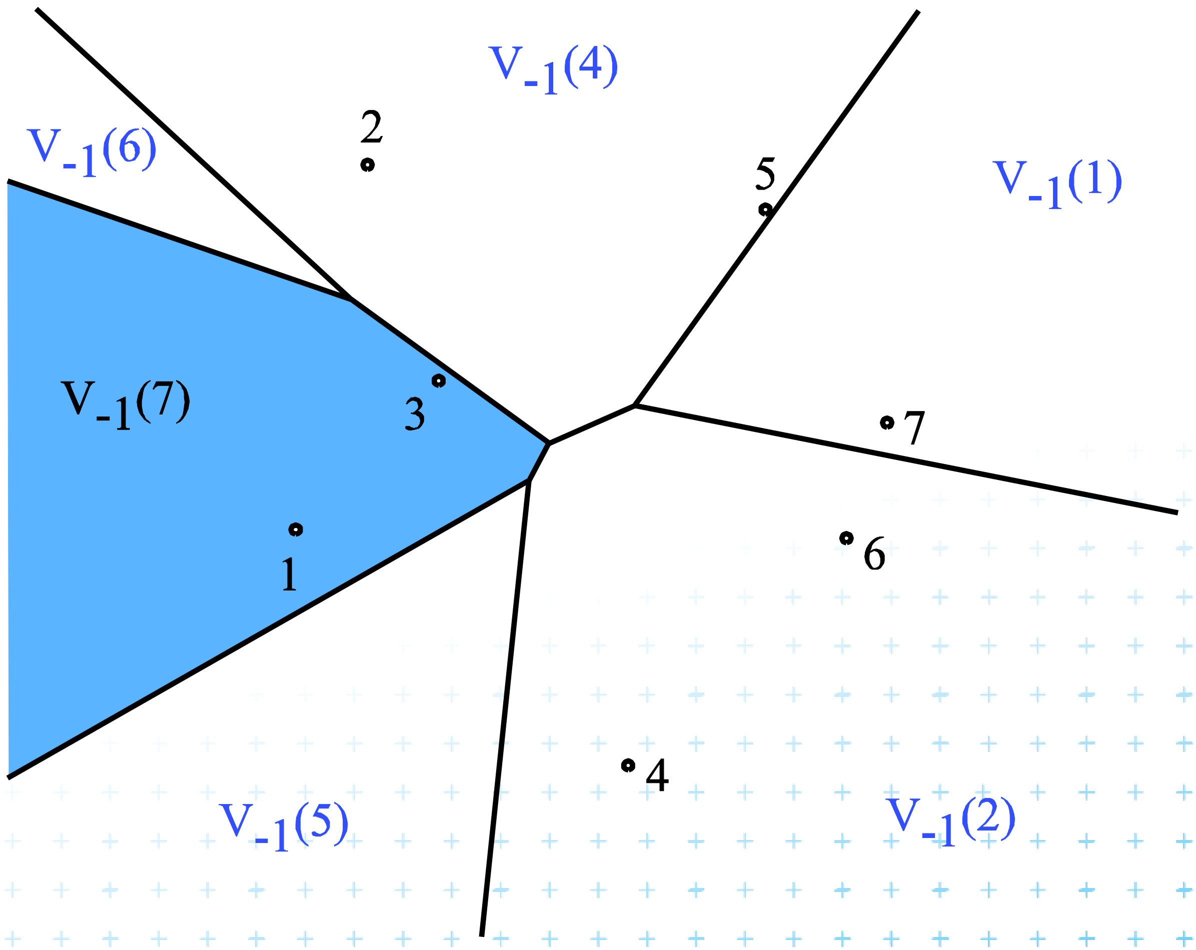


V-1(4)
V-1(6)
2
5
V-1(1)
3
7
1
6
4
V-1(5)
V-1(2)

Farthest-point Voronoi diagram

$V_{-1}(p_i)$ cell
= set of points in the plane farther from p_i than from any other site

$Vor_{-1}(P)$ diagram
= partition of the plane formed by the farthest point Voronoi regions, their edges, and vertices



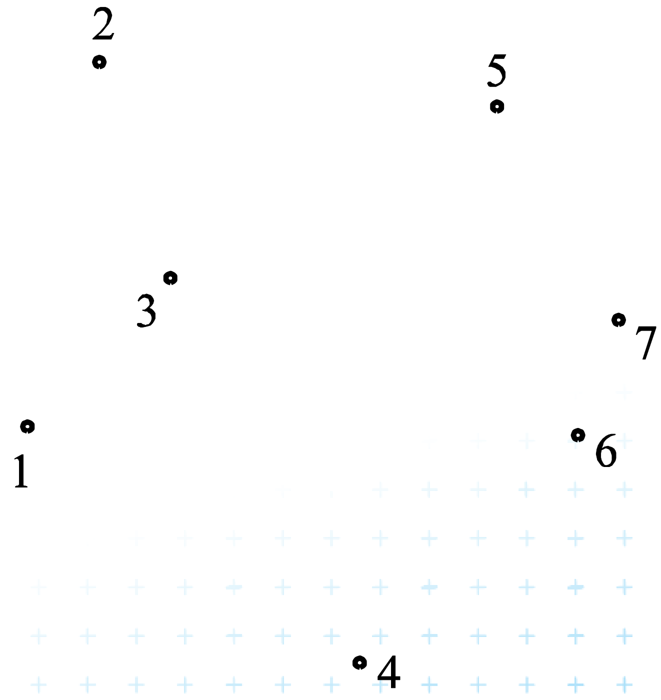
[Nandy]

Farthest-point Voronoi region (cell)

Computed as intersection of halfplanes, but we take “other sides” of bisectors

Construction of $V_{-1}(7)$

$$V_{-1}(y) = \bigcap_{x=1}^{n} h(y, x),\ y \neq x$$



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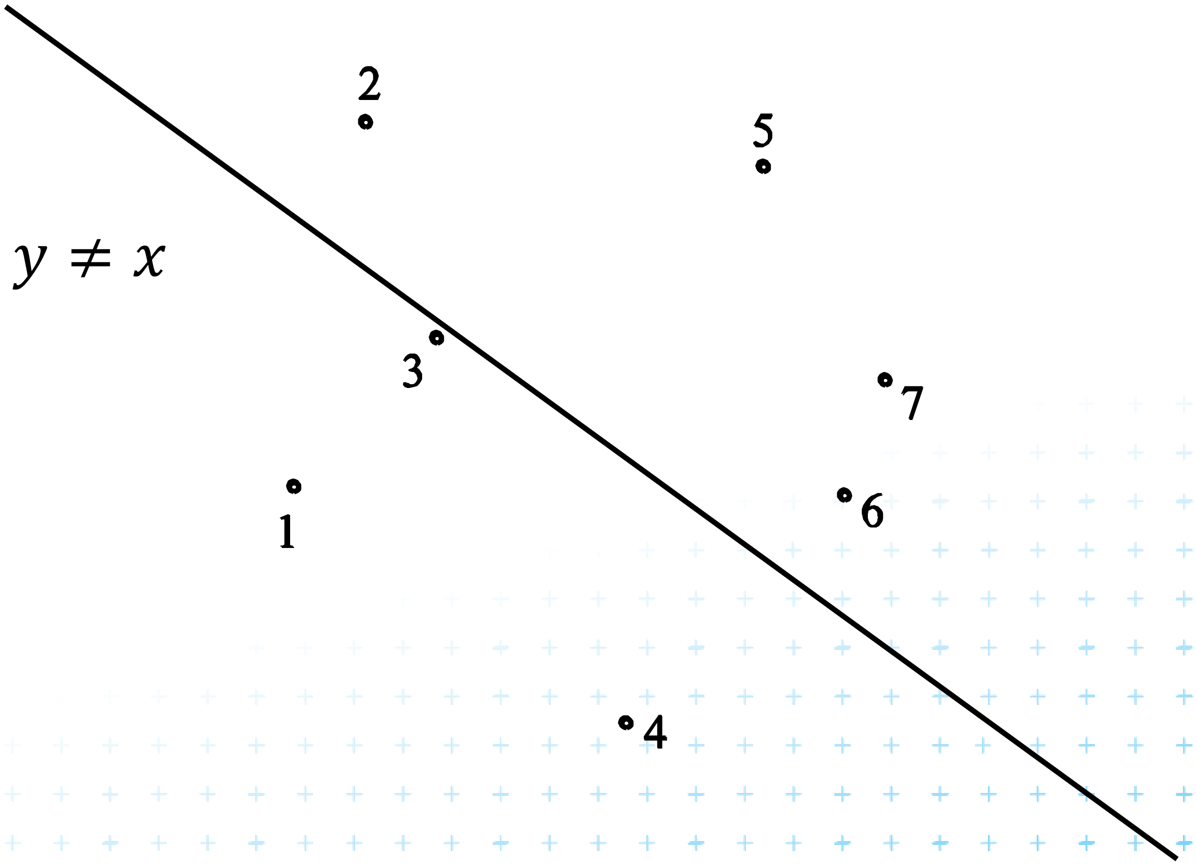

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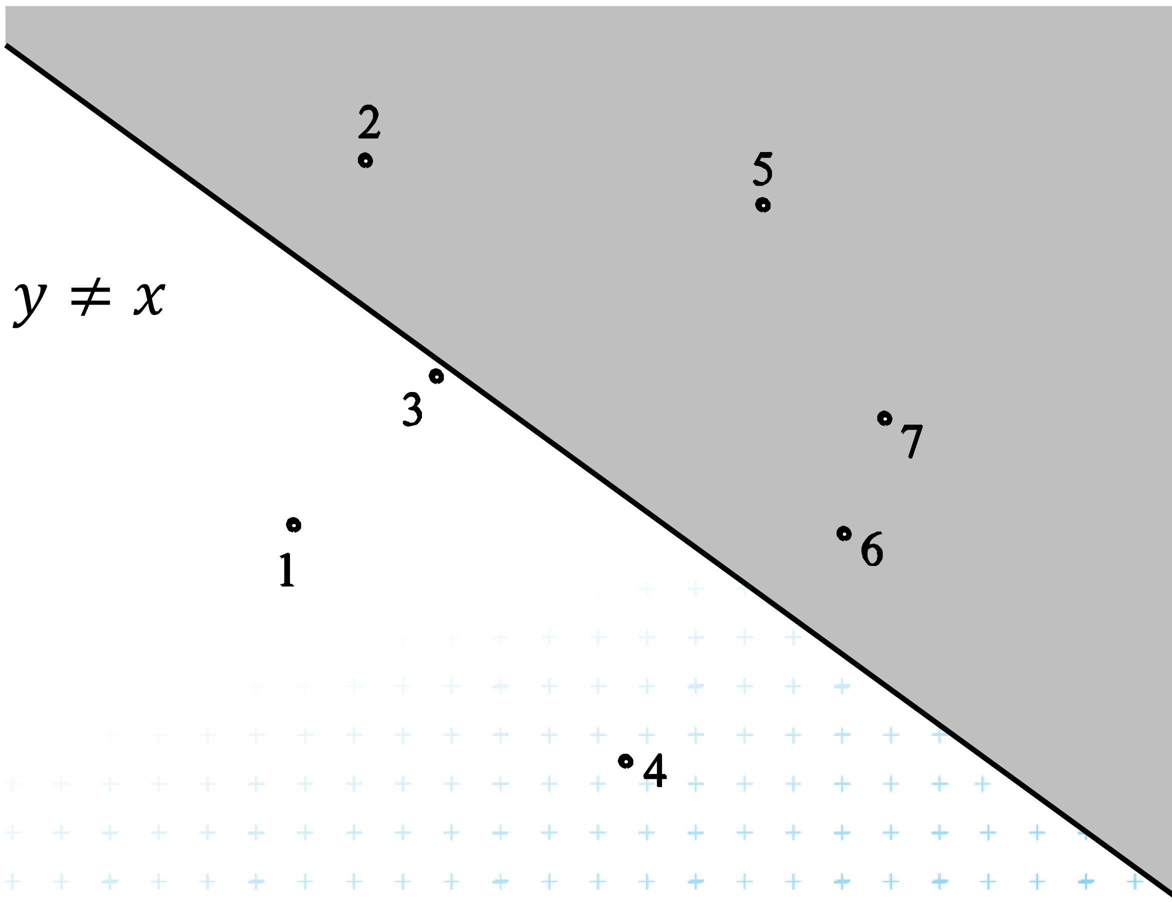

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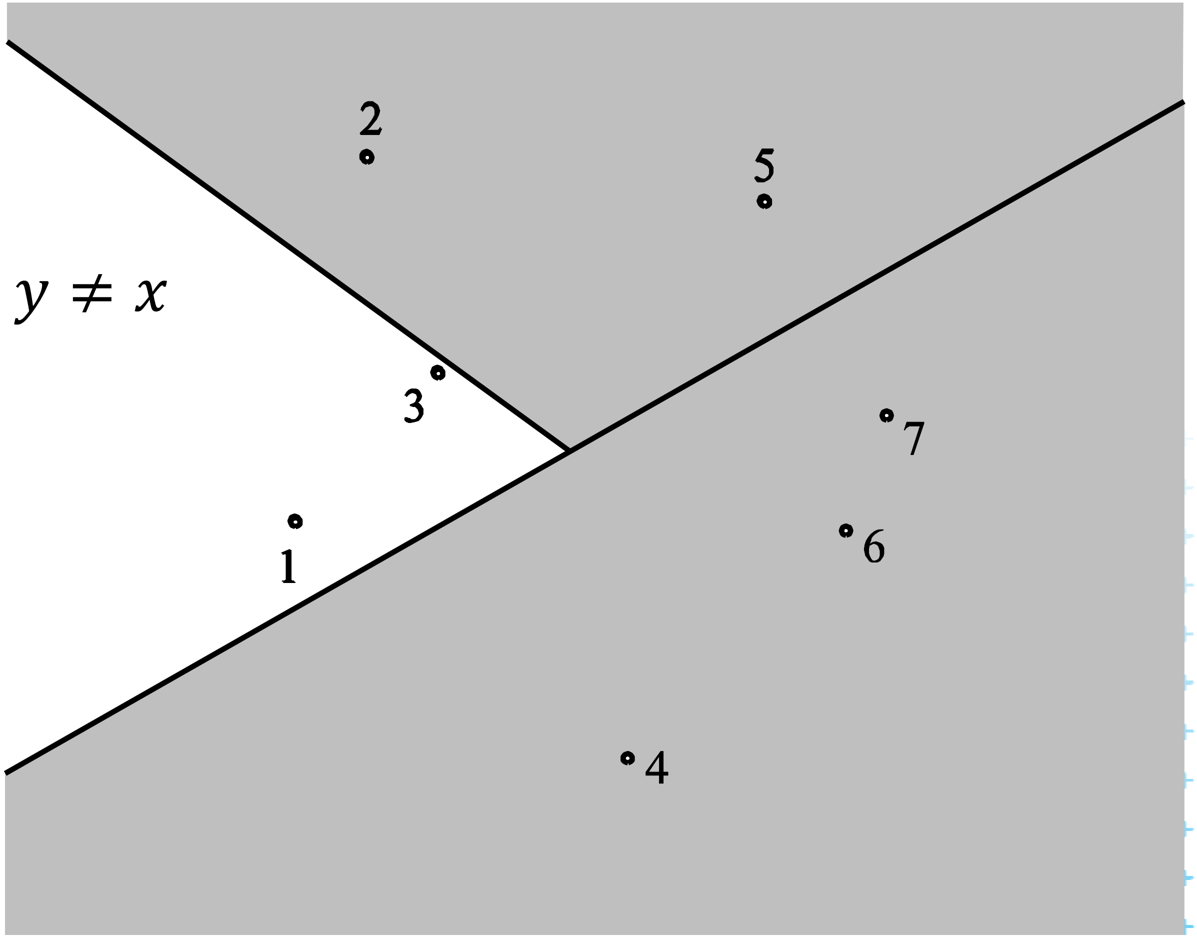

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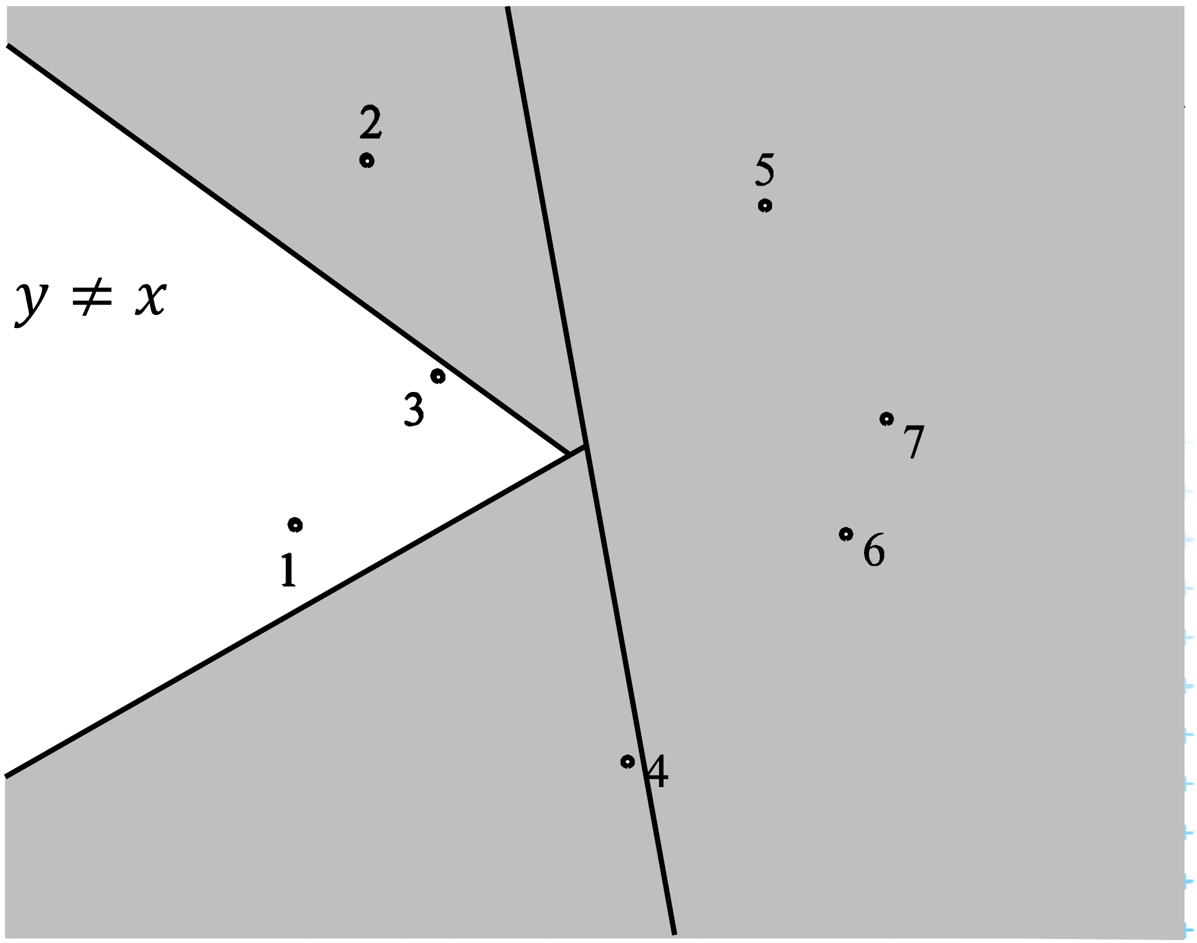

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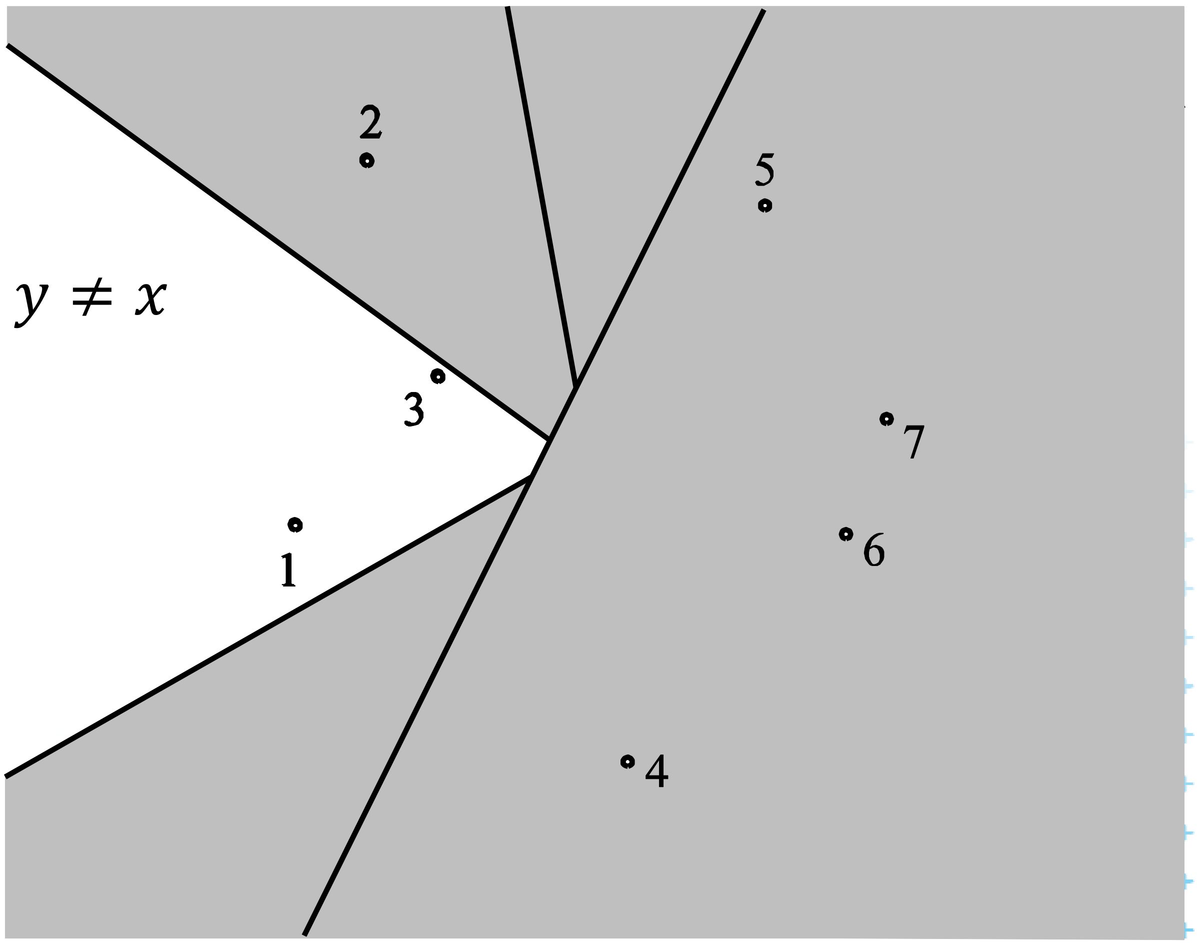

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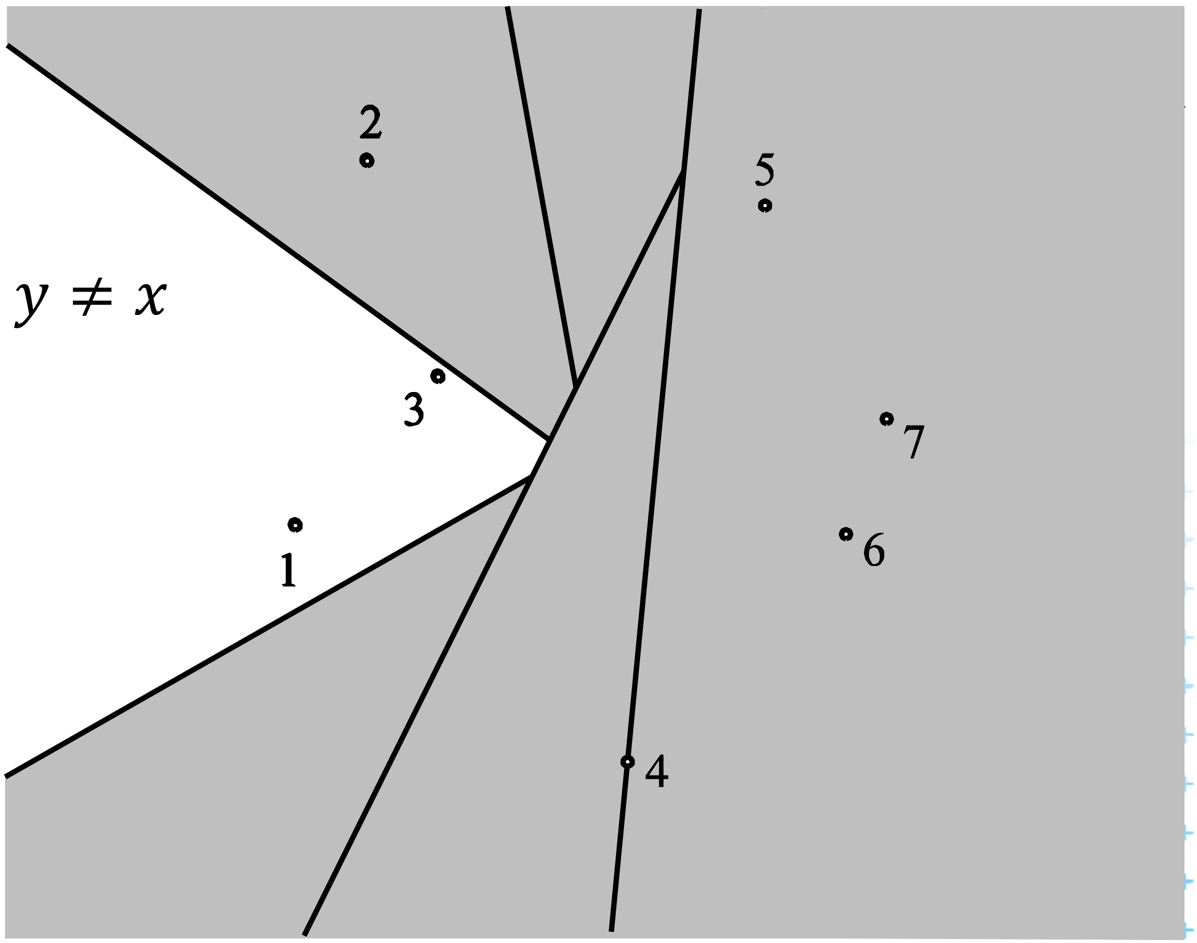

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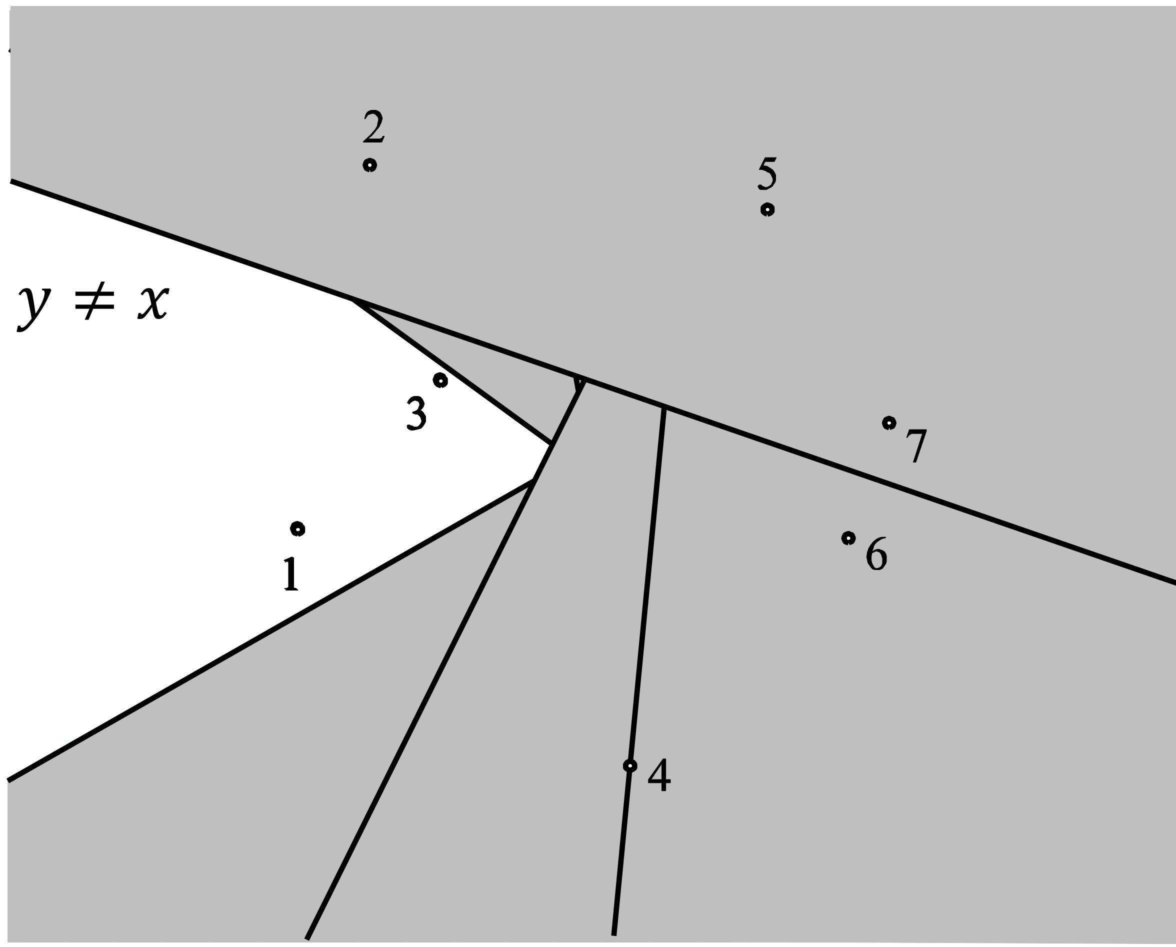


[Nandy]

Farthest-point Voronoi region (cell)

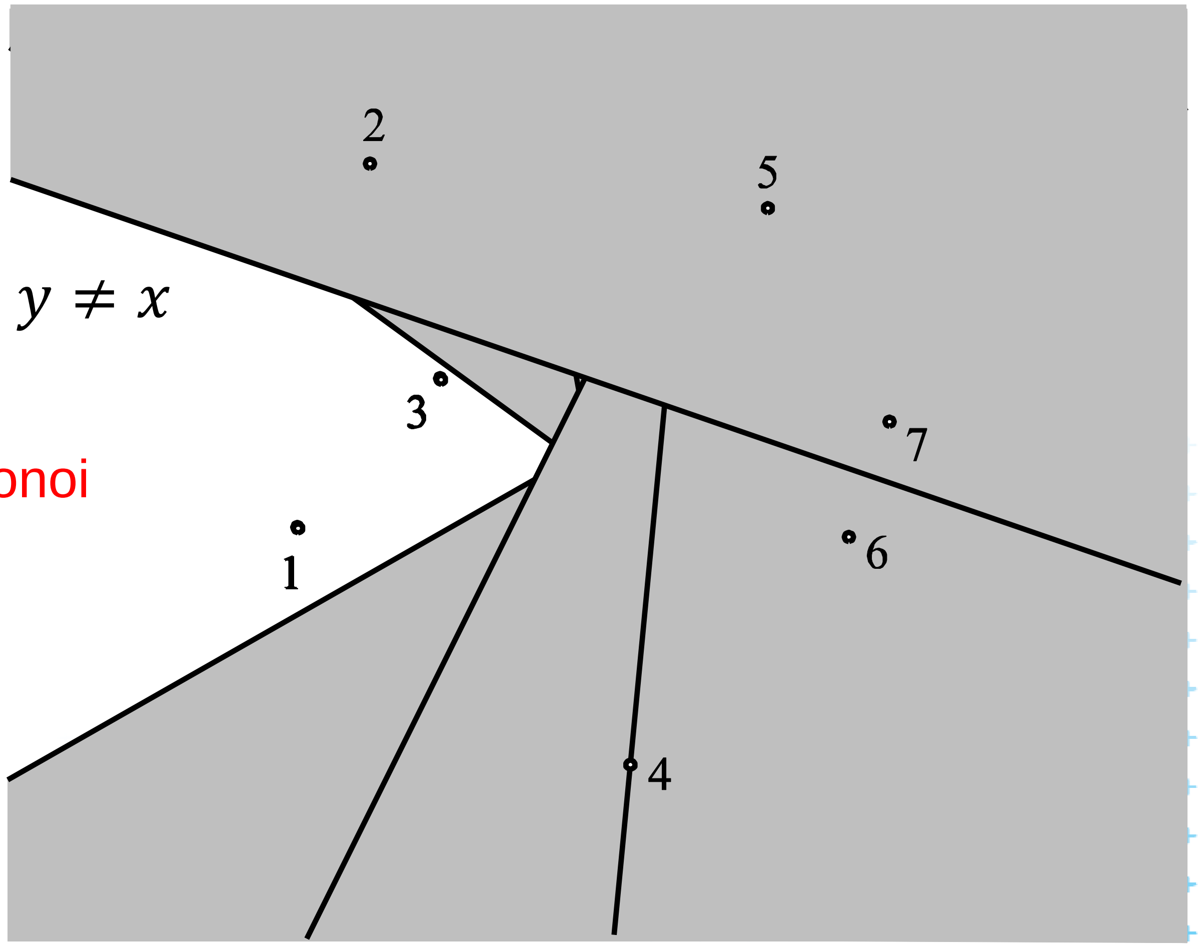
Computed as intersection of halfplanes, but we take "other sides" of bisectors

Construction of $V_{-1}(7)$

$$V_{-1}(y) = \cap_{x=1}^{n} h(y, x),\ y \neq x$$

Property

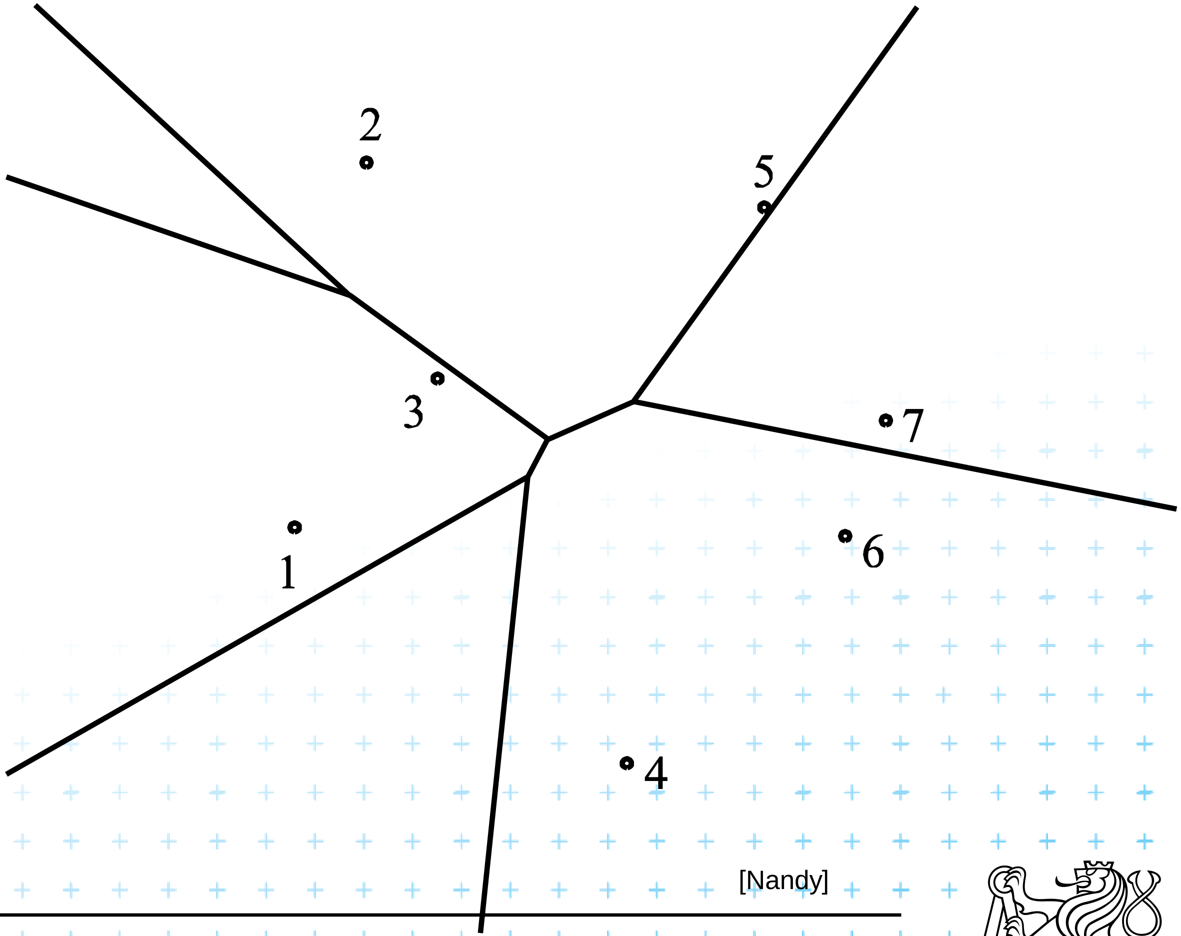
The farthest point Voronoi regions are convex and unbounded



[Nandy]

Farthest-point Voronoi region

Properties:

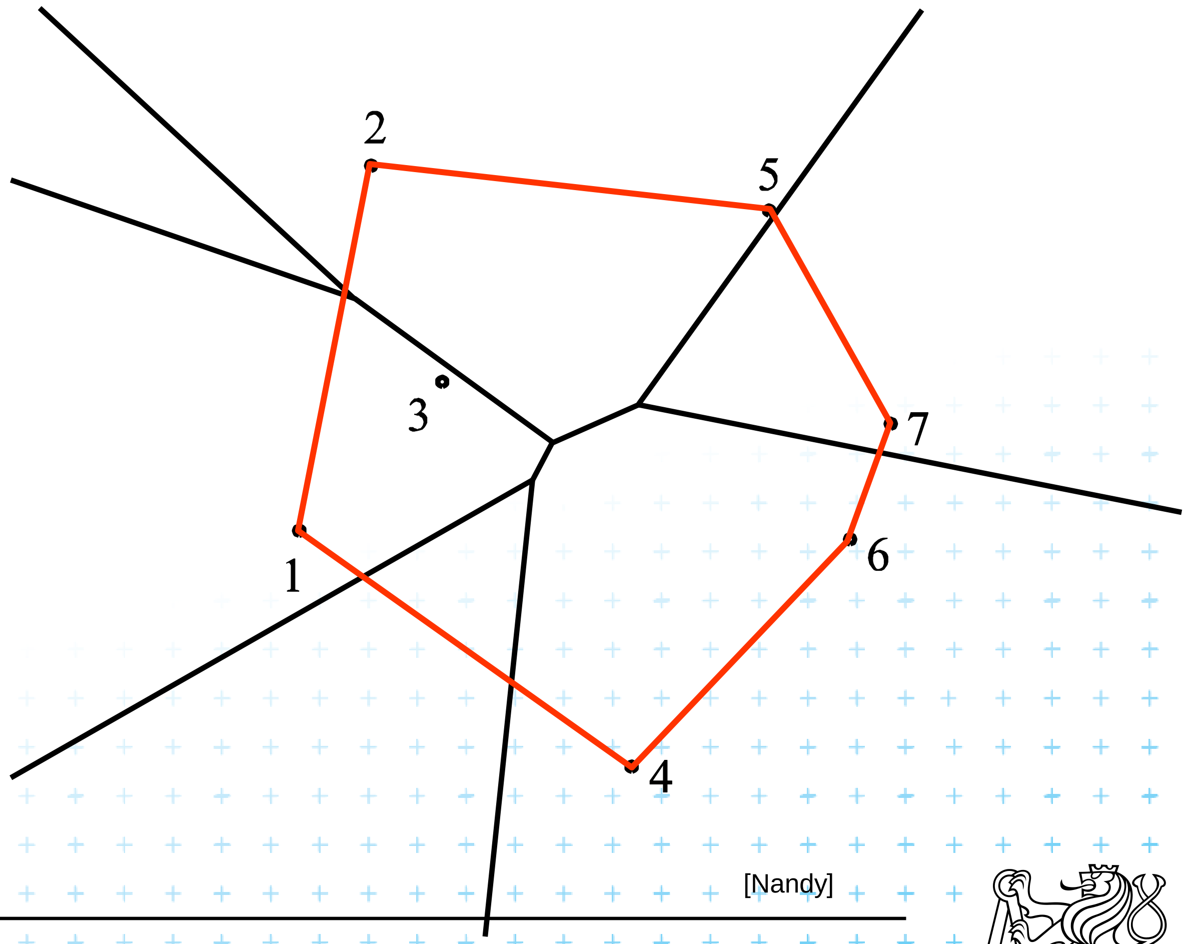


[Nandy]

Farthest-point Voronoi region

Properties:

- Only vertices of the convex hull have their cells in farthest Voronoi diagram



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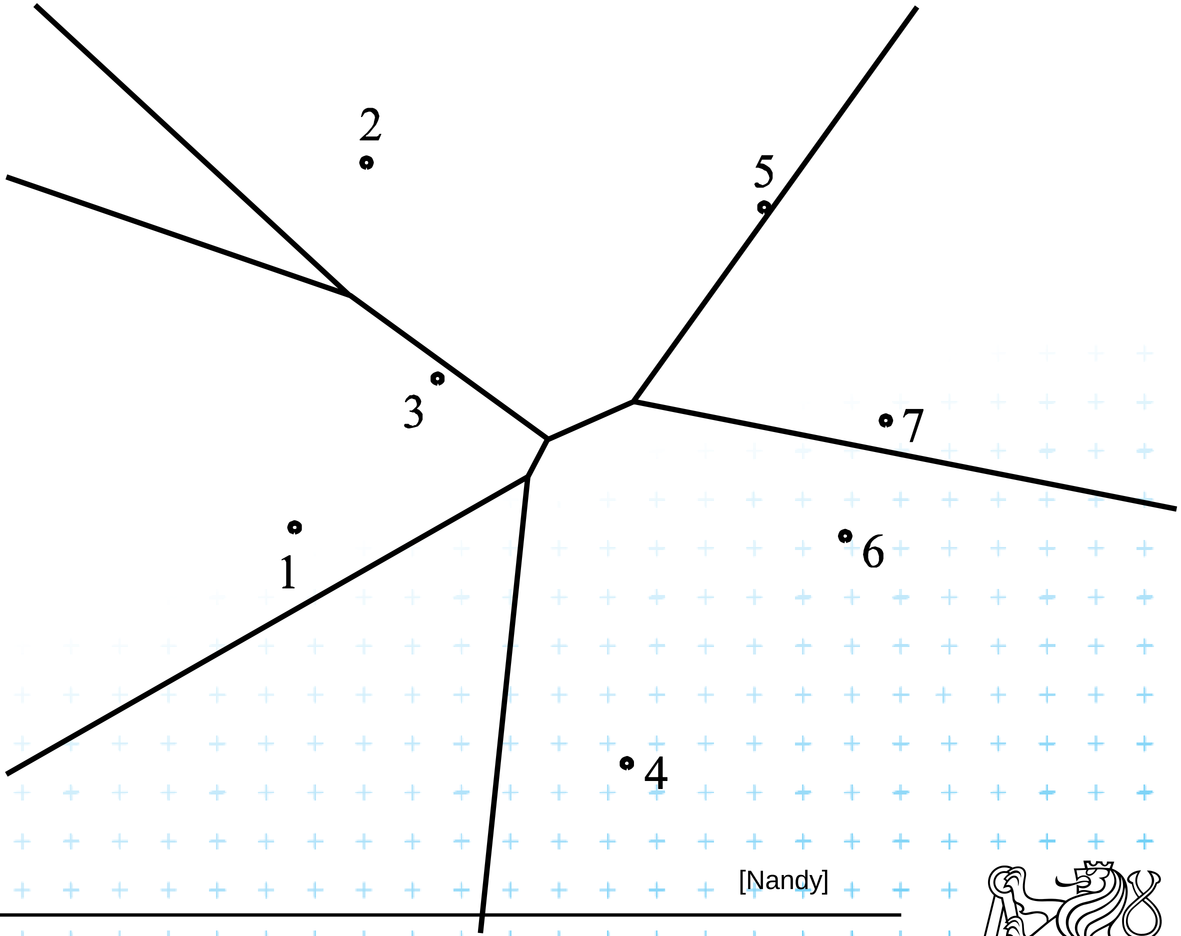

[Nandy]

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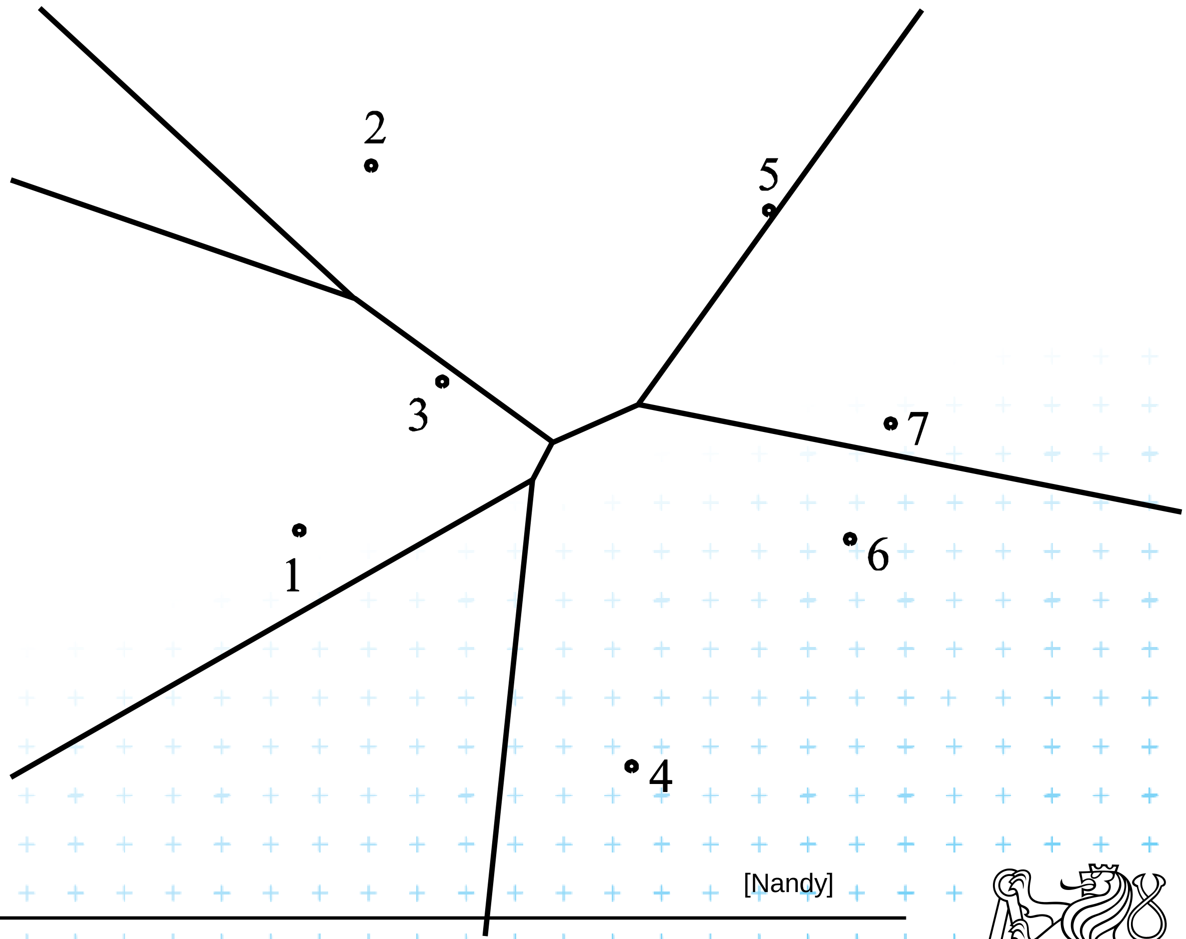

[Nandy]

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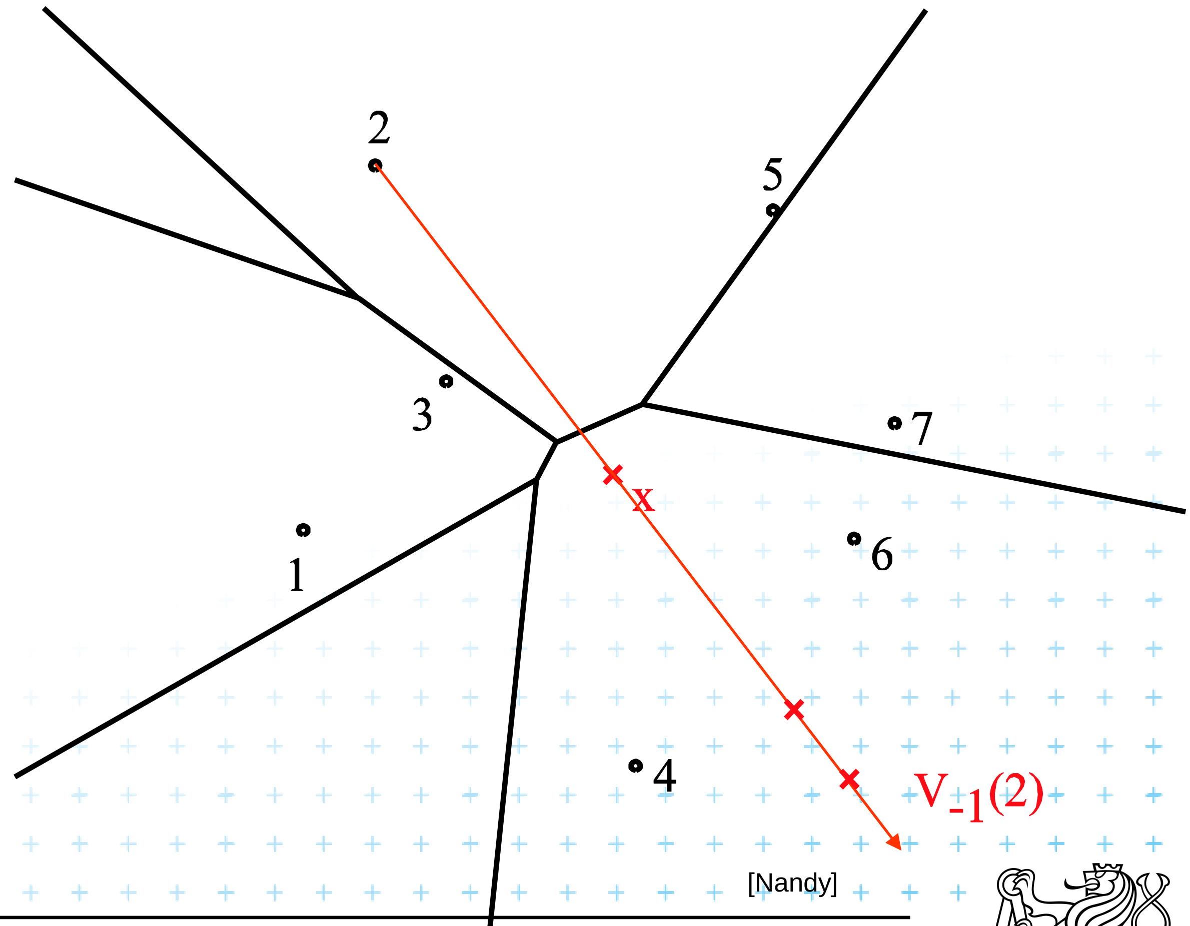

[Nandy]

Farthest-point Voronoi region

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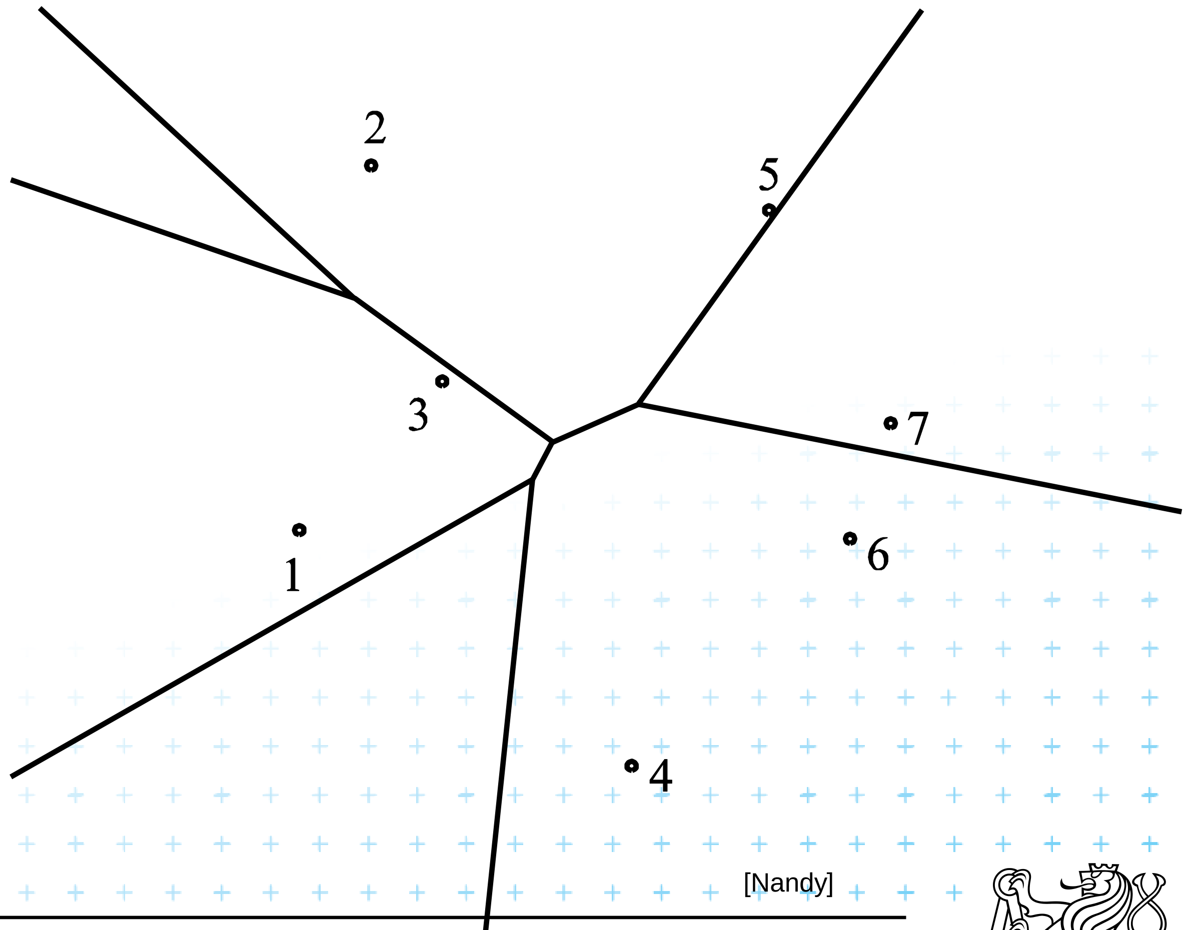
- The farthest point Voronoi regions are unbounded

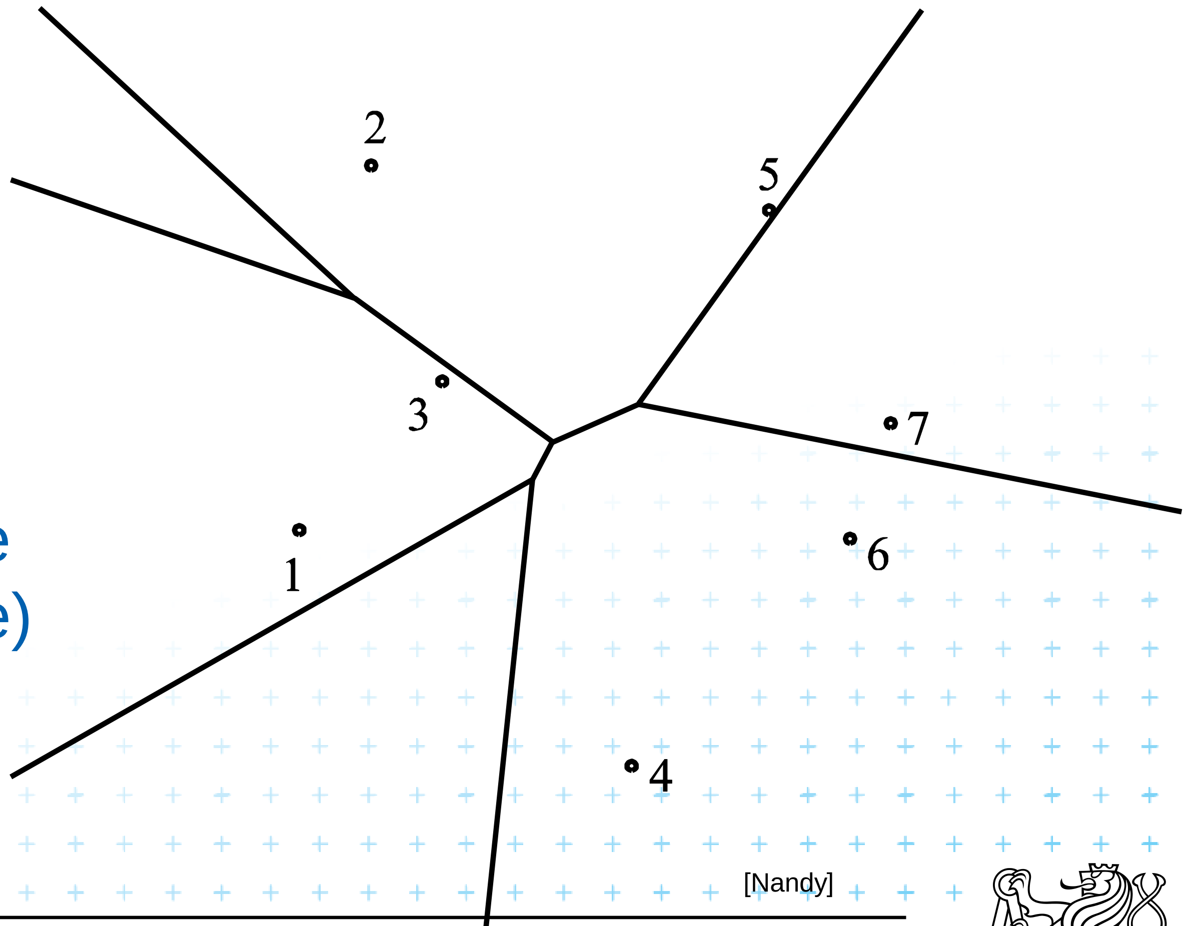


[Nandy]

Farthest-point Voronoi region

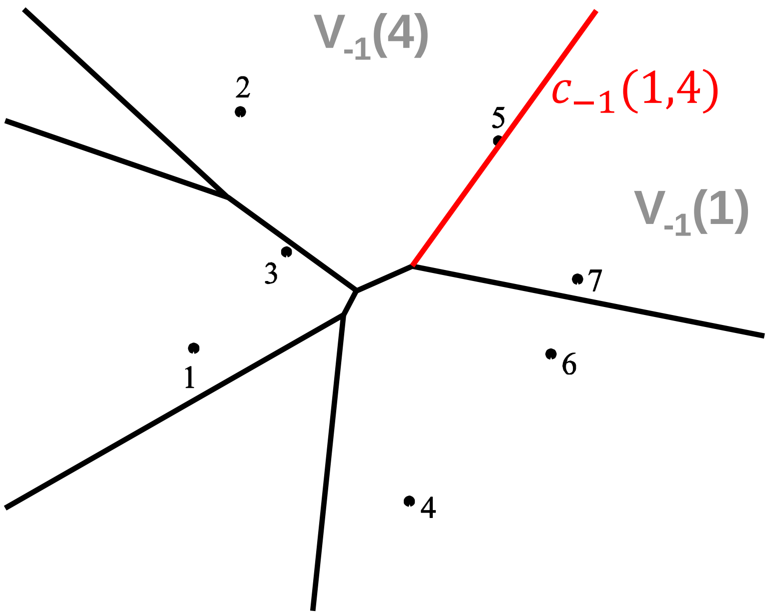
Properties:

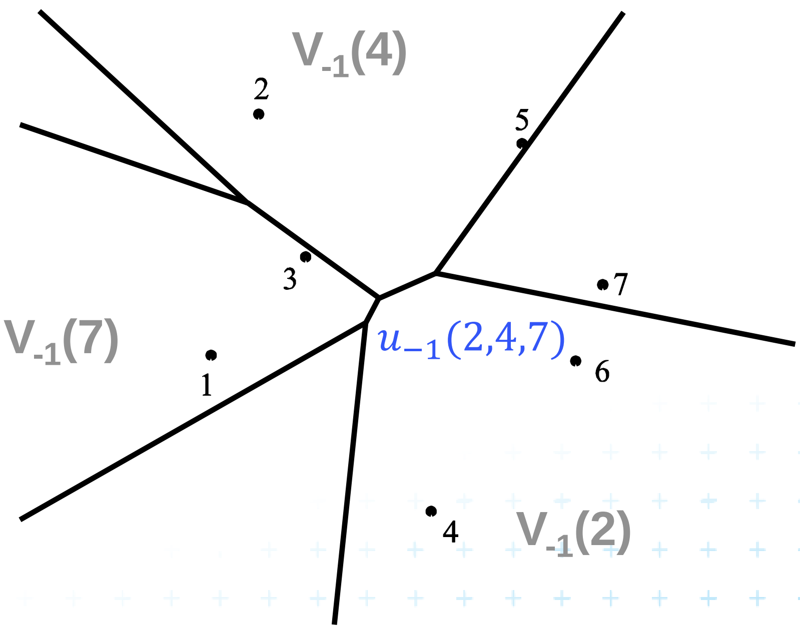
- Only vertices of the convex hull have their cells in farthest Voronoi diagram
- The farthest point Voronoi regions are unbounded
- The farthest point Voronoi edges and vertices form a tree (in the graph sense)



[Nandy]

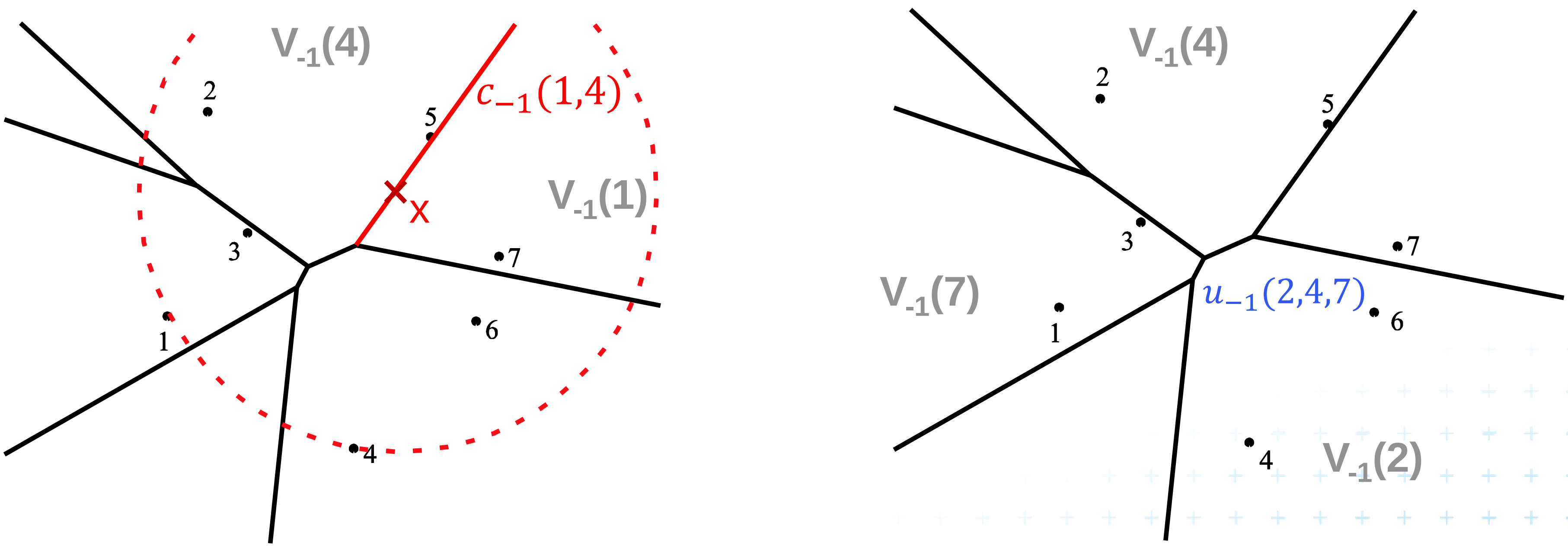
Farthest point Voronoi edges and vertices





edge : set of points equidistant
from 2 sites and closer to
all the other sites

Farthest point Voronoi edges and vertices



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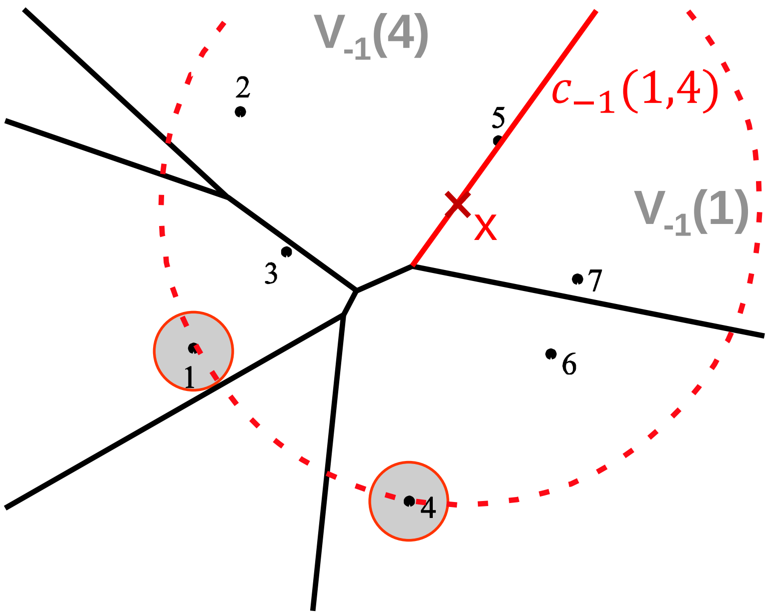
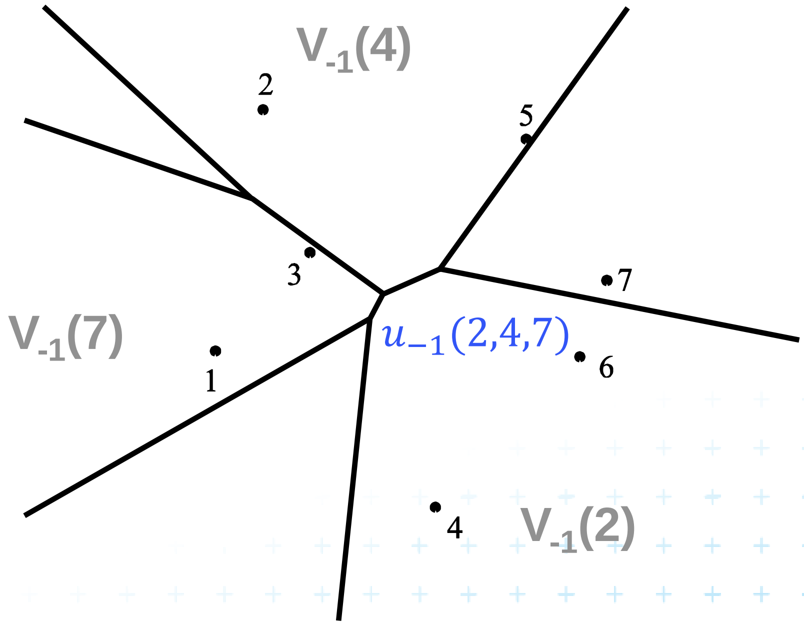
[Nandy]

Farthest point Voronoi edges and vertices





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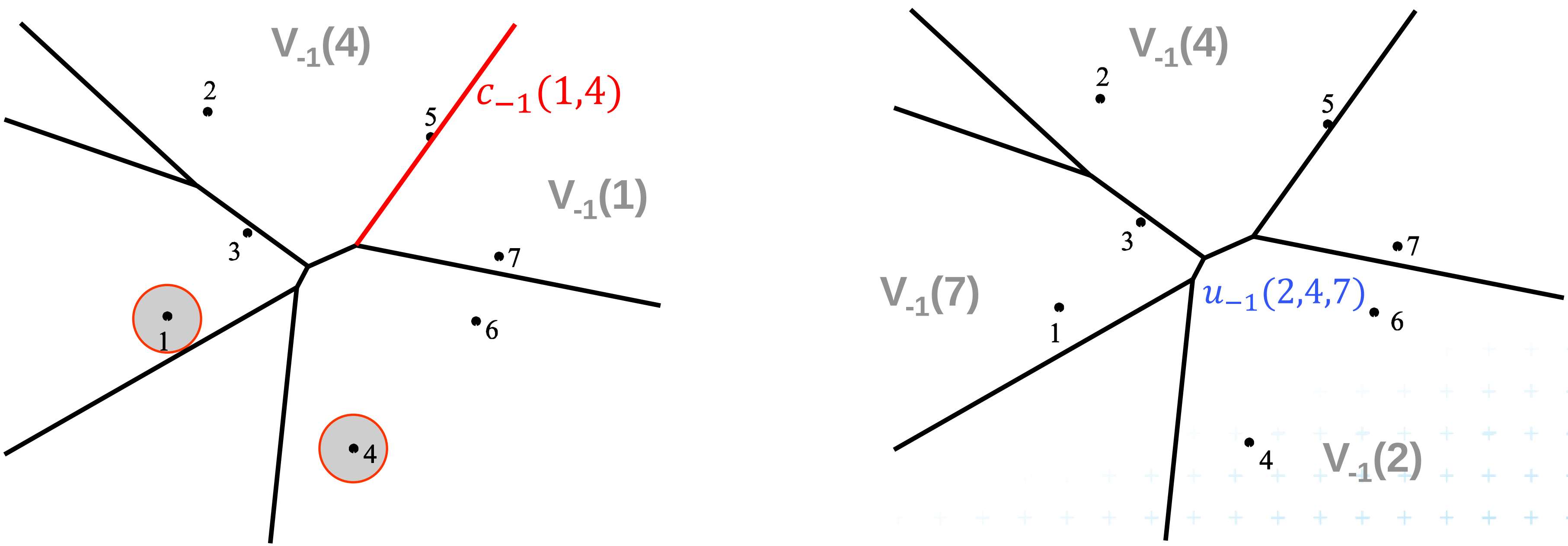
Farthest point Voronoi edges and vertices



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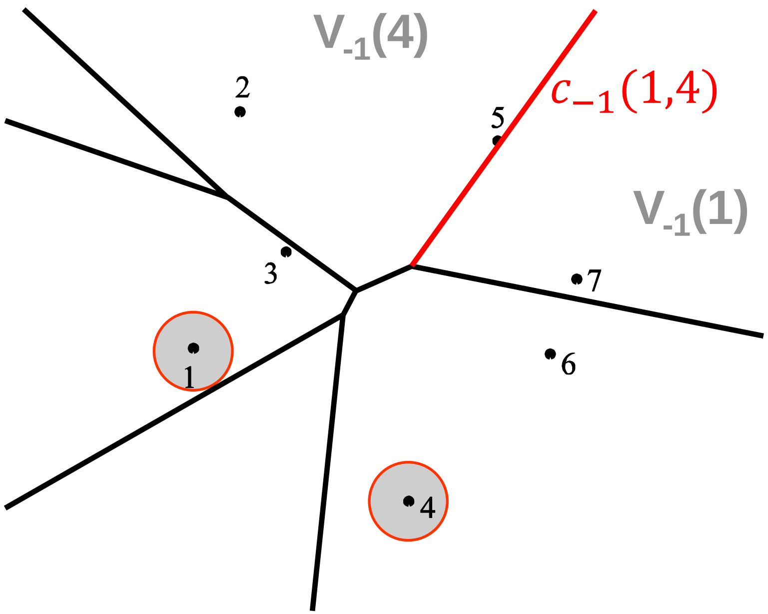
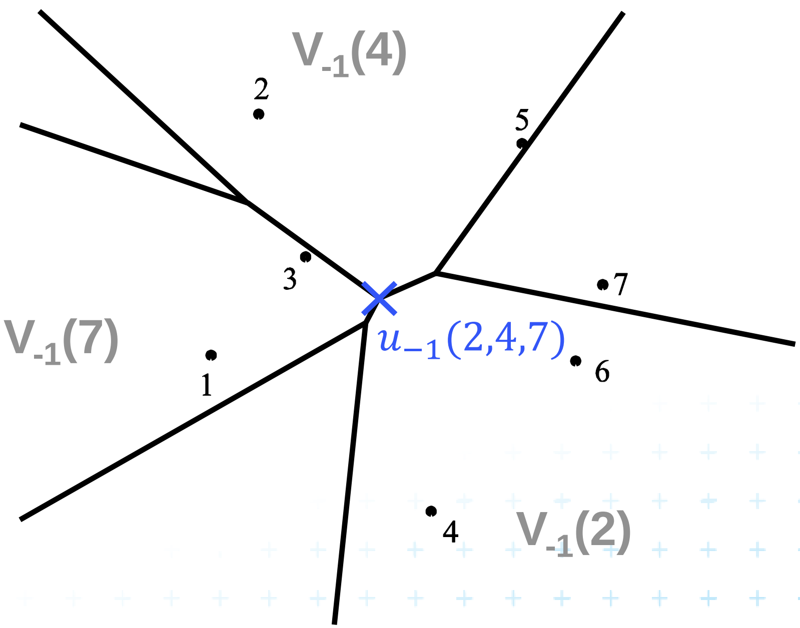
[Nandy]

Farthest point Voronoi edges and vertices





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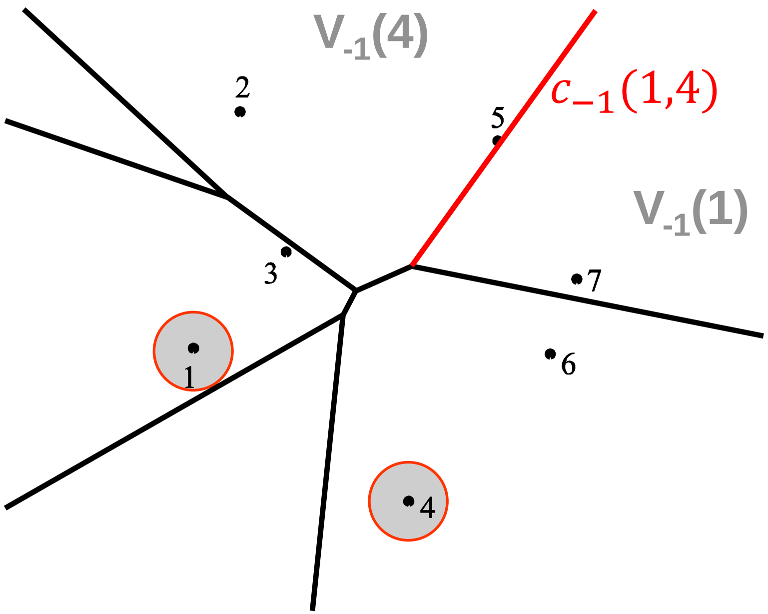
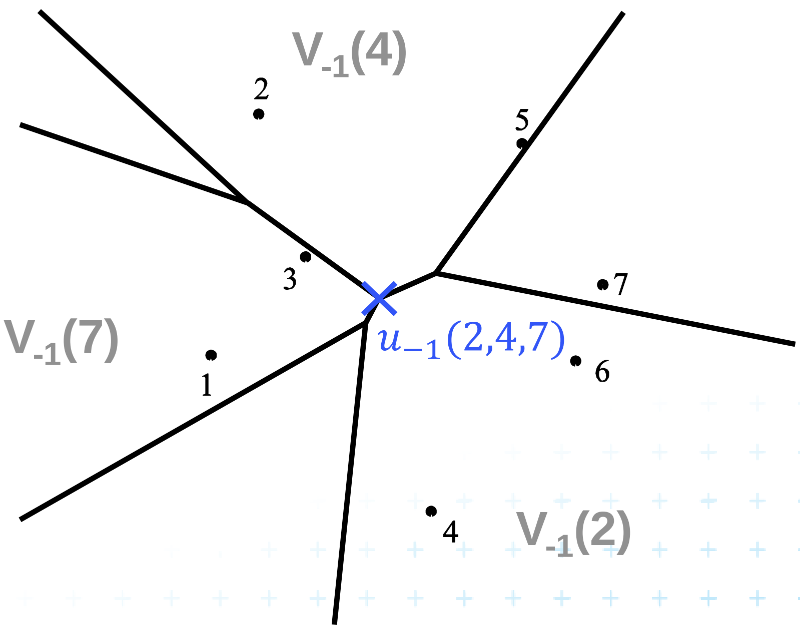


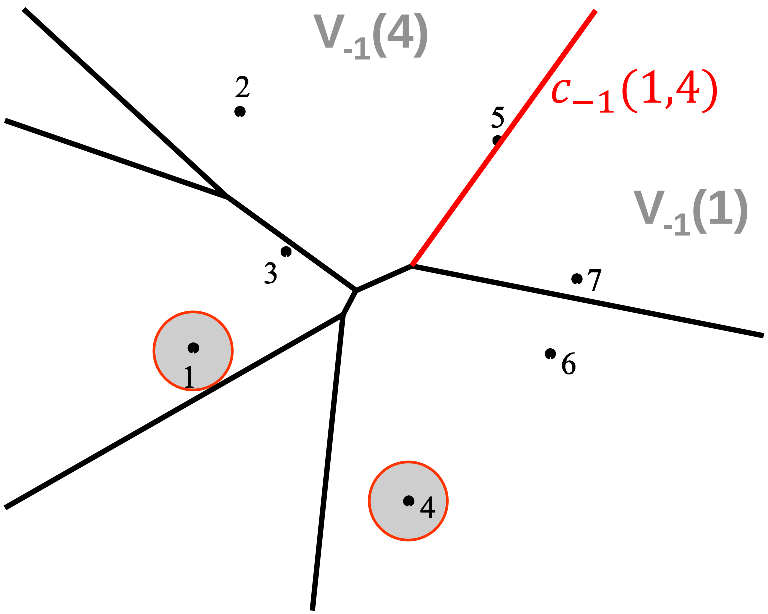


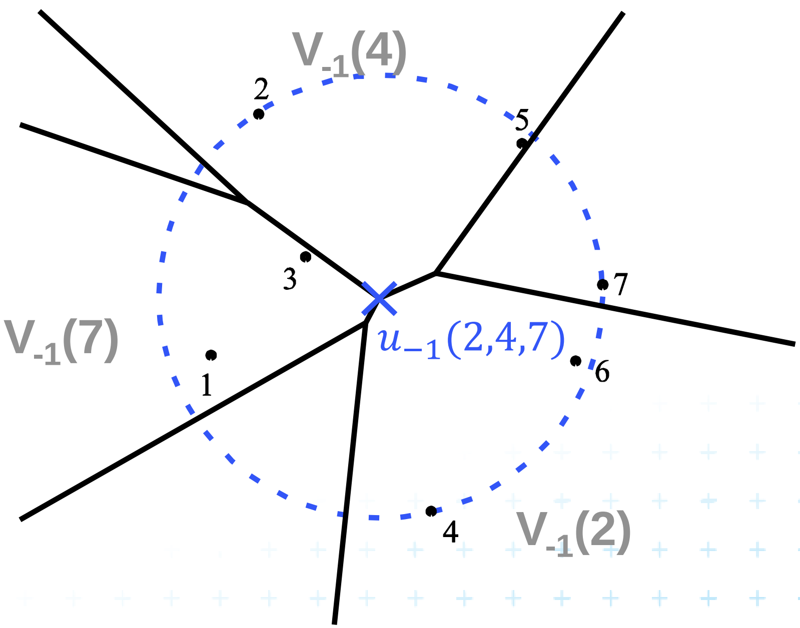
edge : set of points equidistant from 2 sites and closer to all the other sites

vertex : point equidistant from at least 3 sites and closer to all the other sites – Enclosing circle

[Nandy]

Farthest point Voronoi edges and vertices





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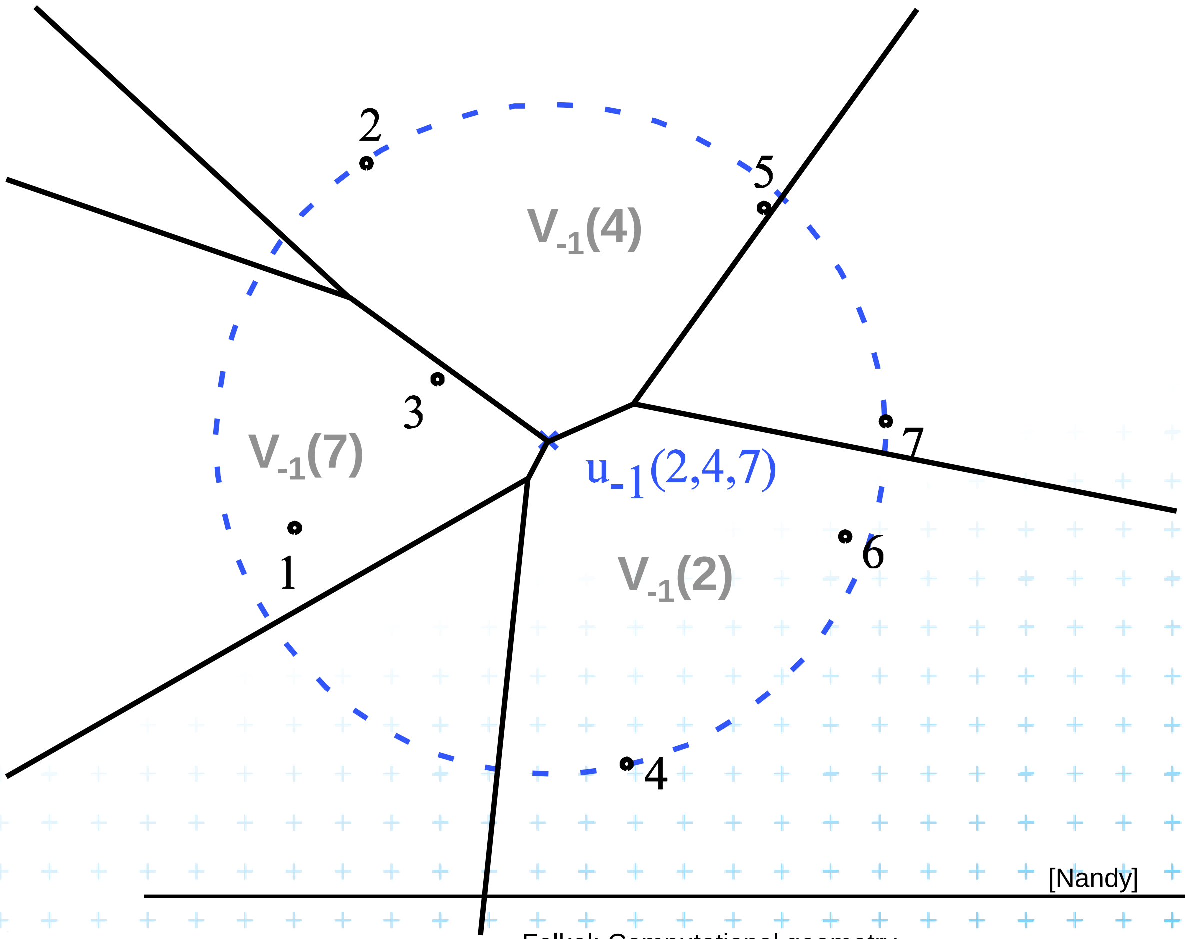
– Enclosing circle

[Nandy]

Application of $Vor_{-1}(P)$: Smallest enclosing circle

- Construct $Vor_{-1}(P)$ and find minimal circle with center in $Vor_{-1}(P)$ vertices or on edges



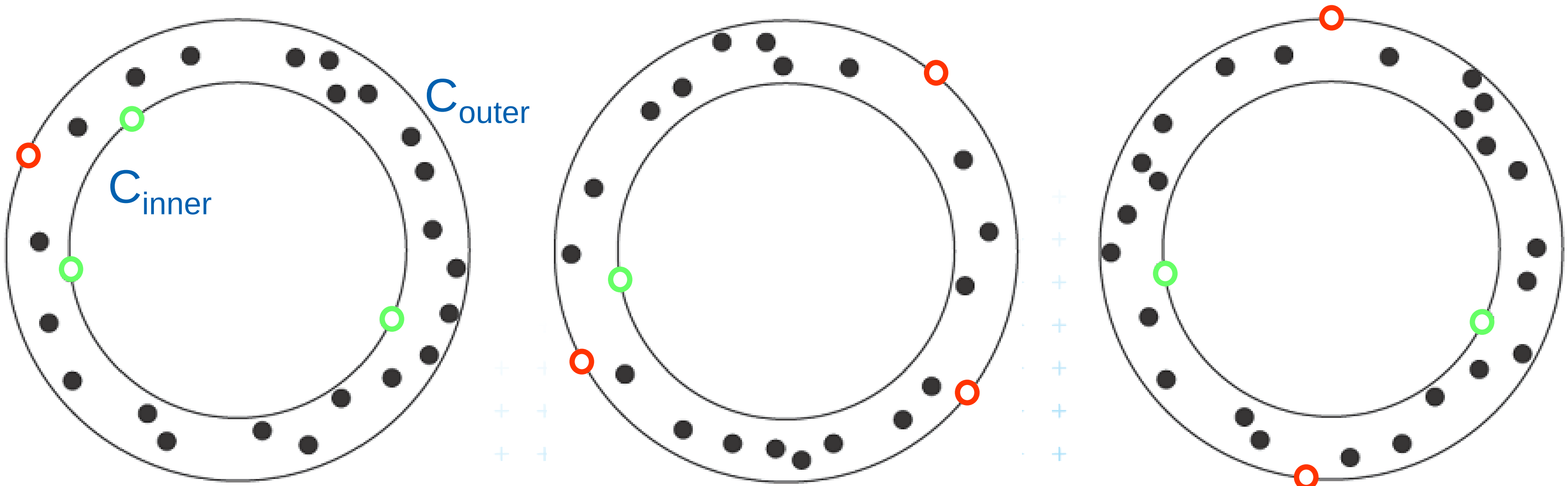
[Nandy]

Farthest-point Voronoi diagrams example

Roundness of manufactured objects

- Input: set of measured points in 2D
- Output: width of the smallest-width annulus mezikruží s nejmenší šířkou (region between two concentric circles C_{inner} and C_{outer})

Three cases to test – one will win:



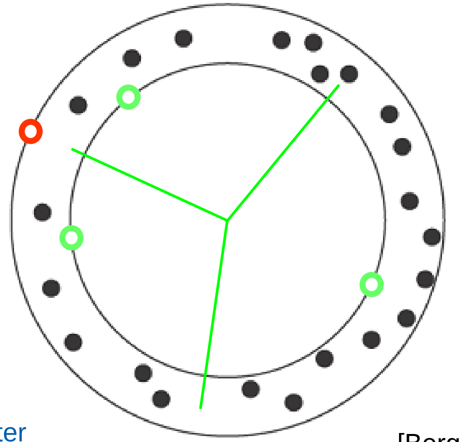
a) 3 in – 1 out   b) 1 point in – 3 out   c) 2 in – 2 out

[Berg]

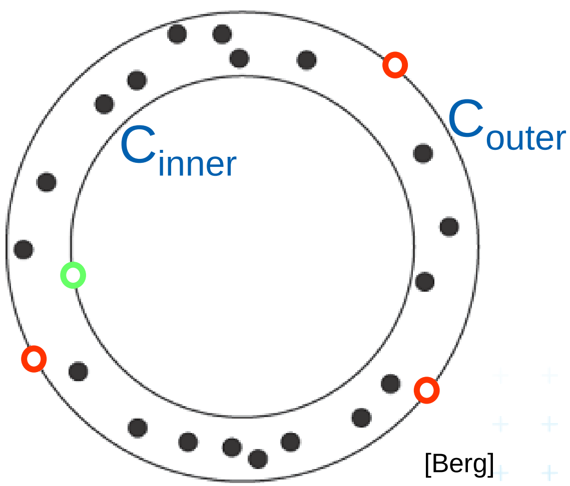
Smallest width annulus – cases with 3 pts

a) C_{inner} contains at least 3 points

- Center is the *vertex of normal Voronoi diagram* (1st order VD)
- The remaining point on C_{outer} in O(n) for each vertex
  - ⇒ not the largest (inscribed) empty circle - as discussed on seminar as we must test all VD vertices in combination with point on C outer
  - ⇒ $O(n^2)$



3 in – 1 out



1 point in – 3 out

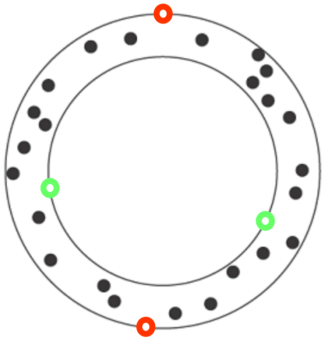
b) C_{outer} contains at least 3 points

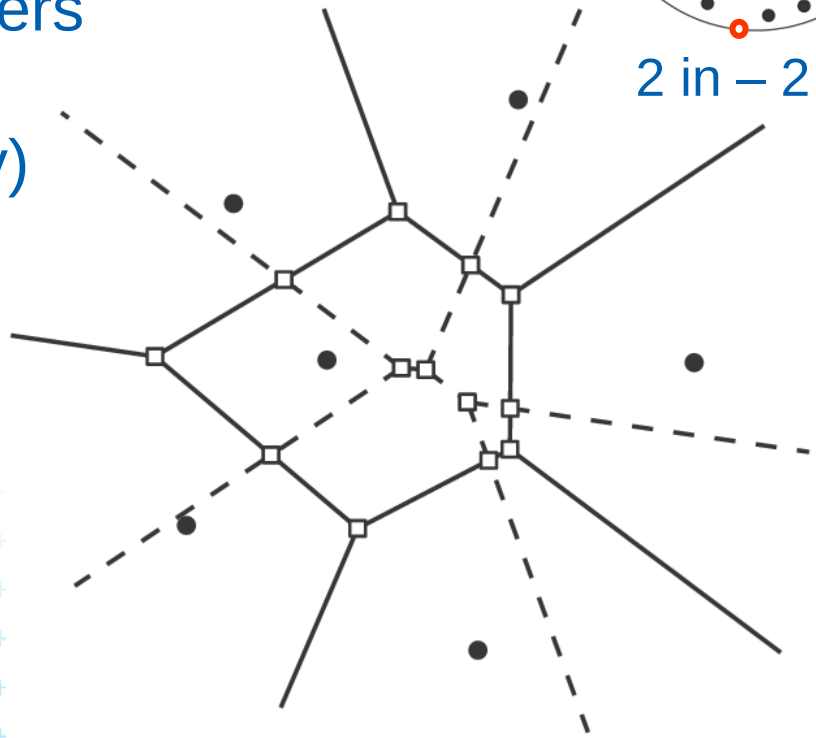
- Center is the *vertex of the farthest Voronoi diagram*
- The remaining point on C_{inner} in O(n)
  - ⇒ not the smallest enclosing circle - as discussed on seminar as we must test all vertices **in combination** with point on C inner
  - ⇒ $O(n^2)$

Smallest width annulus – case with 2+2 pts

c) C_{inner} and C_{outer} contain 2 points each

- Generate vertices of overlay of Voronoi (__) and farthest-point Voronoi (- - -) diagrams
  => $O(n^2)$ candidates for centers
  (we need only vertices, not the complete overlay)
- annulus computed in O(1) from center and 4 points (same for all 3 cases)
- $O(n^2)$





[Berg]

Smallest width annulus – case with 2+2 pts

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- annulus computed in O(1) from center and 4 points (same for all 3 cases)
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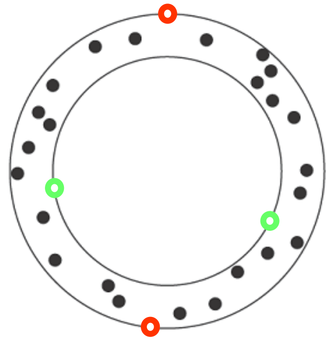
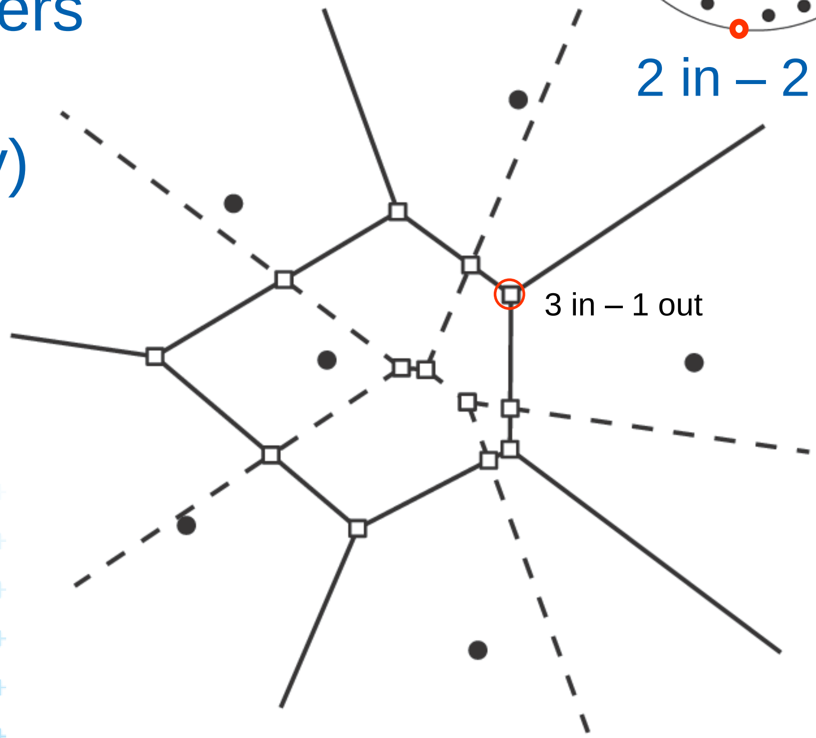


2 in – 2 out



[Berg]

Smallest width annulus – case with 2+2 pts

c) C_{inner} and C_{outer} contain 2 points each

- Generate vertices of overlay of Voronoi (__) and farthest-point Voronoi (- - -) diagrams
  => $O(n^2)$ candidates for centers
  (we need only vertices, not the complete overlay)
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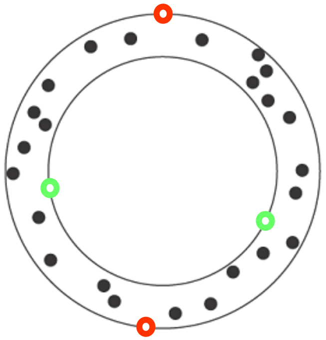
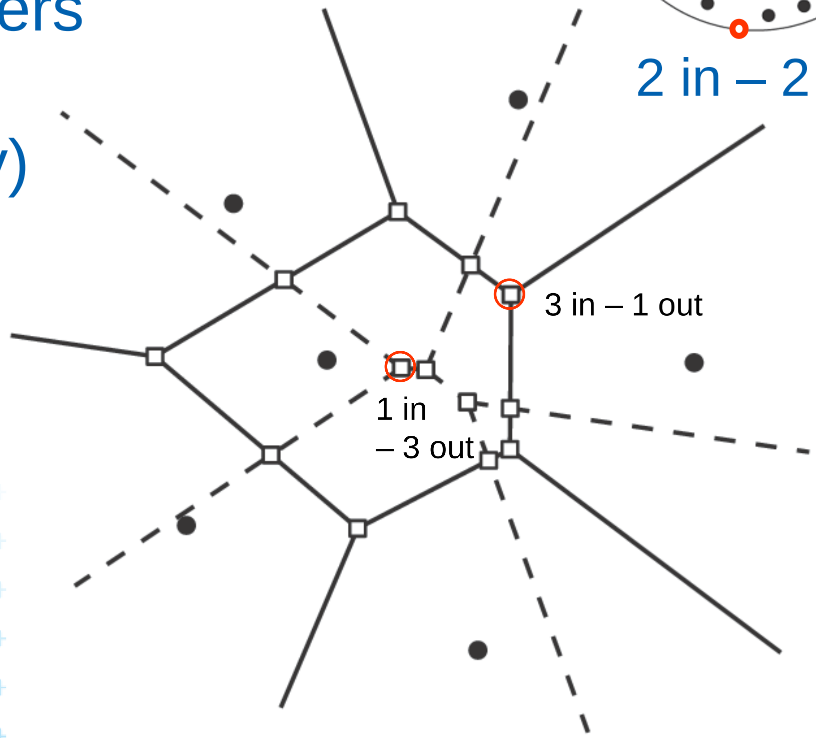


2 in – 2 out



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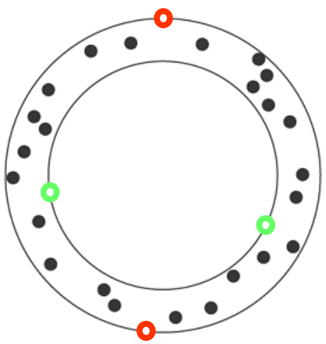
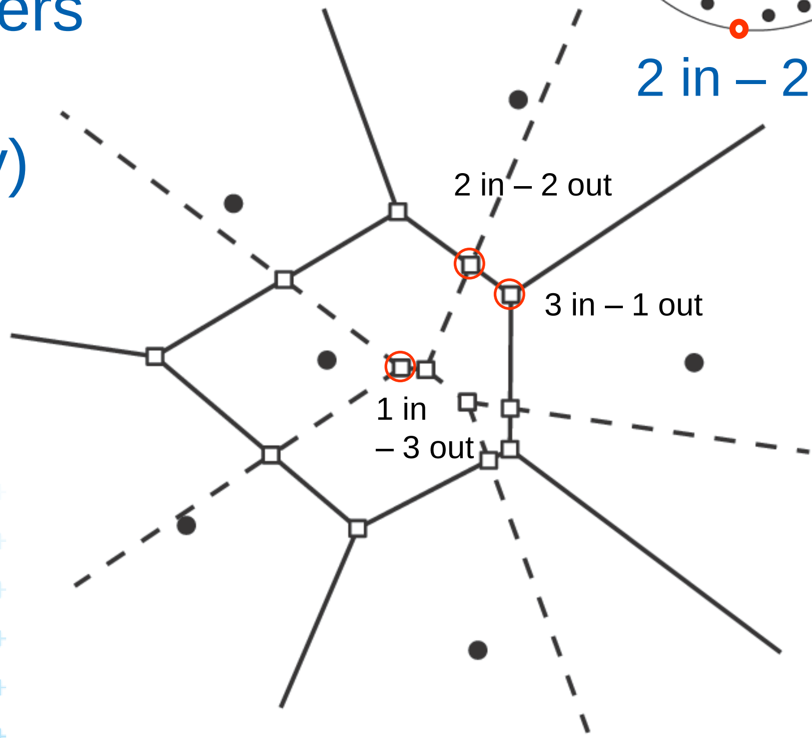


2 in – 2 out



[Berg]

Smallest width annulus

Smallest-Width-Annulus

Input: Set P of n points in the plane

Output: Smallest width annulus center and radii r and R (roundness)

1. Compute Voronoi diagram Vor(P)
   and farthest-point Voronoi diagram $Vor_{-1}(P)$ of P
2. For each vertex of Vor(P) (r) determine the *farthest point* (R) from P
   => $O(n)$ sets of four points defining candidate annuli – case a)
3. For each vertex of $Vor_{-1}(P)$ (R) determine the *closest point* (r) from P
   => $O(n)$ sets of four points defining candidate annuli – case b)
4. For every pair of edges Vor(P) and $Vor_{-1}(P)$ test if they intersect
   => another set of four points defining candidate annulus – c)
5. For all candidates of all three types
   chose the smallest-width annulus

$O(n^2)$ time using $O(n)$ storage

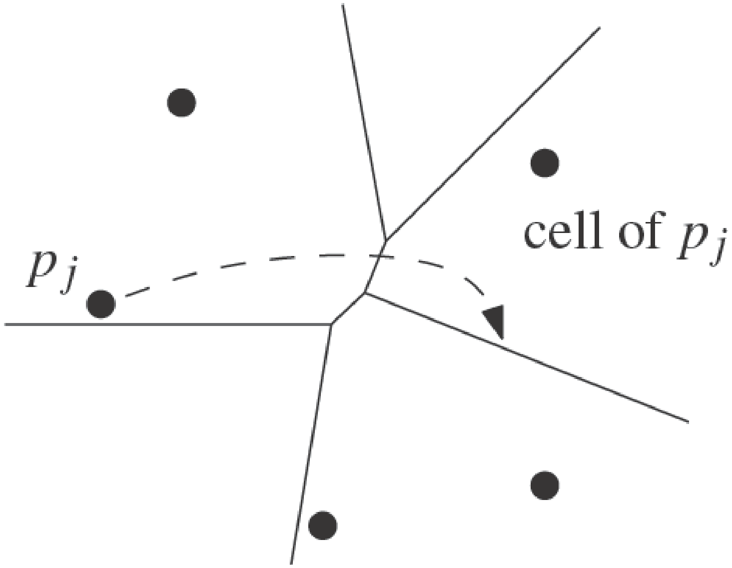
1. $O(n \log n)$
2. $O(n^2)$
3. $O(n^2)$
4. $O(n^2)$
5. $O(n^2)$

DCGI

Order n-1 VD construction

Modified DCEL for farthest-point Voronoi d

- Half-infinite edges -> we adapt DCEL
- Half-edges with origin in infinity
  - Special vertex-like record for origin in infinity
  - Store direction instead of coordinates
  - Next(e) or Prev(e) pointers undefined
- For each inserted site p_j
  - store a pointer to the most CCW half-infinite half-edge of its cell in DCEL



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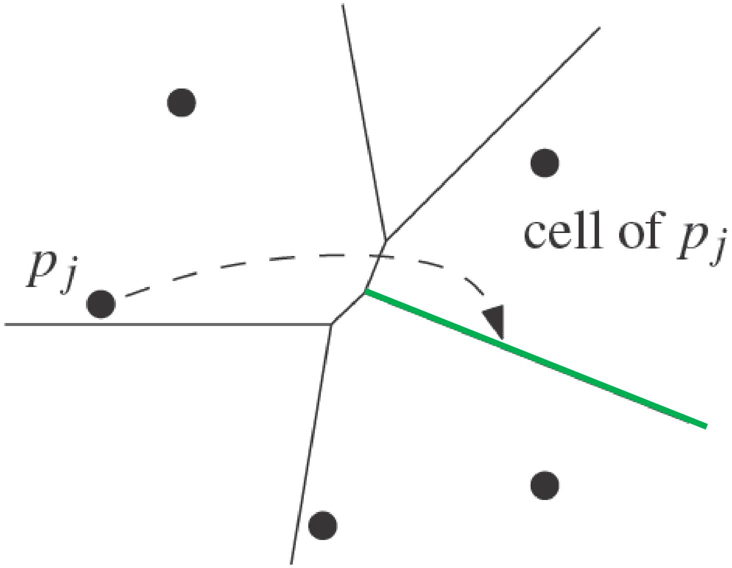

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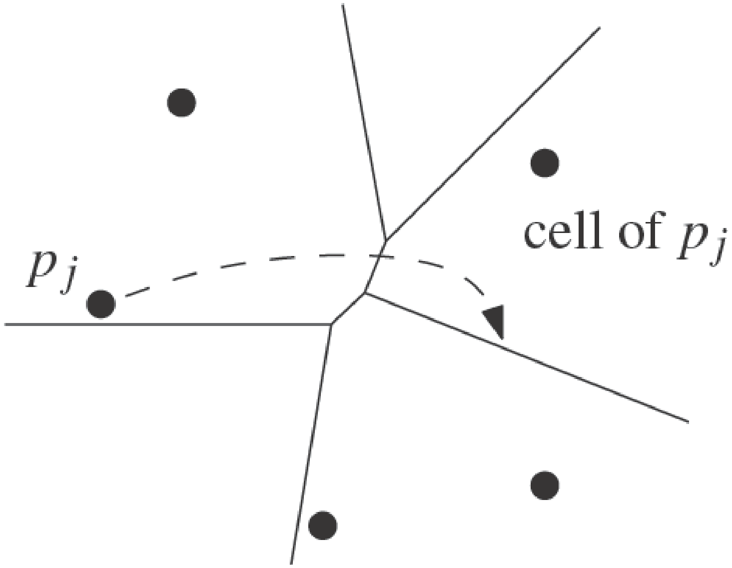
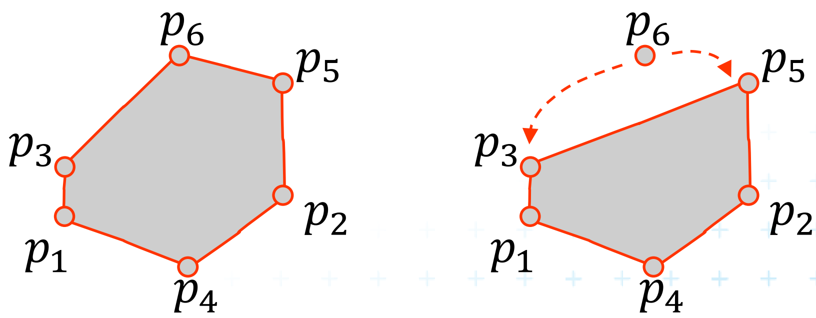


Idea of the algorithm

1. Create the convex hull
   and number the CH points randomly
2. Remove the points starting in the last of this random order and store *cw*(p_i) and *ccw*(p_i) points at the time of removal.
3. Include the points back and compute V_{-1}



p_i	$ccw(p_i)$	$cw(p_i)$
p_6	p_3	p_5
p_5	p_3	p_2
…		

Farthest-point Voronoi d. construction

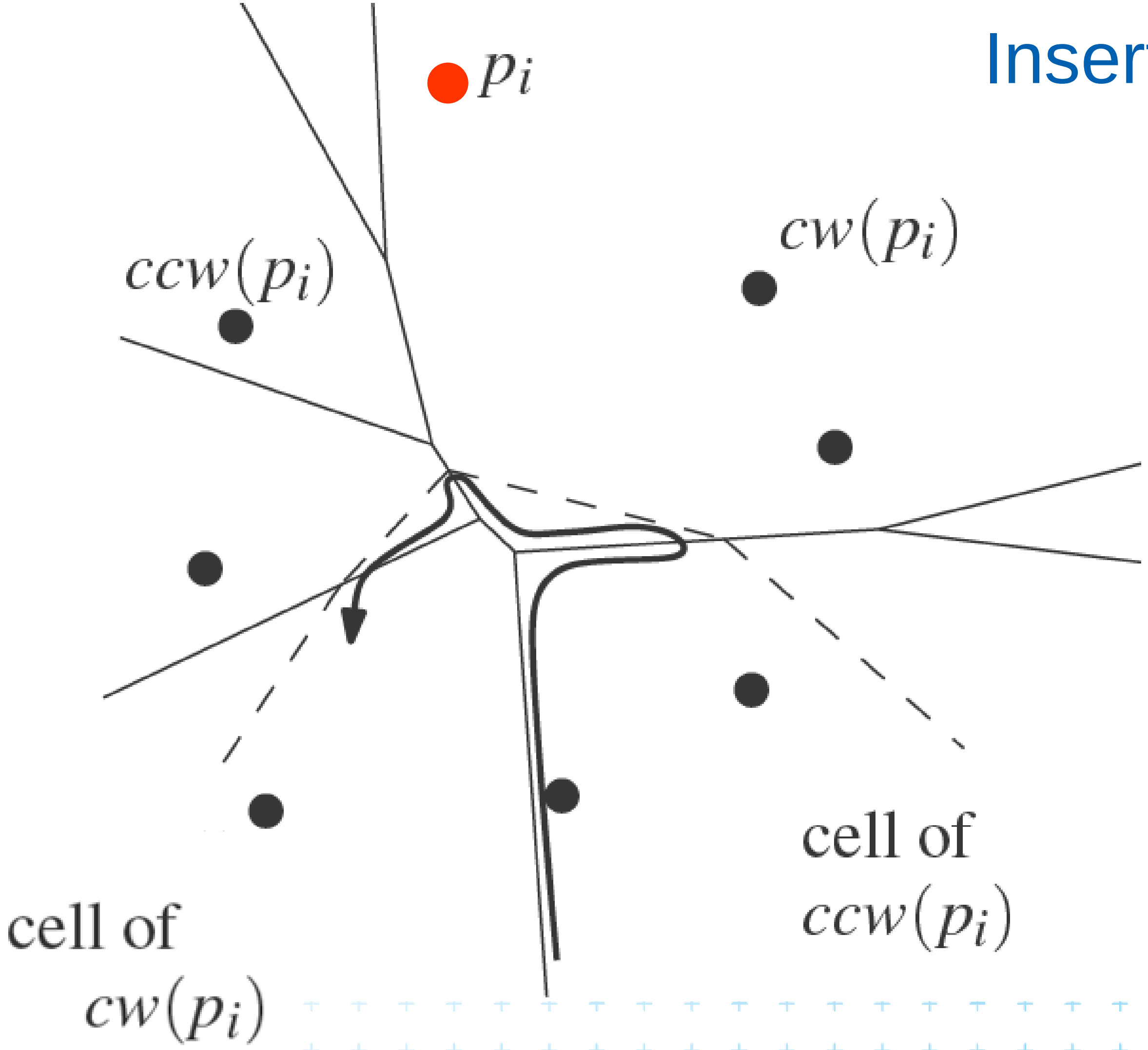
Farthest-pointVoronoi $O(n \log n)$ expected time in $O(n)$ storage

Input: Set of points *P* in plane

Output: Farthest-point VD $\text{Vor}_{-1}(P)$

1. Compute convex hull of *P*
2. Put points in CH(*P)* of *P* in random order $p_1, \ldots, p_h$
3. Remove $p_h, \ldots, p_4$ from the cyclic order (around the CH).
   When removing p_i, store the neighbors: *cw*(p_i) and *ccw*(p_i) at the time of removal. (This is done to know the neighbors needed in step 6.)
4. Compute $\text{Vor}_{-1}(\{p_1, p_2, p_3\})$ as init
5. **for** i = 4 **to** *h* **do**
6. Add site p_i to $\text{Vor}_{-1}(\{p_1, p_2, \ldots, p_{i-1}\})$ between site *cw*(p_i) and *ccw*(p_i)
7. - start at most CCW edge of the cell *ccw*(p_i)
8. - continue CW to find intersection with bisector(*ccw*(p_i), p_i)
9. - trace borders of Voronoi cell p_i in CCW order, add edges
10. - remove invalid edges inside of Voronoi cell p_i

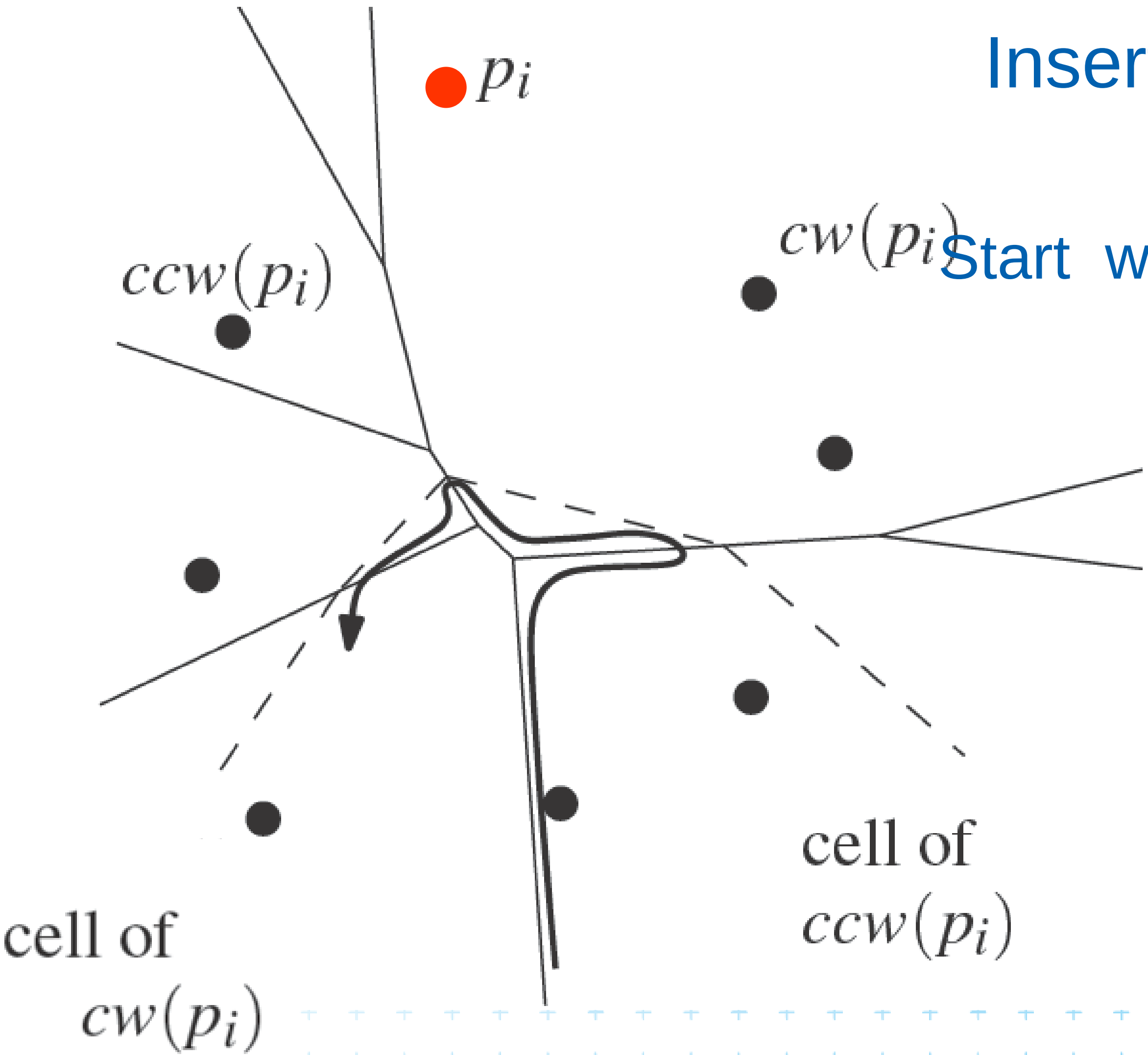
Farthest-point Voronoi d. construction



Insertion of site p_i

)

Farthest-point Voronoi d. construction



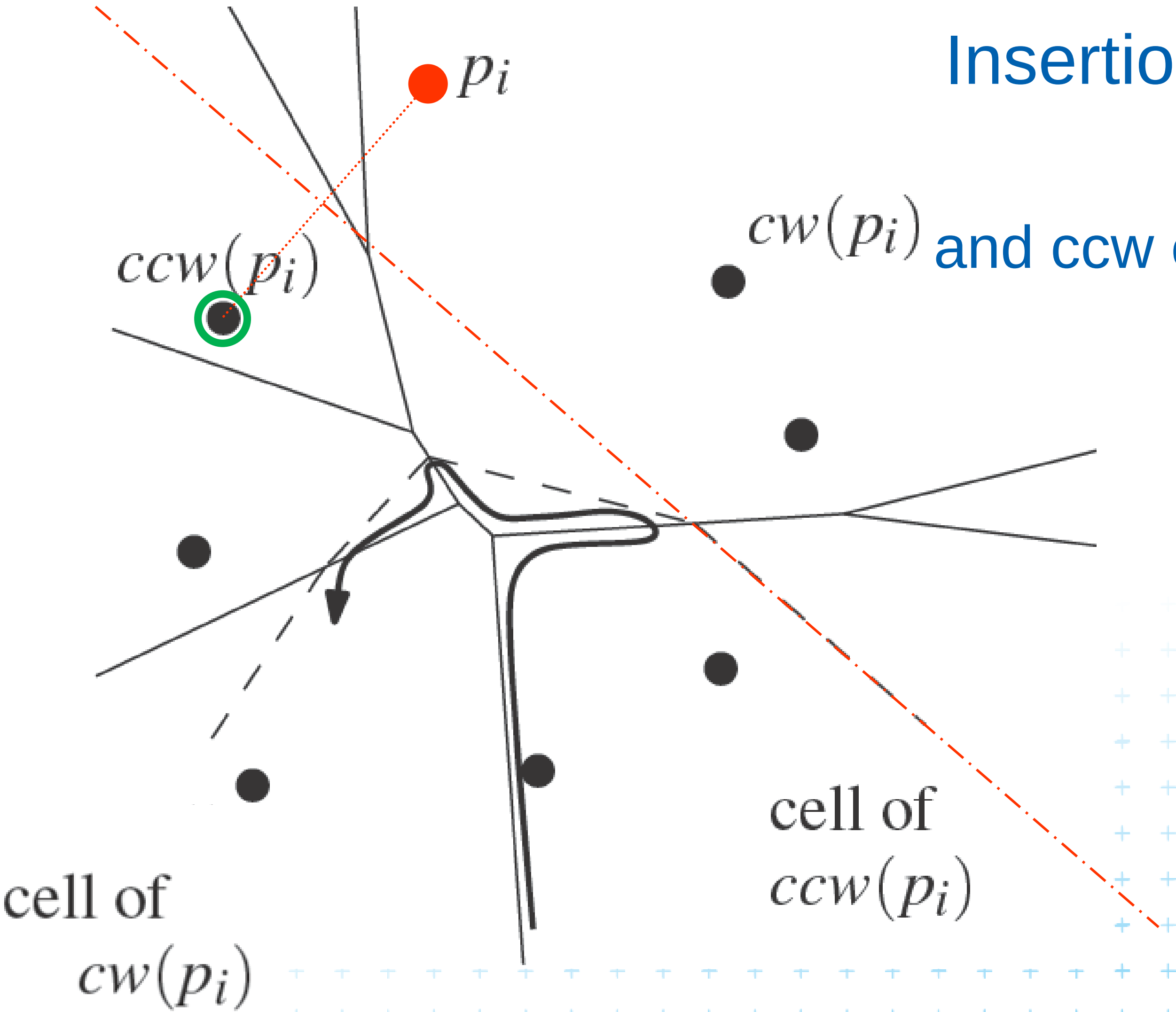
Insertion of site p_i

Start with site ccw(p i) i i i

Farthest-point Voronoi d. construction

Insertion of site p_i

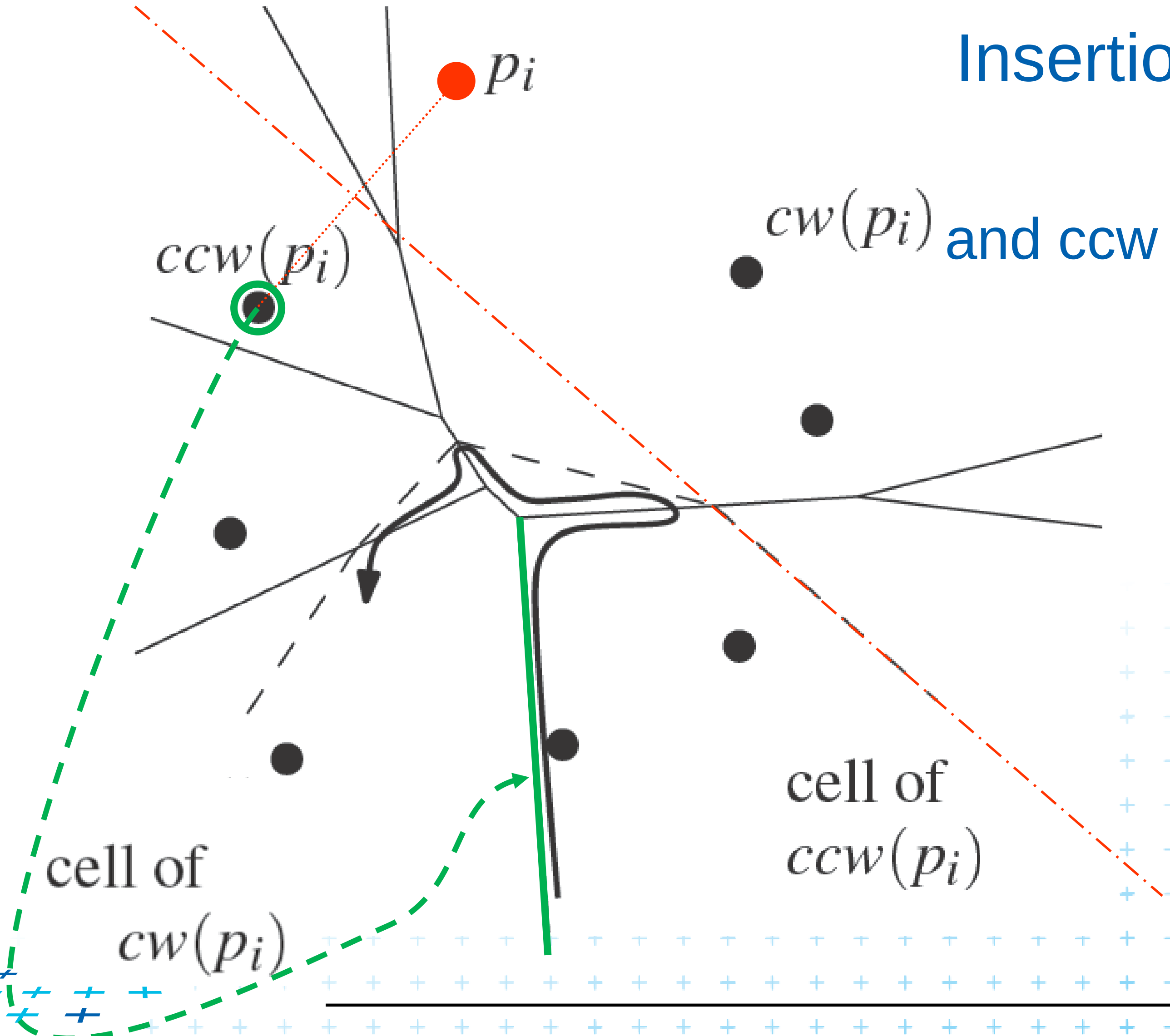
and ccw edge of its cell



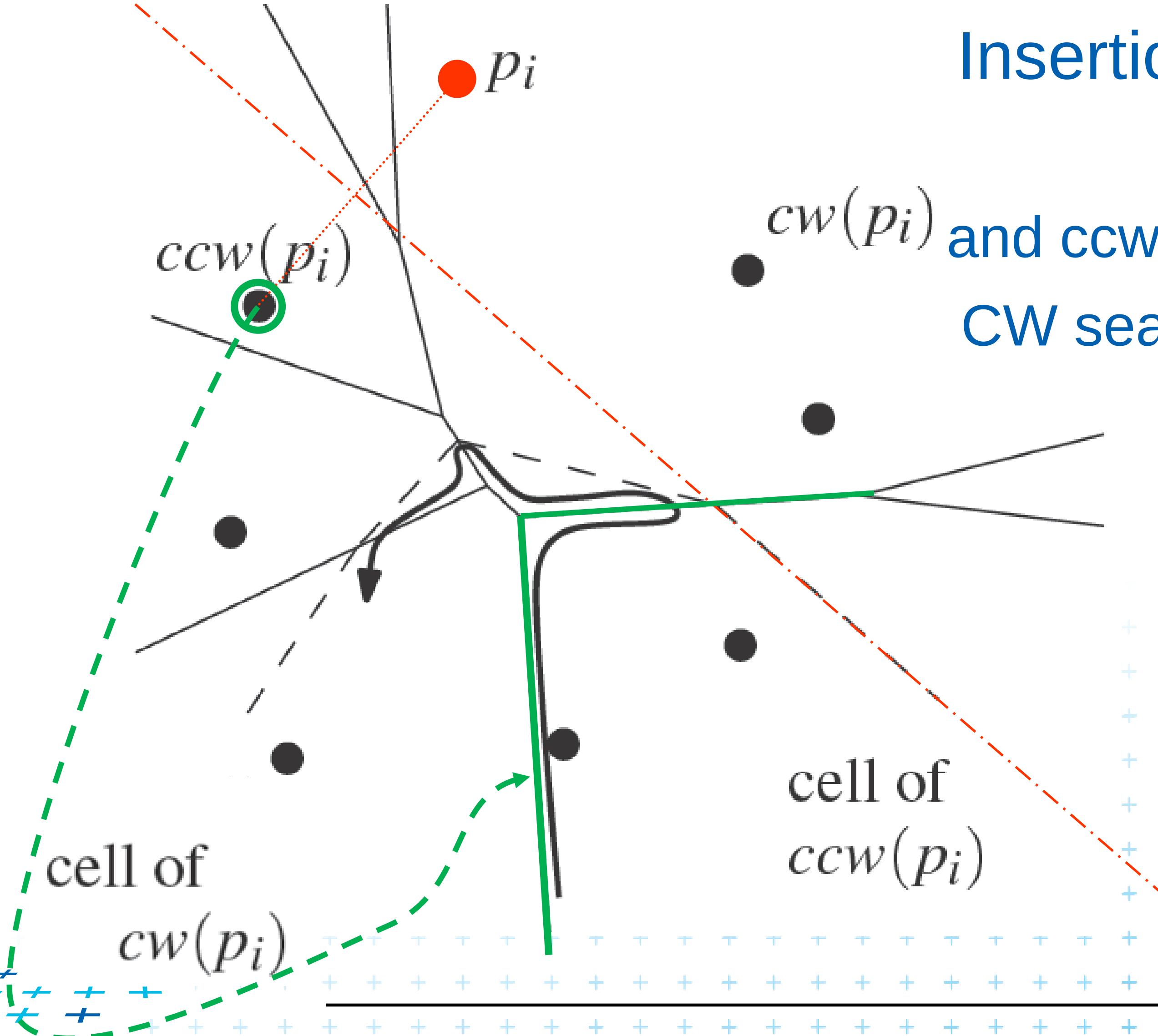
Farthest-point Voronoi d. construction

Insertion of site p_i

$cw(p_i)$ and ccw edge of its cell



Farthest-point Voronoi d. construction

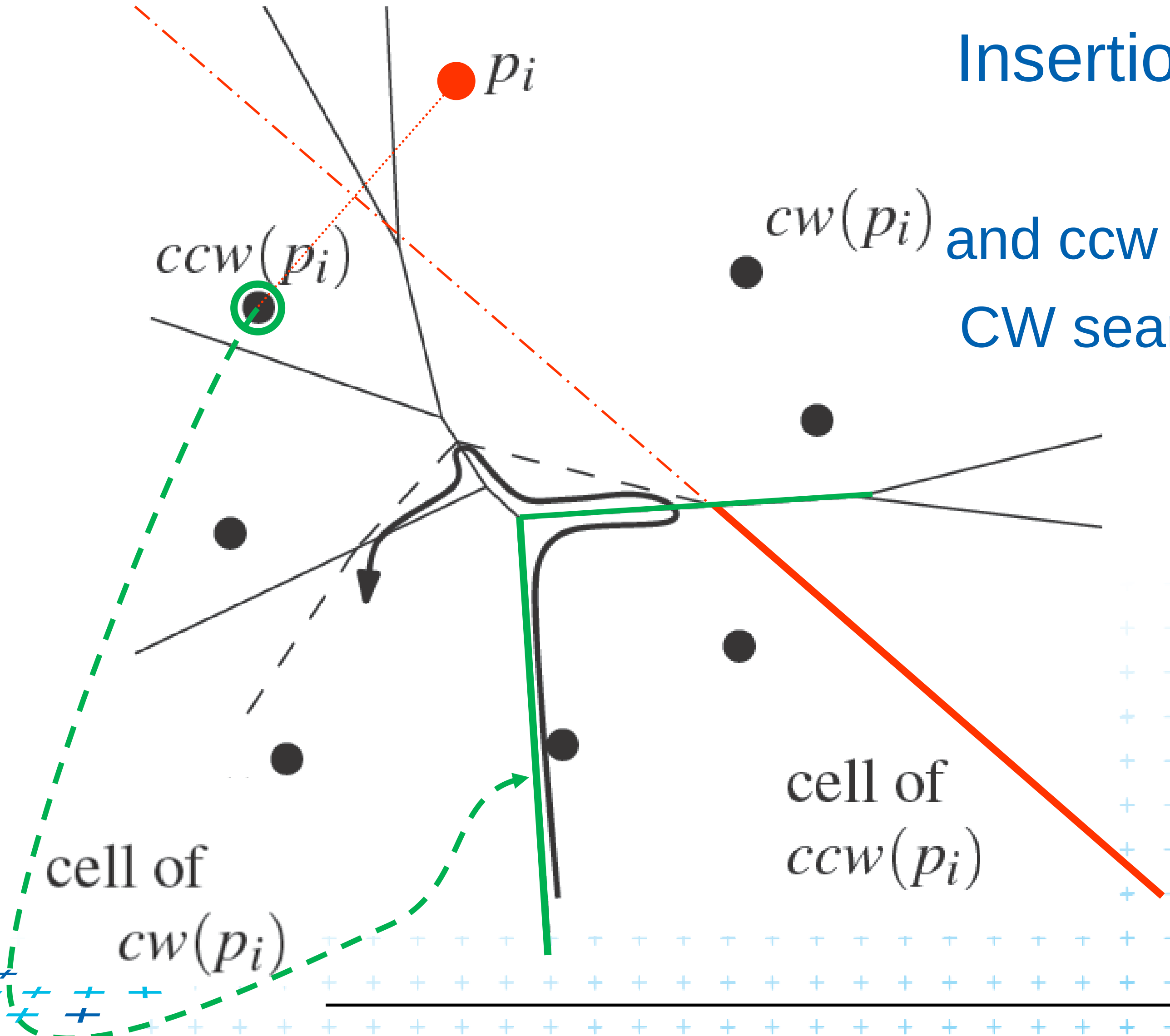


Insertion of site p_i

and ccw edge of its cell

CW search of intersection

Farthest-point Voronoi d. construction



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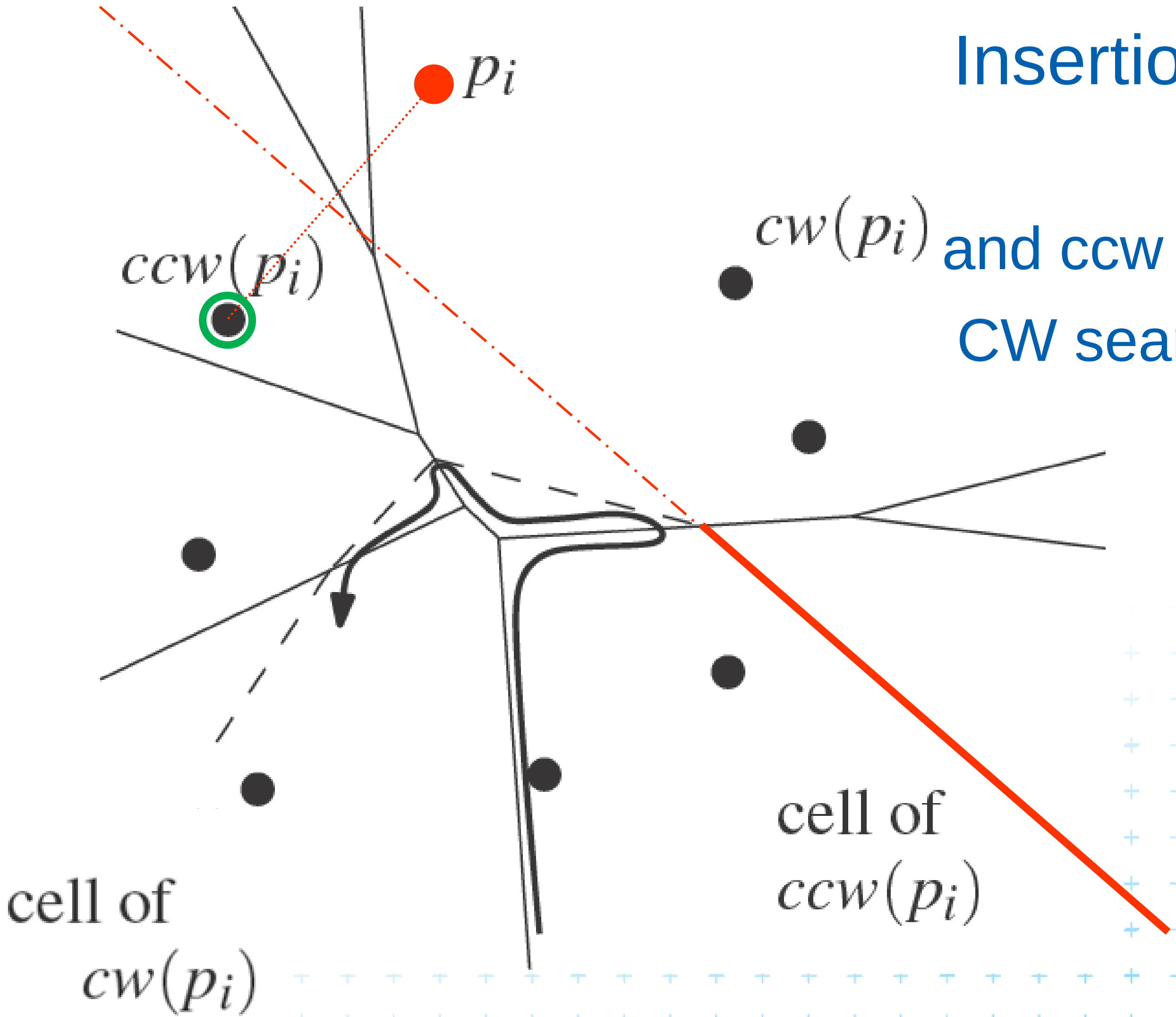
Farthest-point Voronoi d. construction



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Farthest-point Voronoi d. construction

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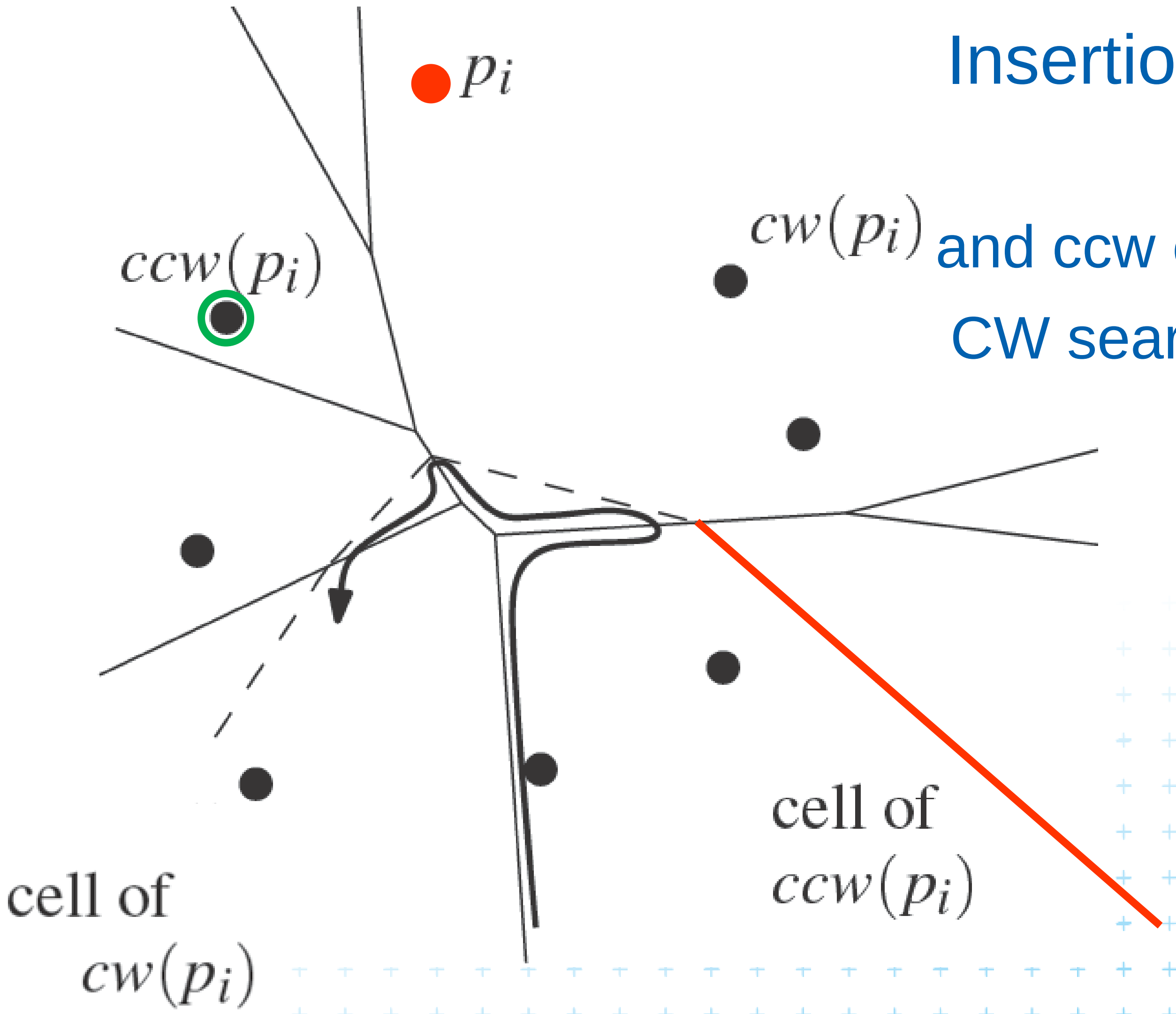

Farthest-point Voronoi d. construction

Insertion of site p_i

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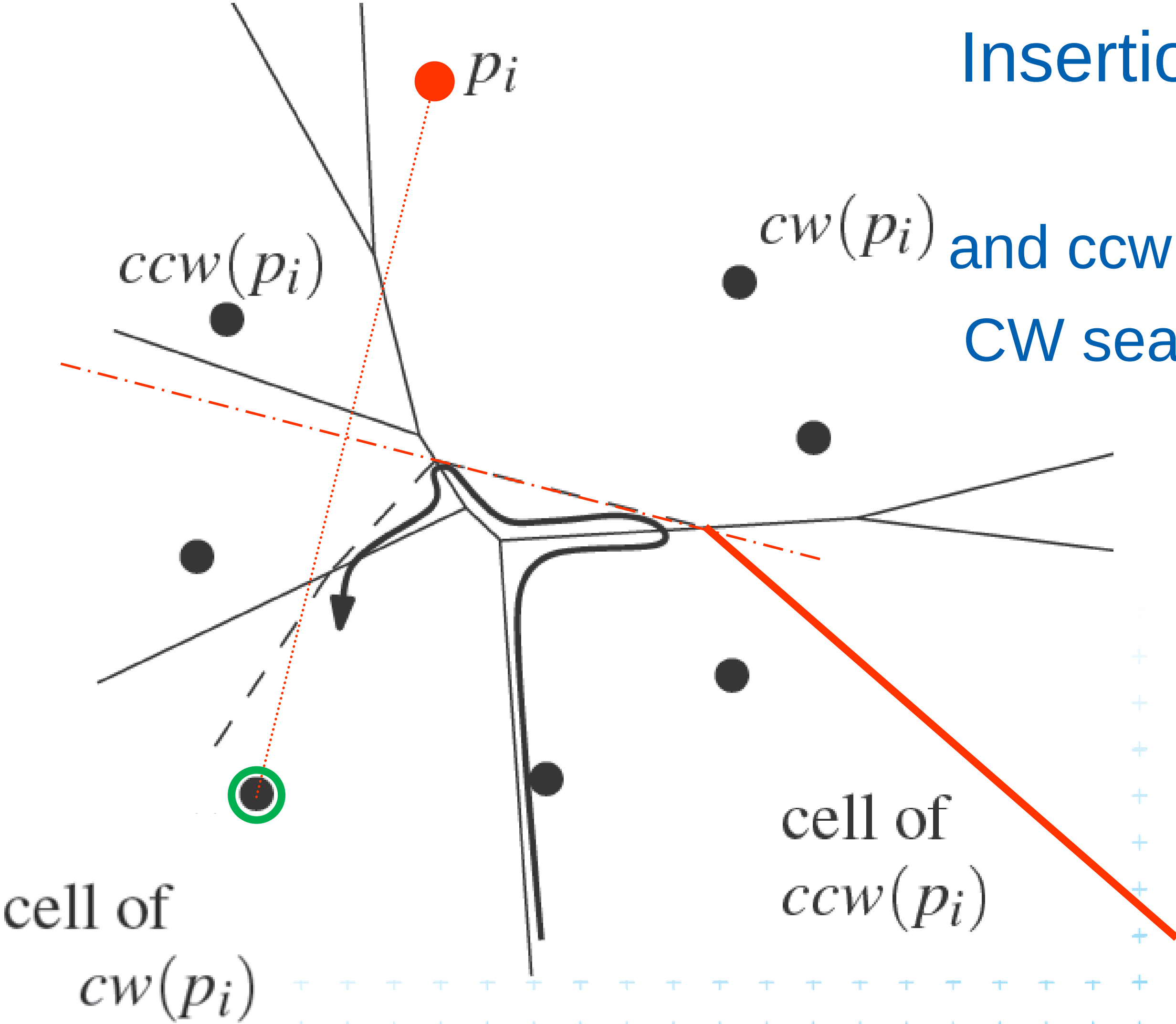
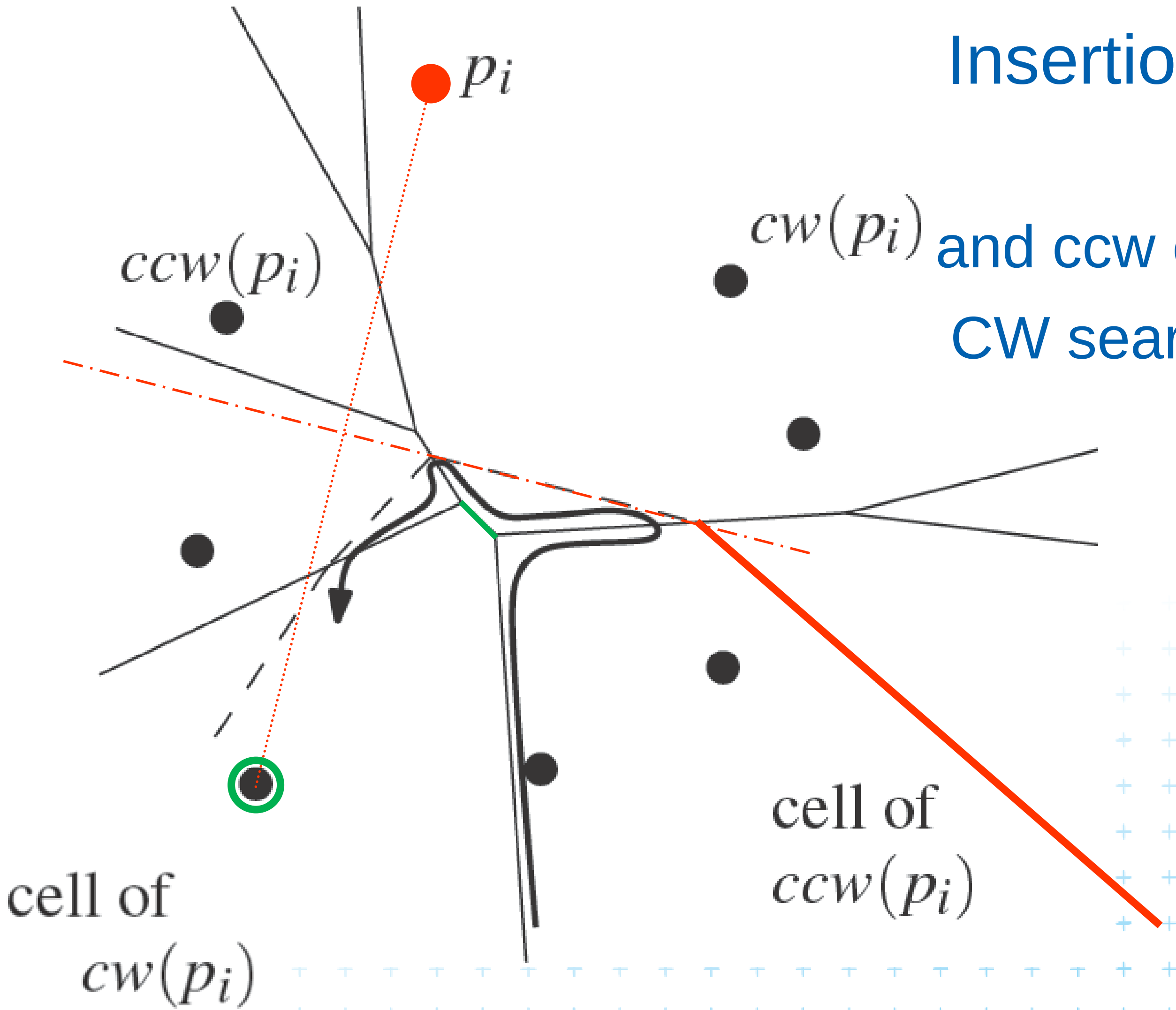
CW search of intersection



Farthest-point Voronoi d. construction



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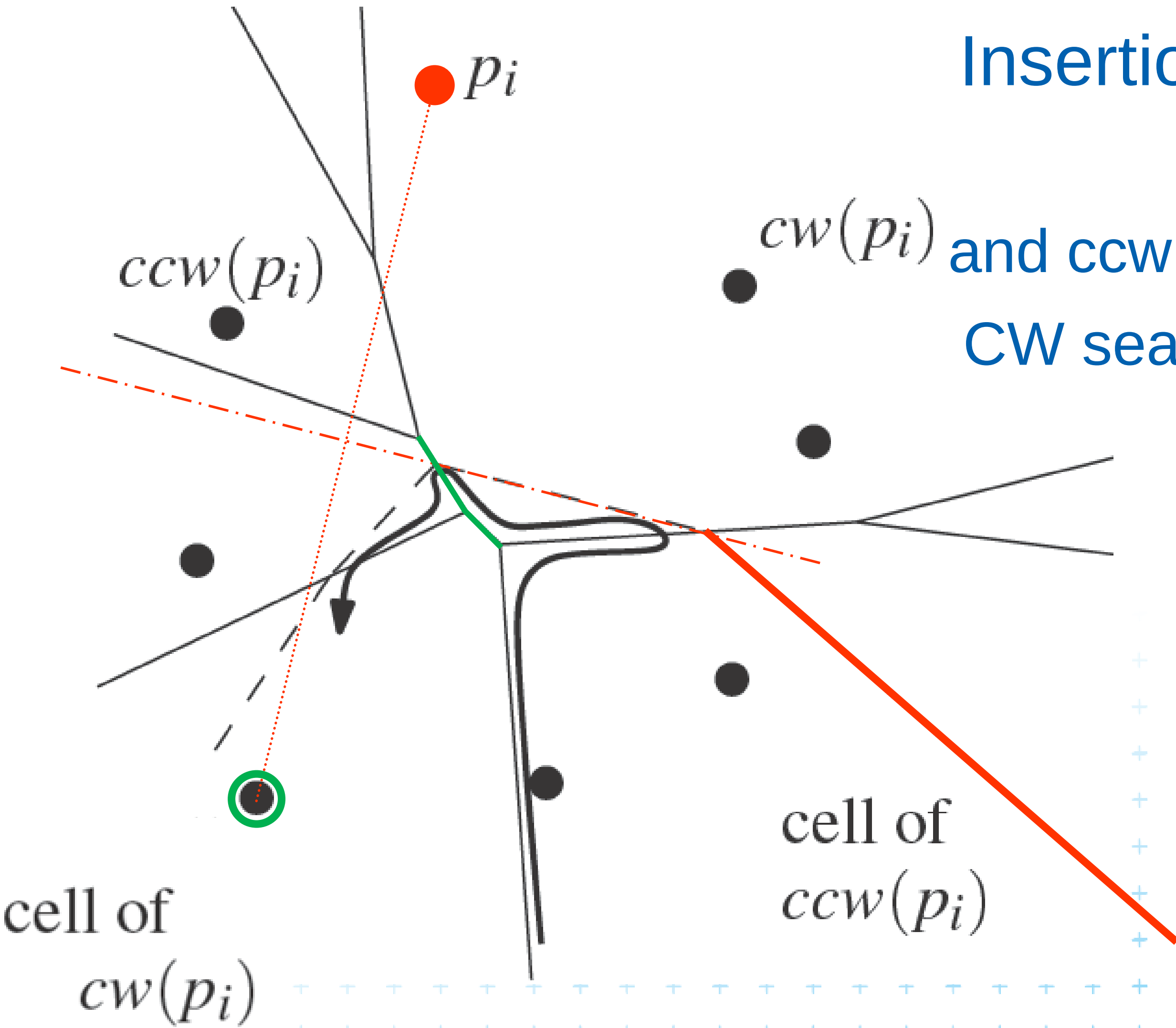
Farthest-point Voronoi d. construction



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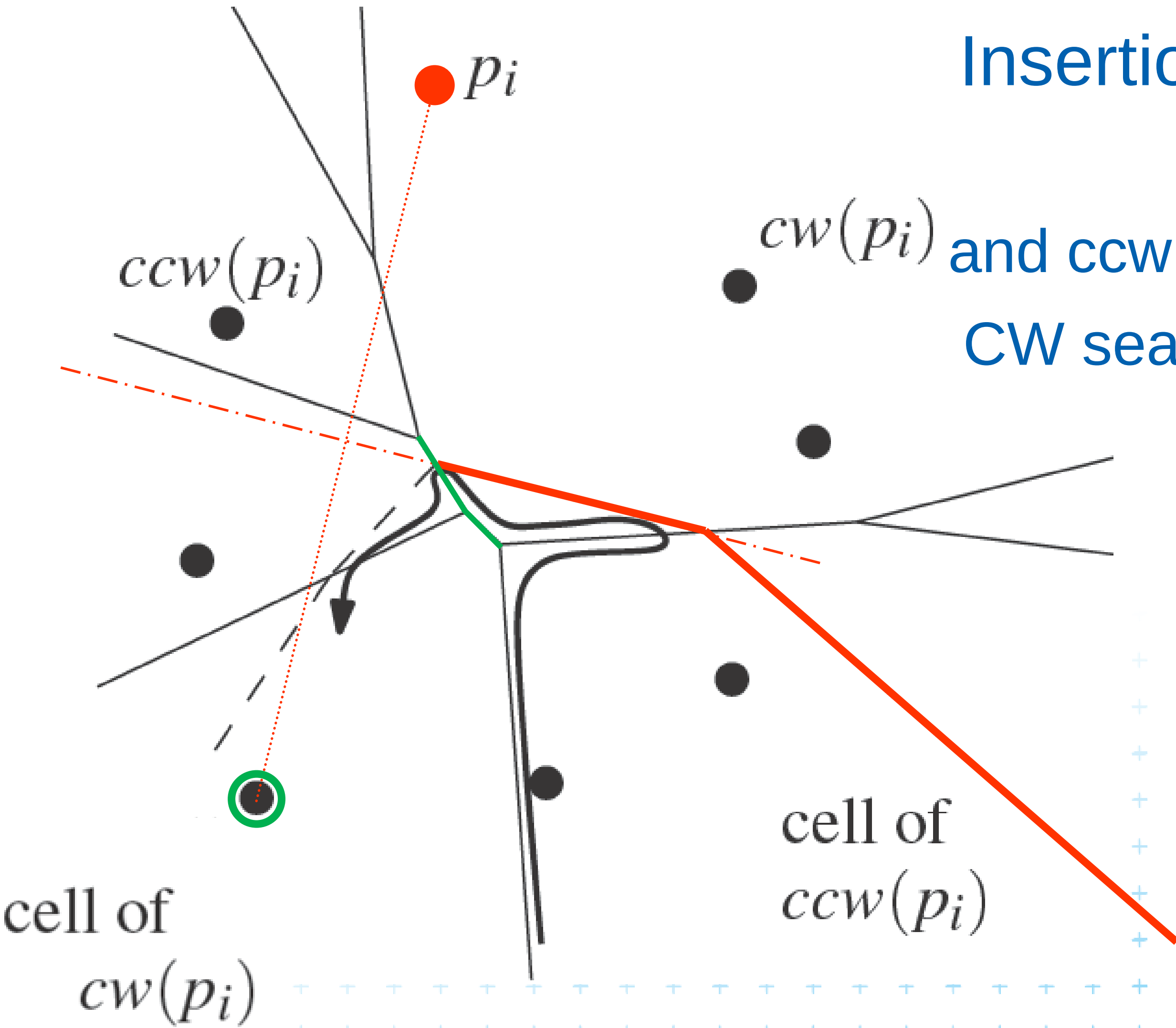

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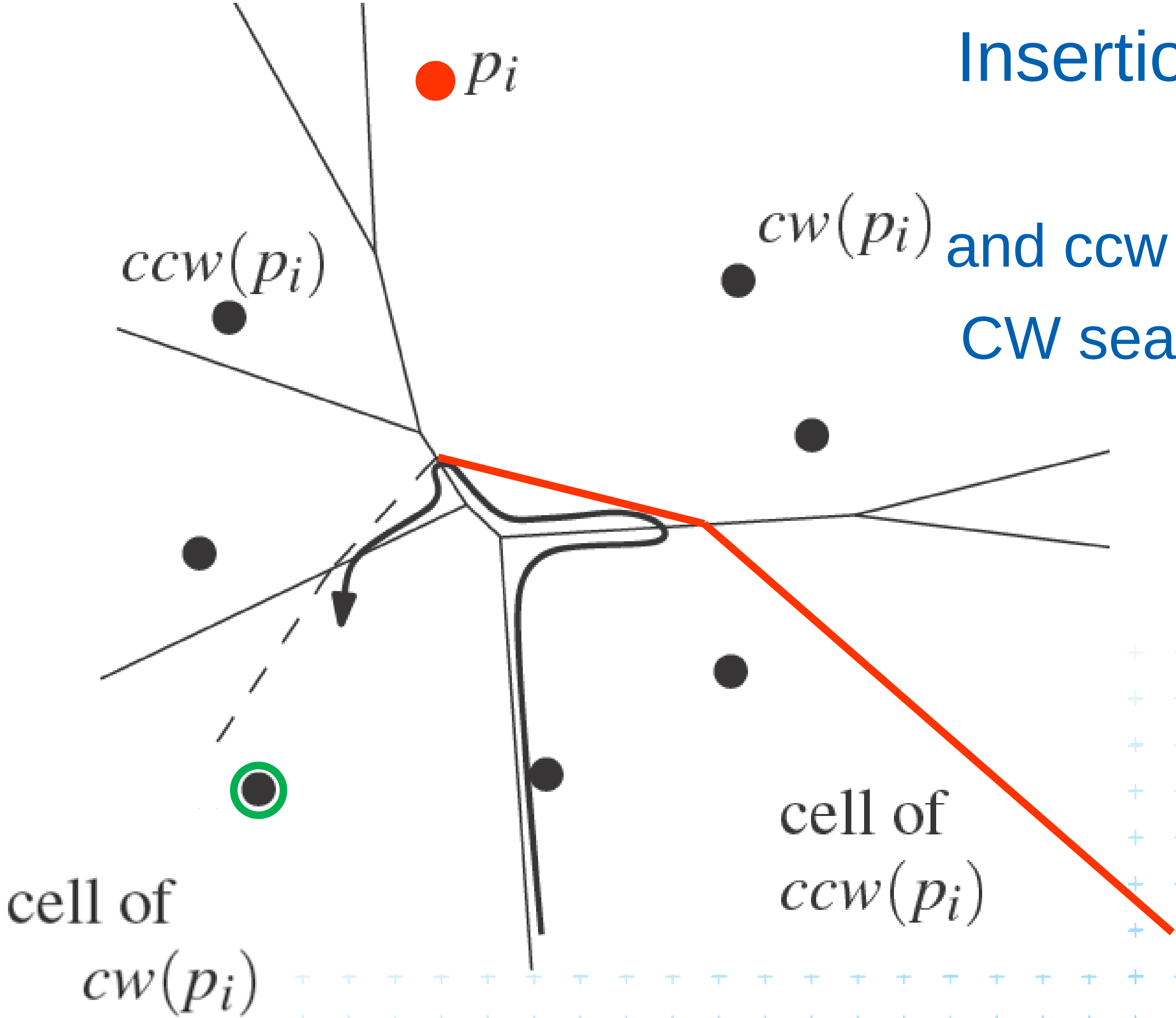

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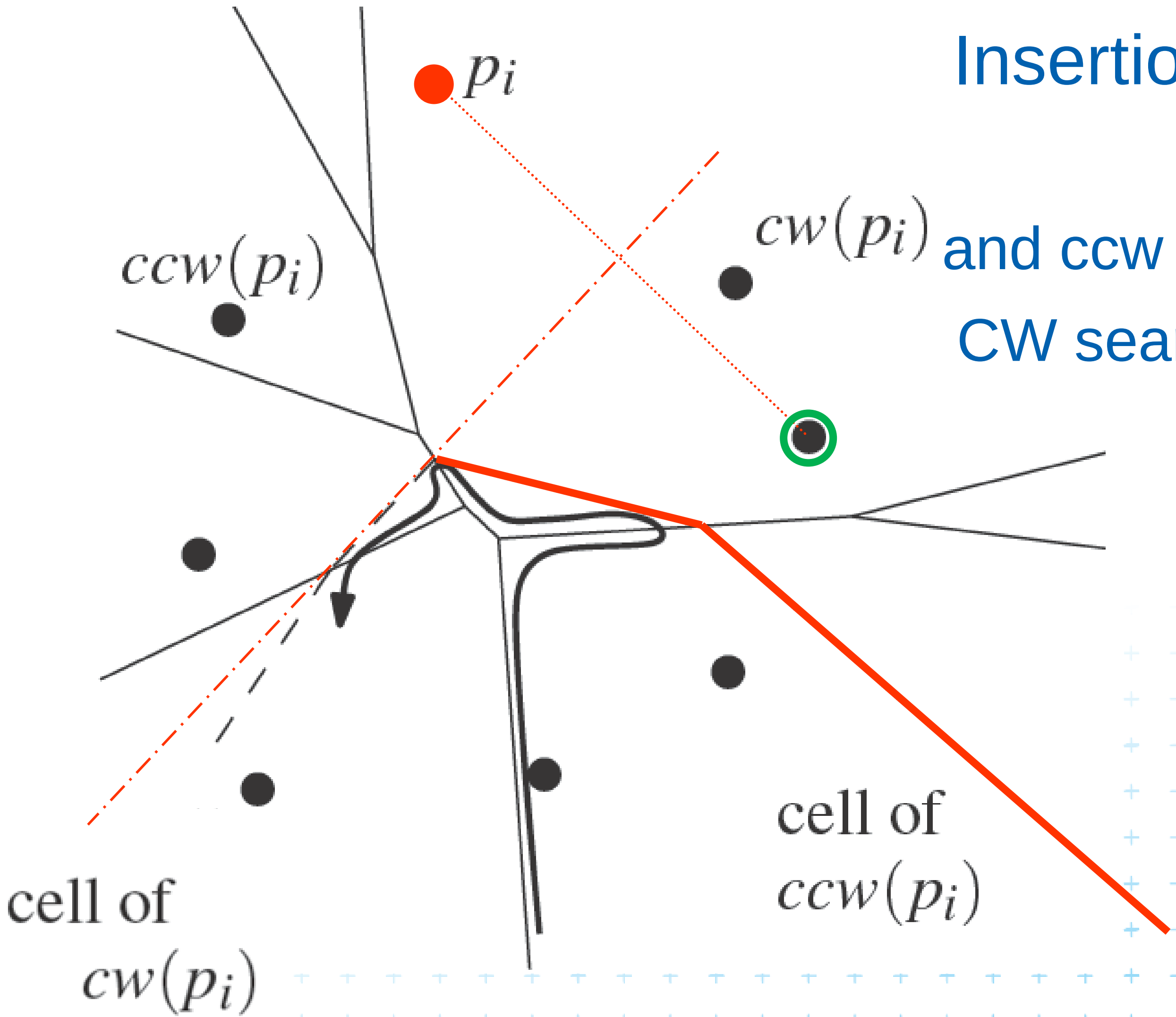
Farthest-point Voronoi d. construction



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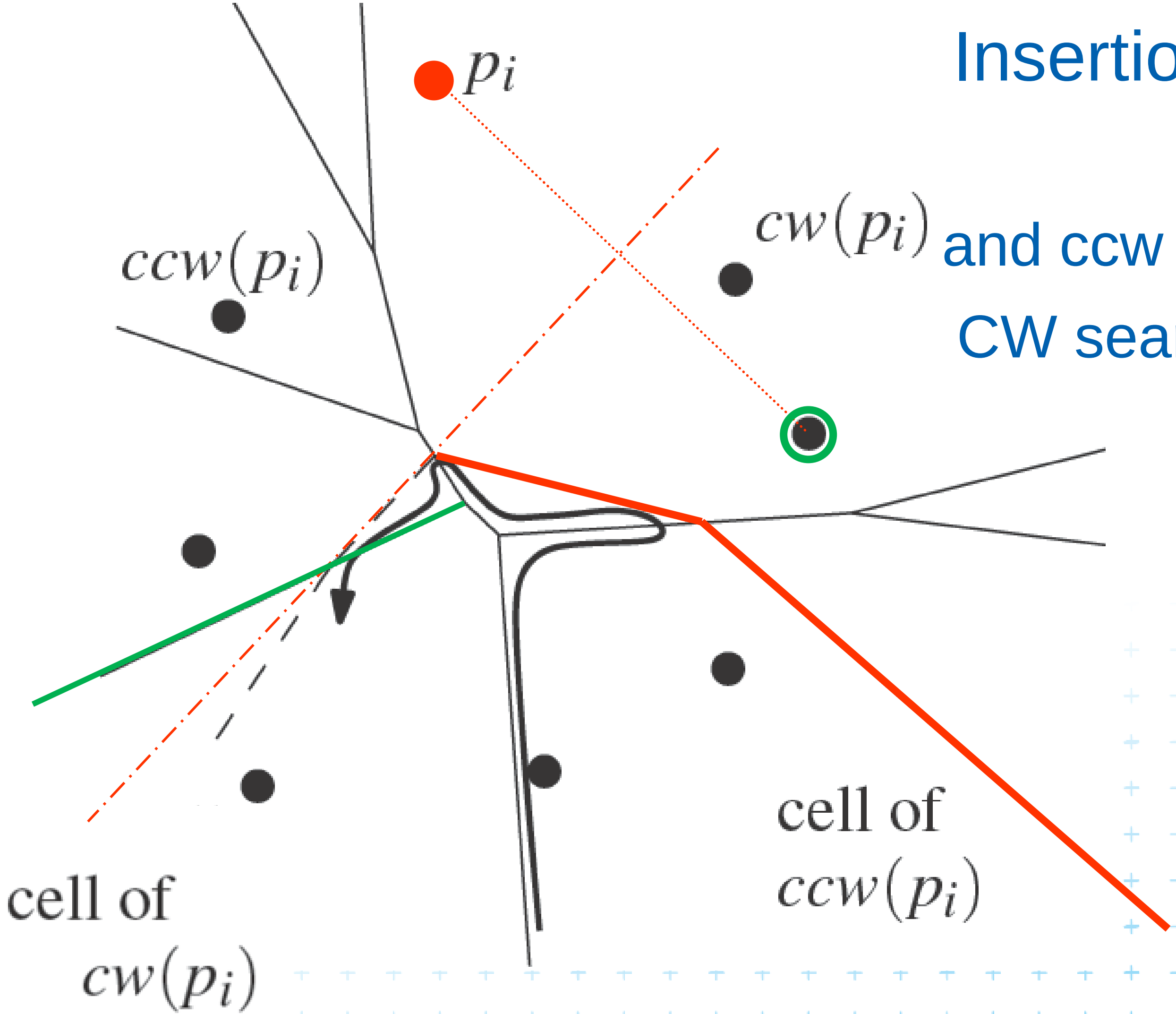
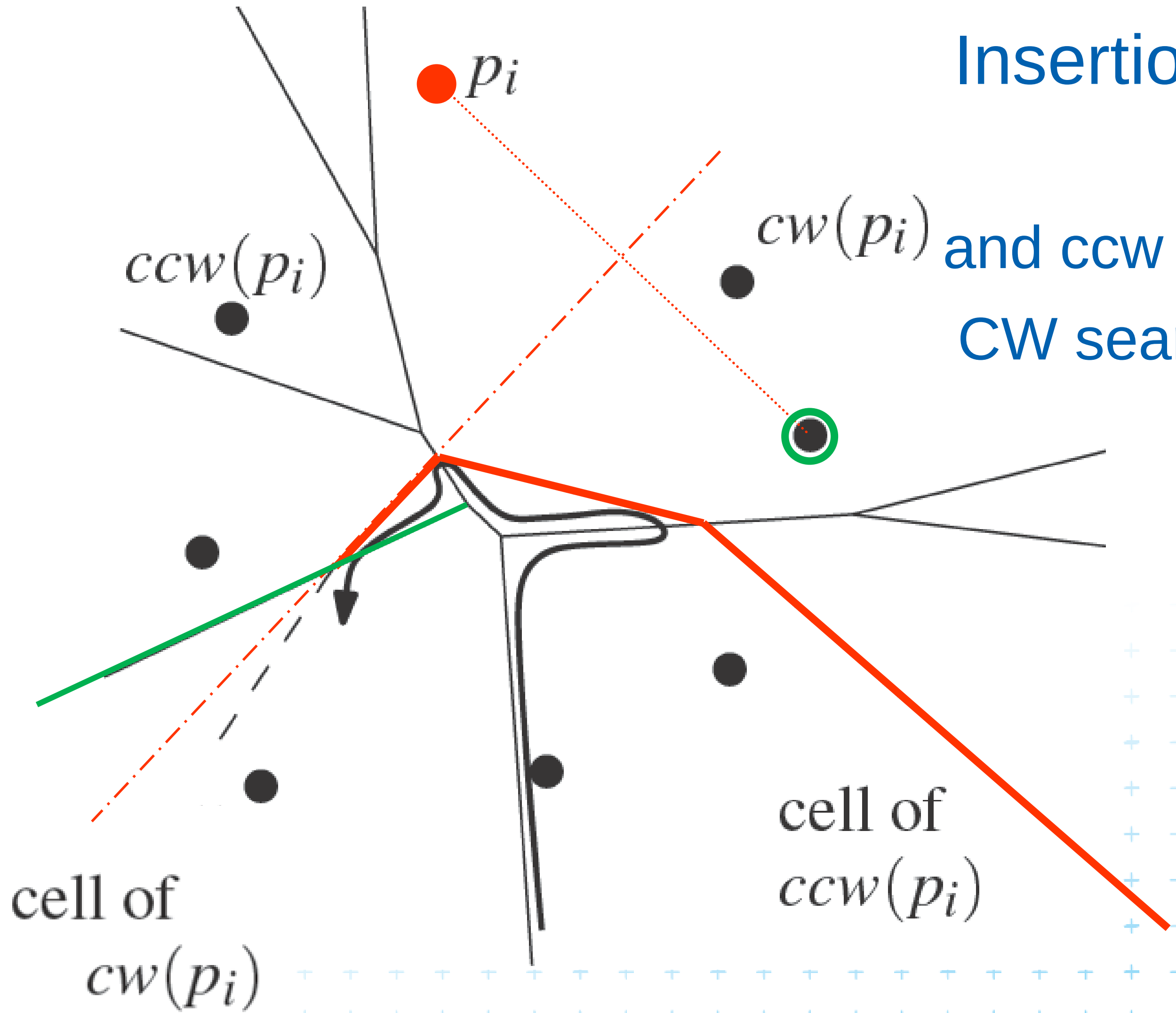


Farthest-point Voronoi d. construction



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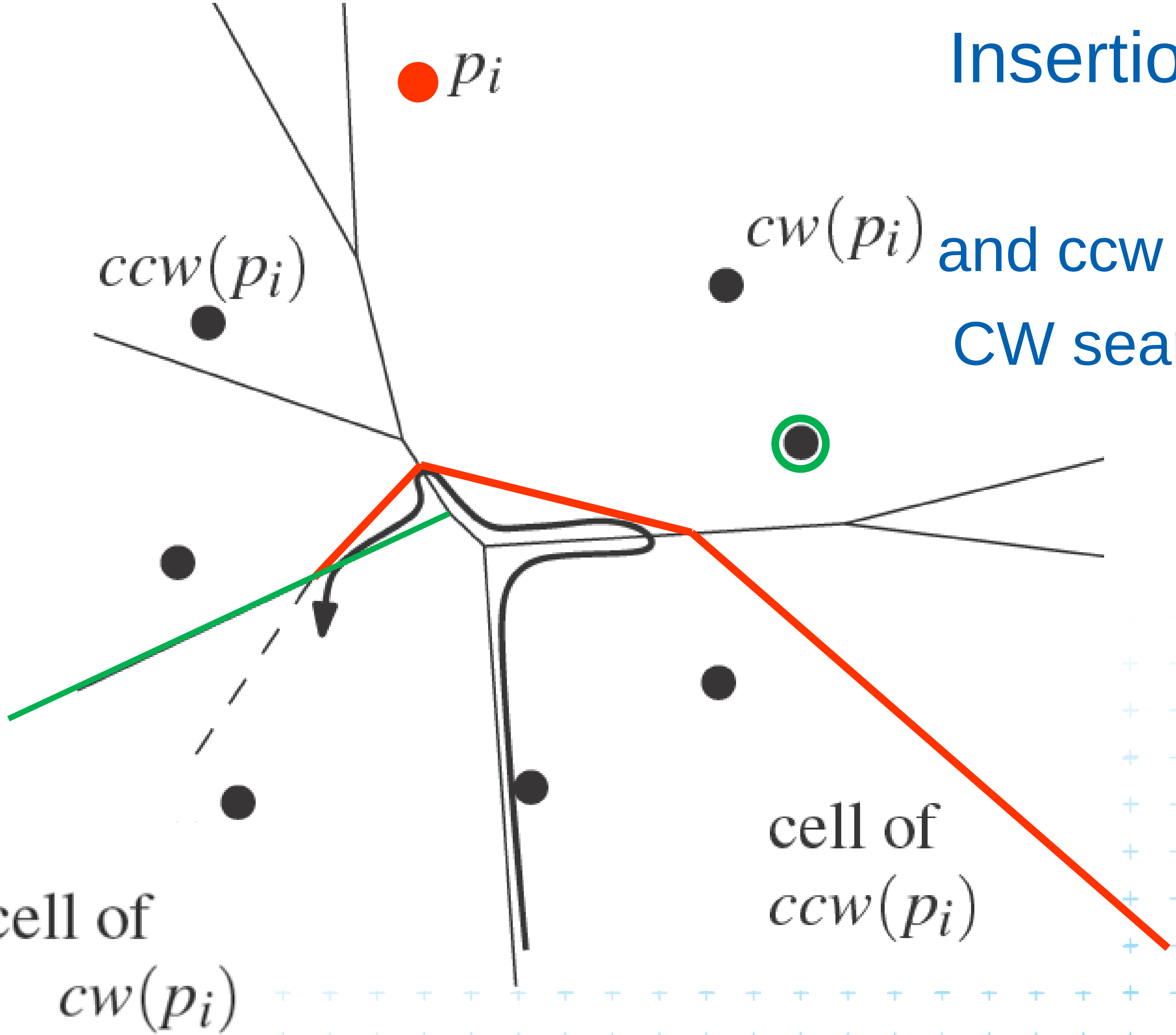

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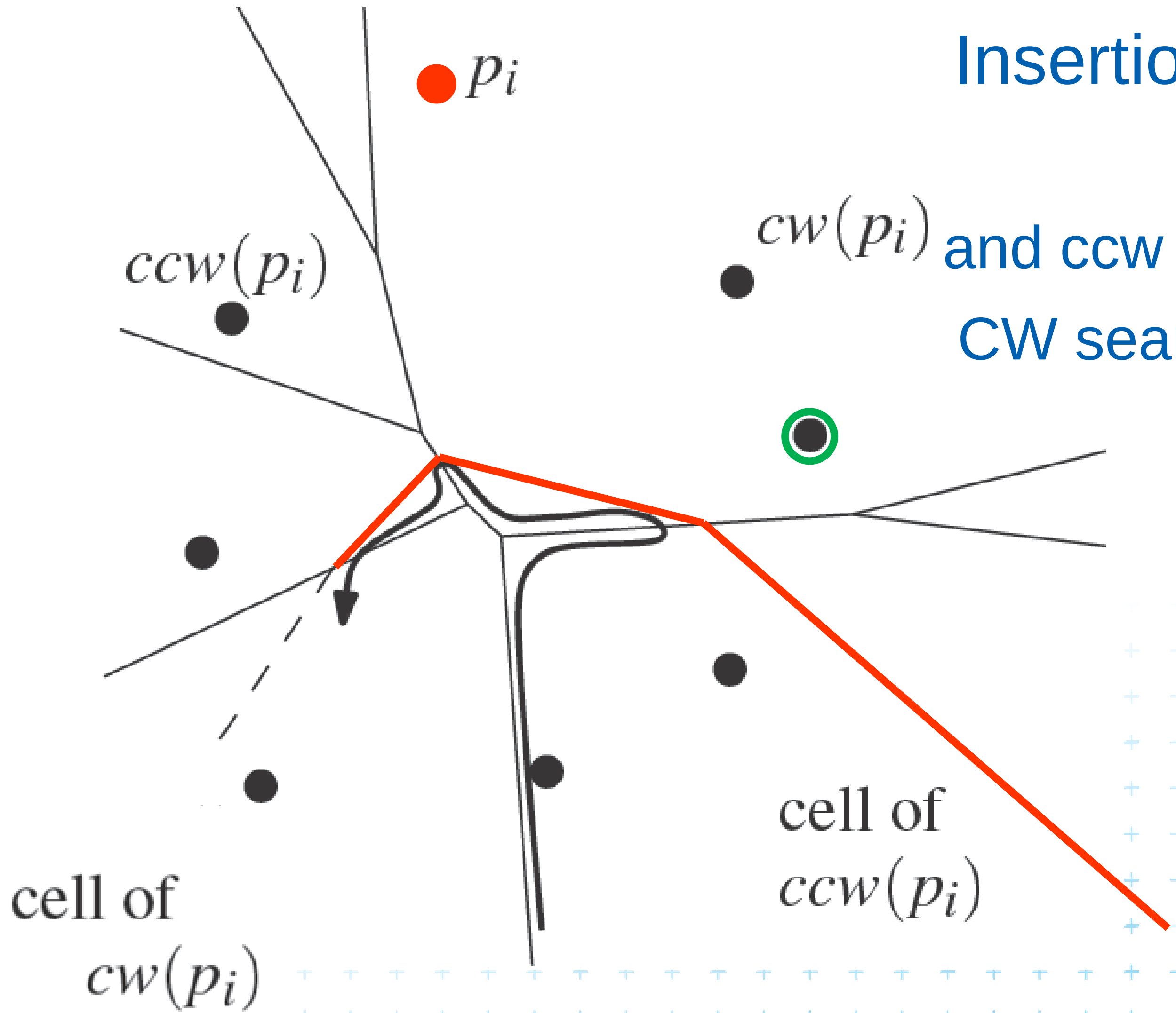

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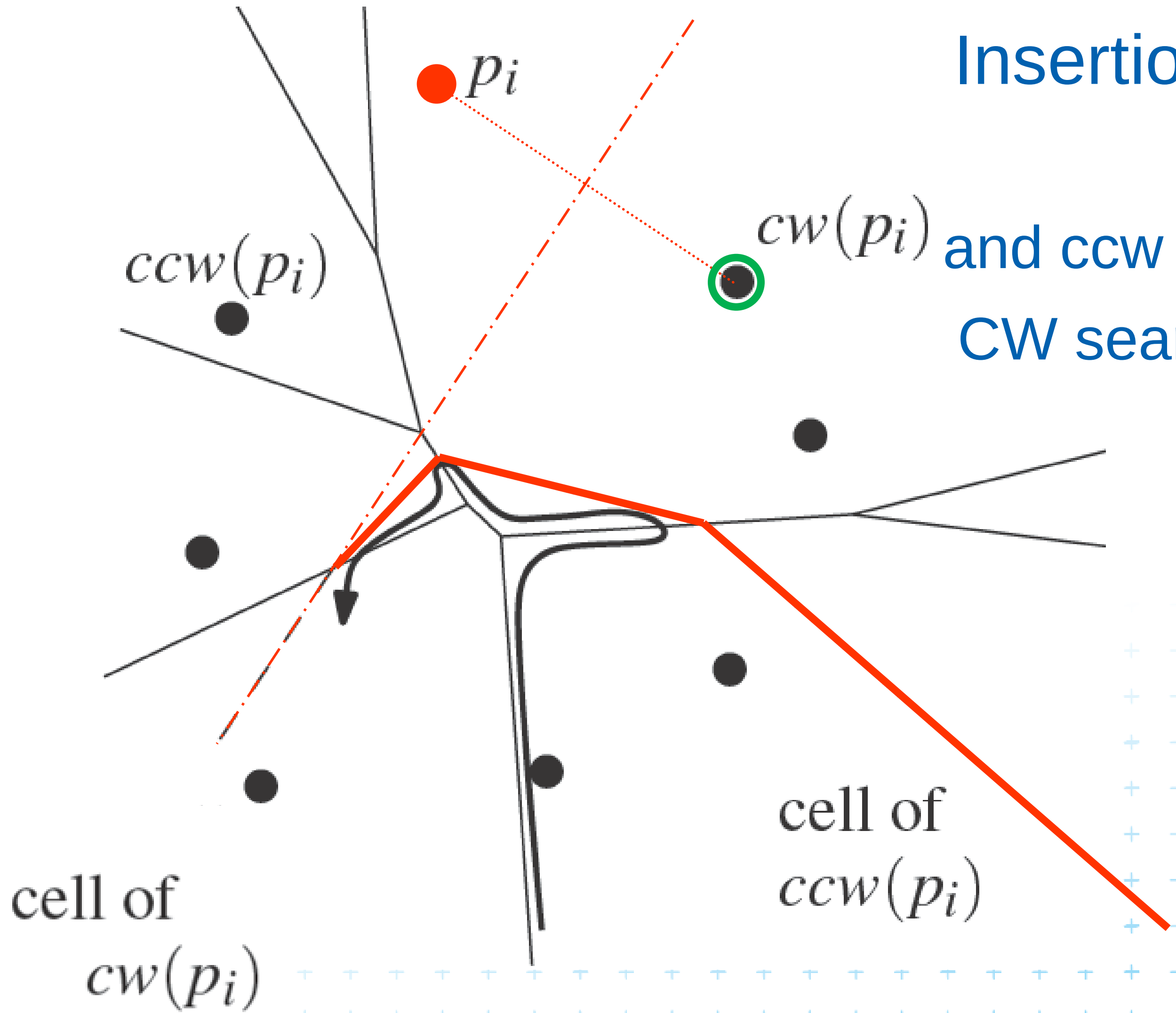

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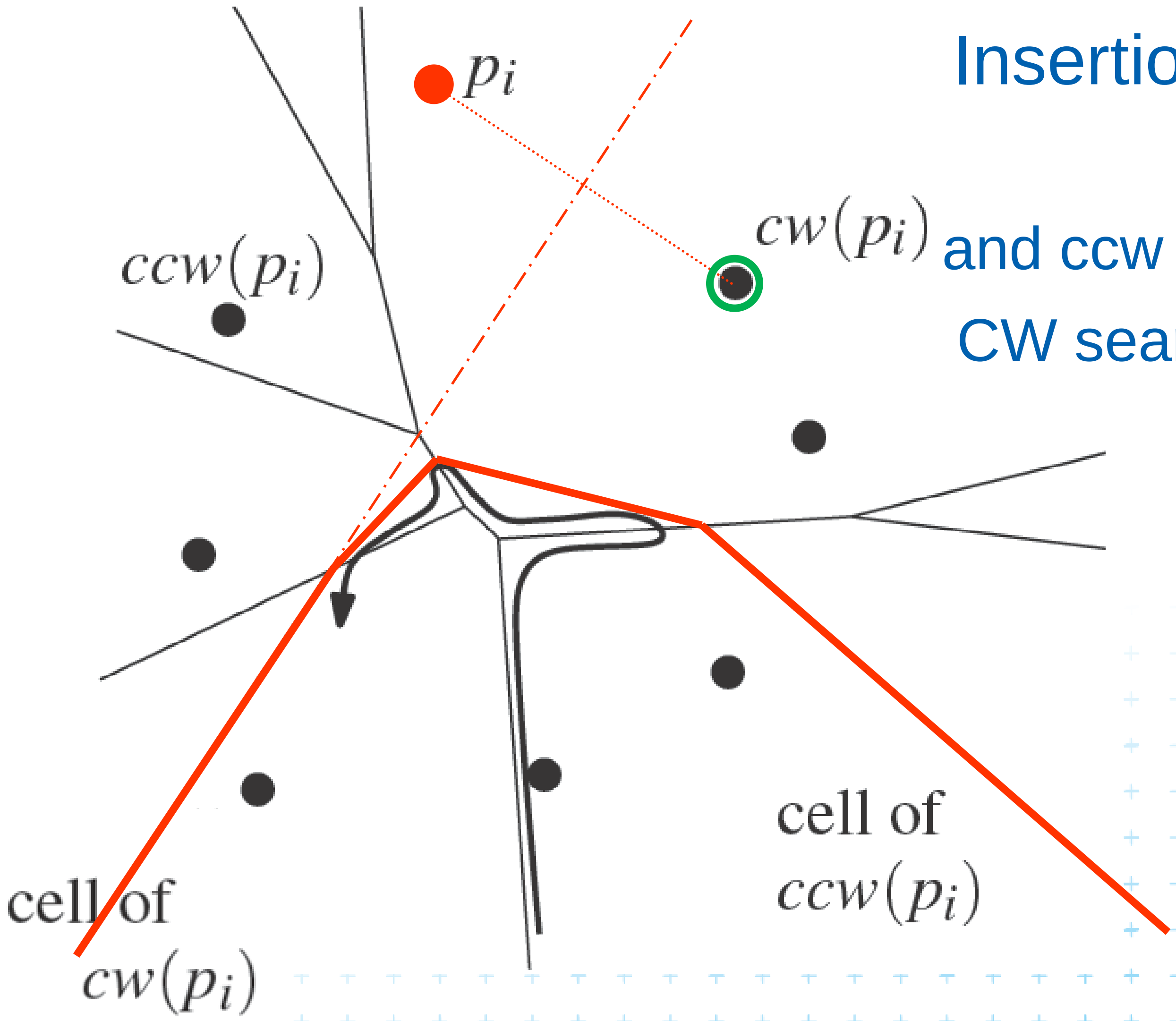

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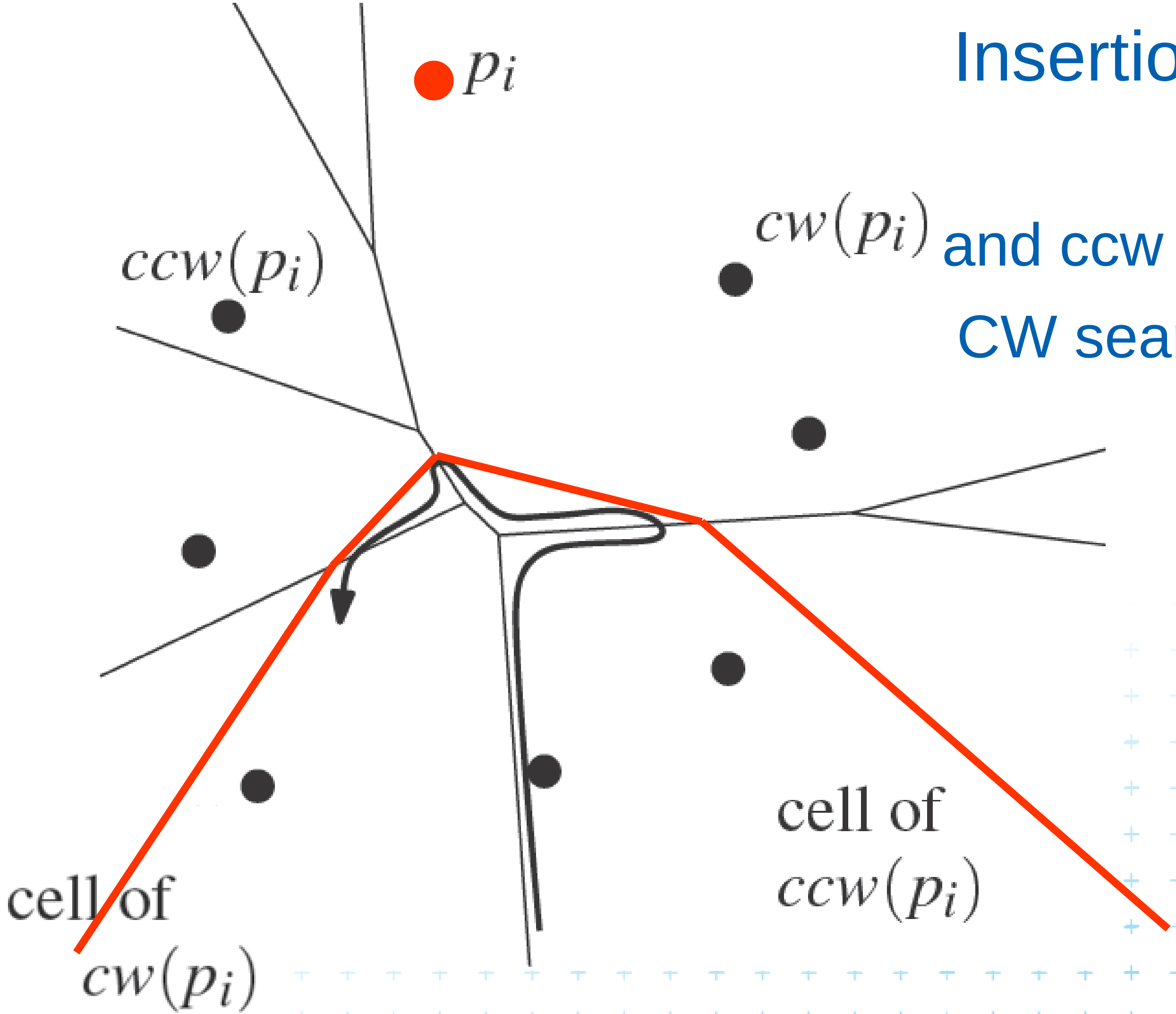
Farthest-point Voronoi d. construction



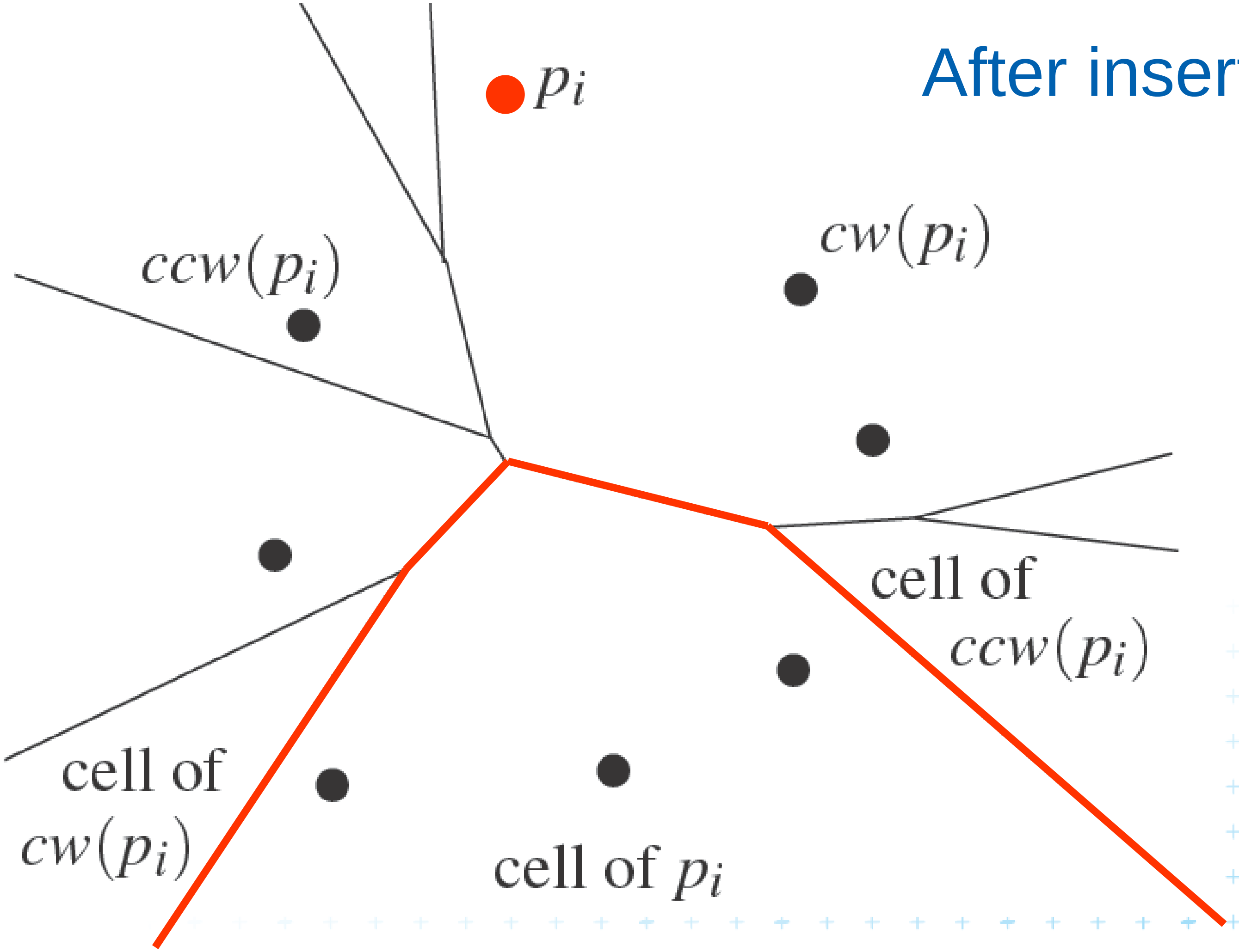
Insertion of site p_i

and ccw edge of its cell

CW search of intersection

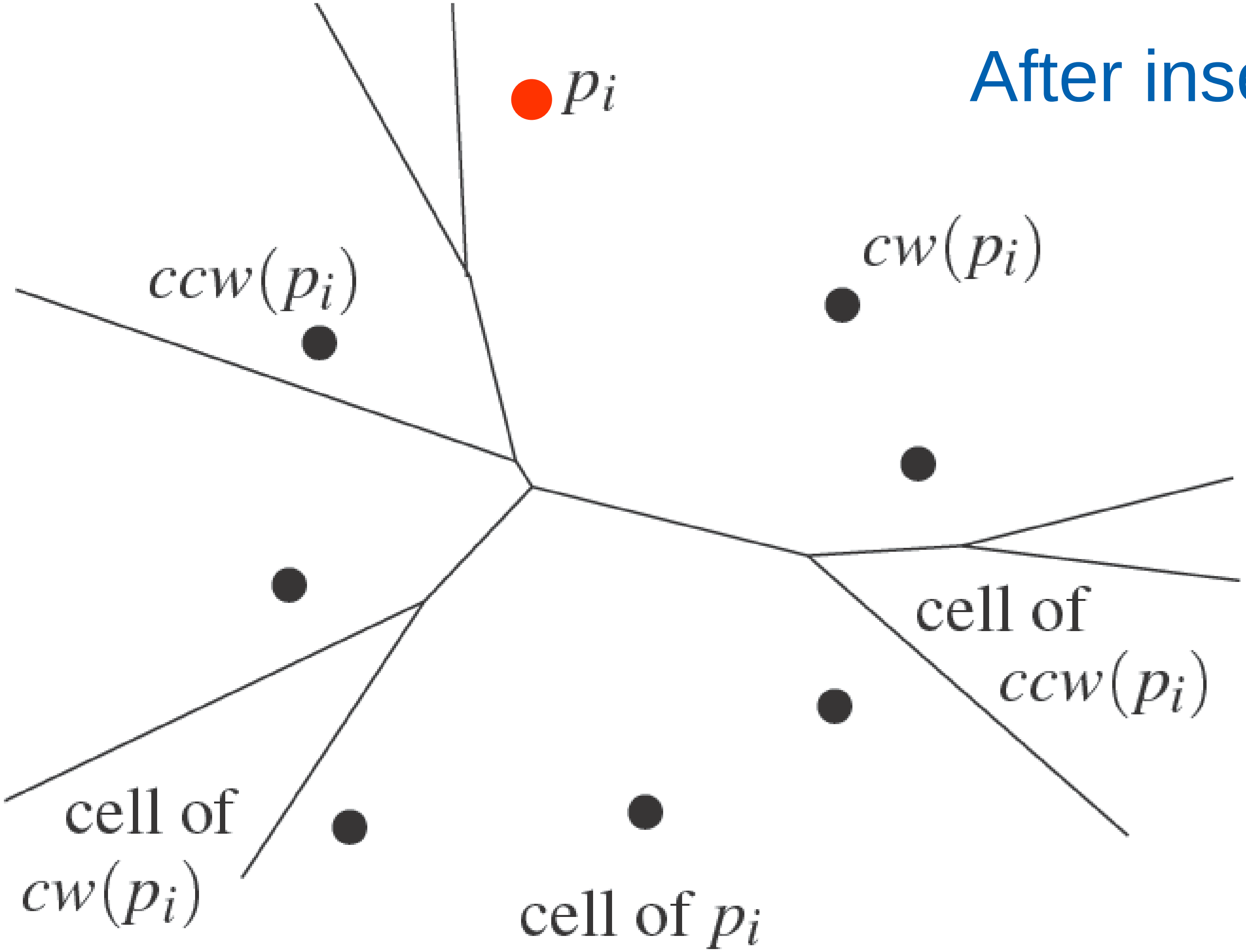
Farthest-point Voronoi d. construction

After insertion of site p_i



Farthest-point Voronoi d. construction

After insertion of site p_i



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