DCGI

DEPARTMENT OF COMPUTER GRAPHICS AND INTERACTION

VORONOI DIAGRAM

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Based on [Berg] and [Mount]

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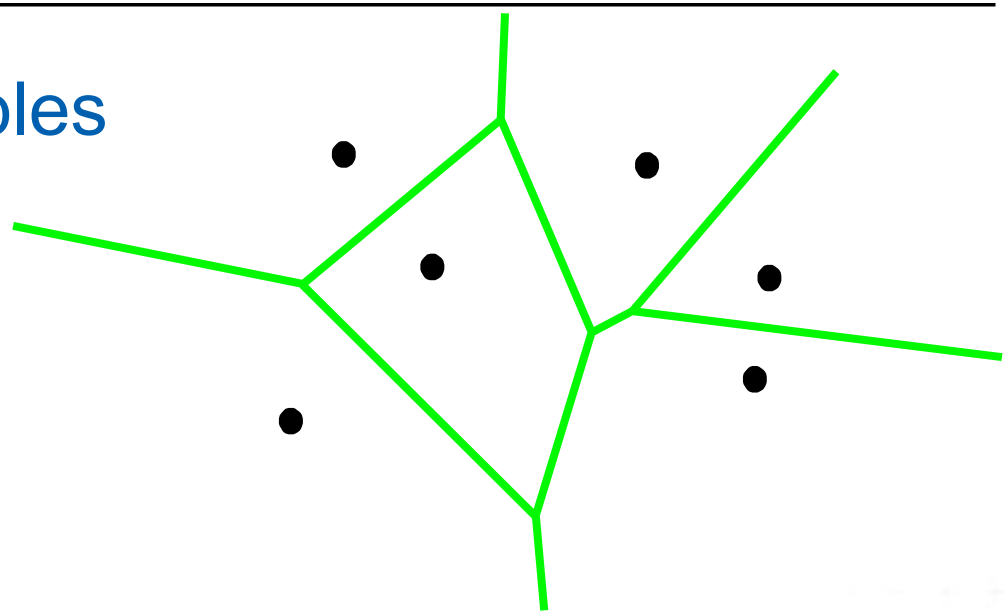
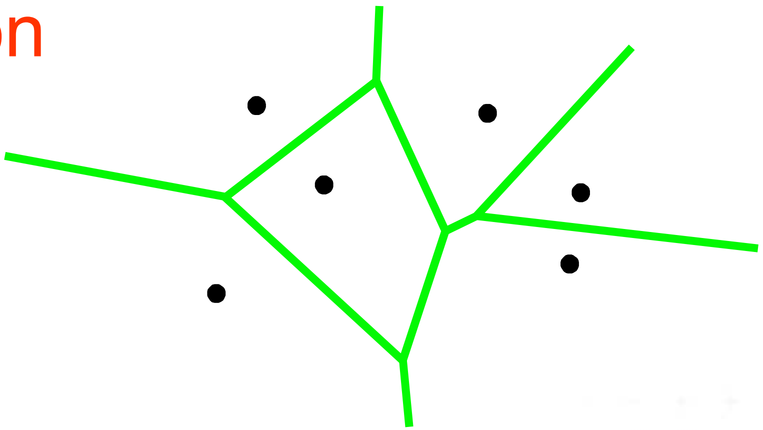
Talk overview

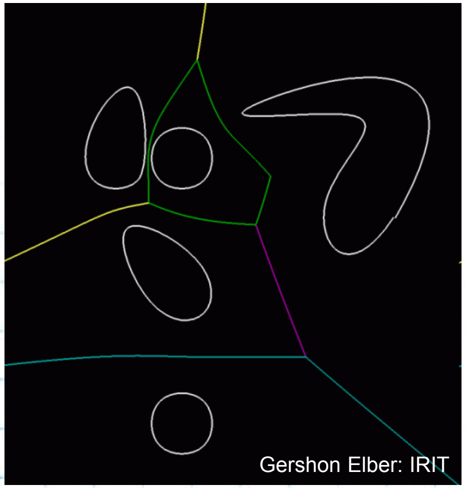
- Definition and examples
- Applications
- Algorithms in 2D
  - D&C O(n log n)
  - Sweep line O(n log n)



Voronoi diagram (VD)

- One of the most important structure in Comp. geom.
- Encodes proximity information
  What is close to what?
- Standard VD – this lecture
  - Set of points - nDim
  - Euclidean space & metric
- Generalizations
  - Set of line segments or curves
  - Different metrics
  - Higher order VD's (furthest point)

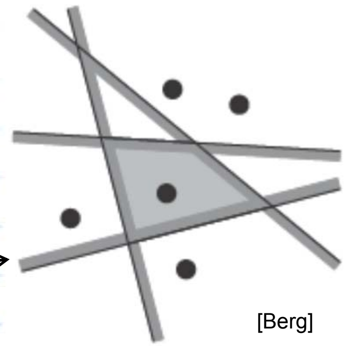




Voronoi cell (for points in plane)

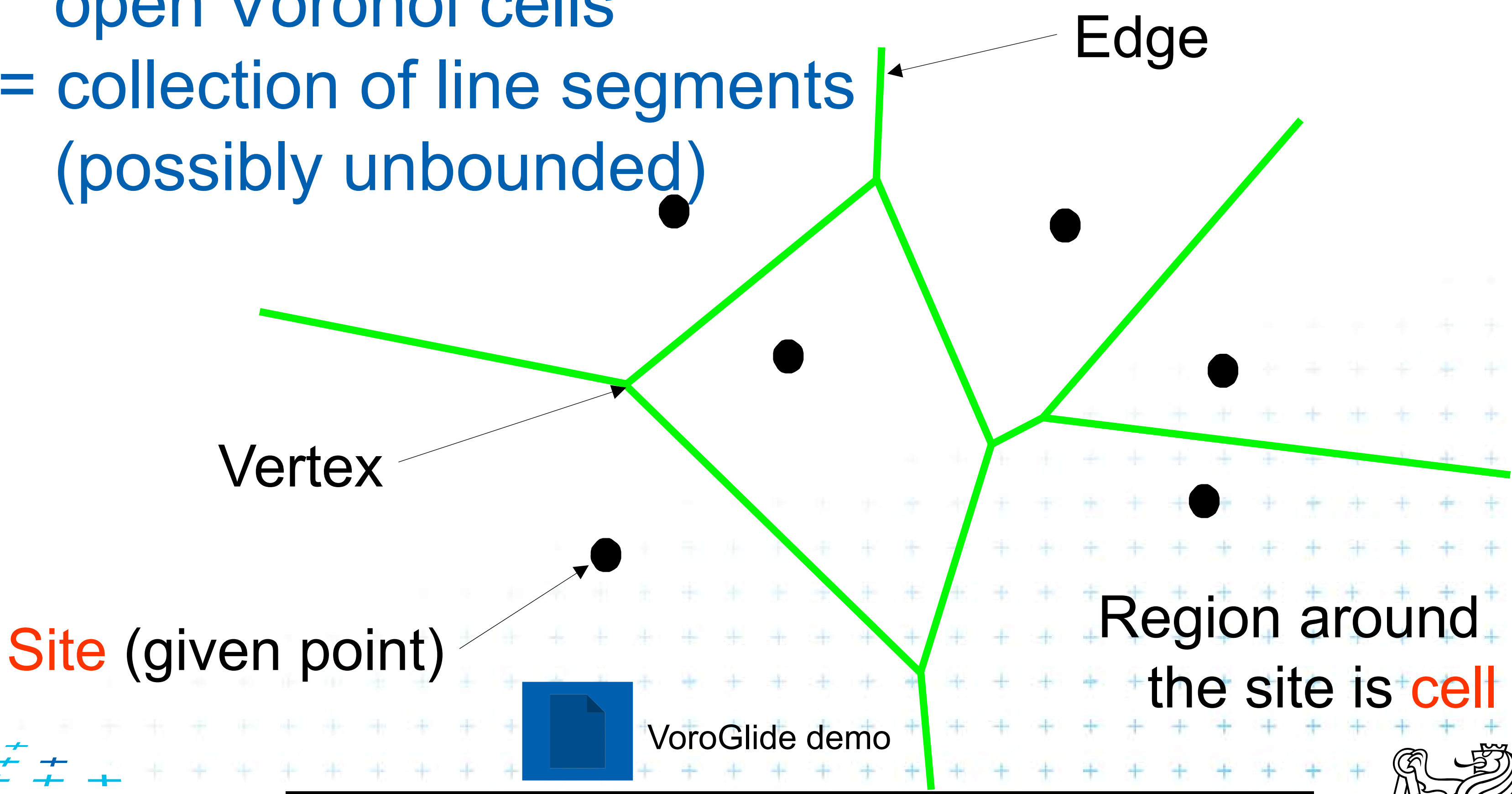
- Let $P = \{p_1, p_2, \ldots, p_n\}$ be a set of points (*sites*) in dDim space … 2D space (plane) here
- Voronoi cell $V(p_i)$ – is open!
  = set of points q closer to p_i than to any other site:
  $$V(p_i) = \{q, \|p_i q\| < \|p_j q\|, \forall j \neq i\}$$, where
  $\|pq\|$ is the Euclidean distance between p and q

  = intersection of open halfplanes
  $$V(p_i) = \bigcap_{j \neq i} h(p_i, p_j)$$
  $h(p_i, p_j)$ = open halfplane
  = set of pts strictly closer to p_i than to p_j


[Berg]

Voronoi diagram (in plane)

- Voronoi diagram Vor(*P*) of points *P*
  = what is left of the plane after removing all the open Voronoi cells
  = collection of line segments (possibly unbounded)



VoroGlide demo

Voronoi diagram examples

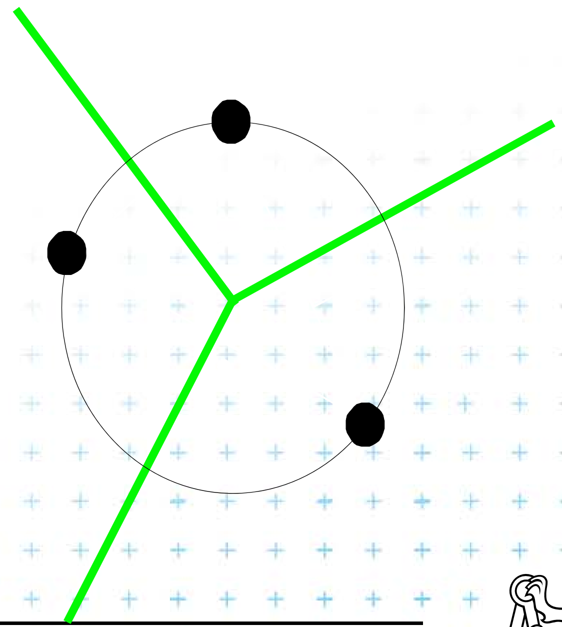
1 point 2 points 3 points

Cell

- The whole plain for 1 point
- Halfplane or strip for collinear points
- Convex (possibly unbounded) polygon

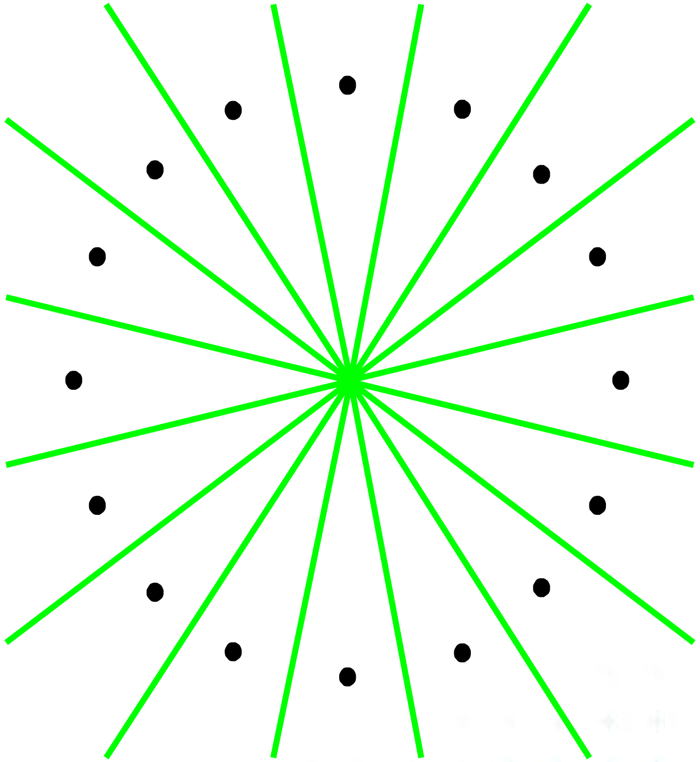
Edges of VD

- || lines for collinear points
- Halflines (for non-collinear CH points)
- Line segments (for bounded cells)

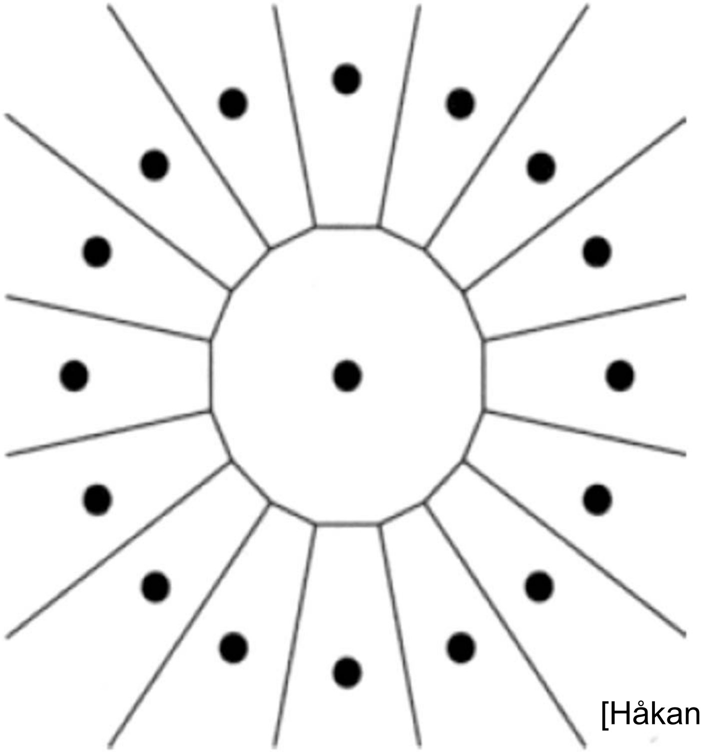


Voronoi diagram examples

16 points



17 points



[Håkan Jonsson]

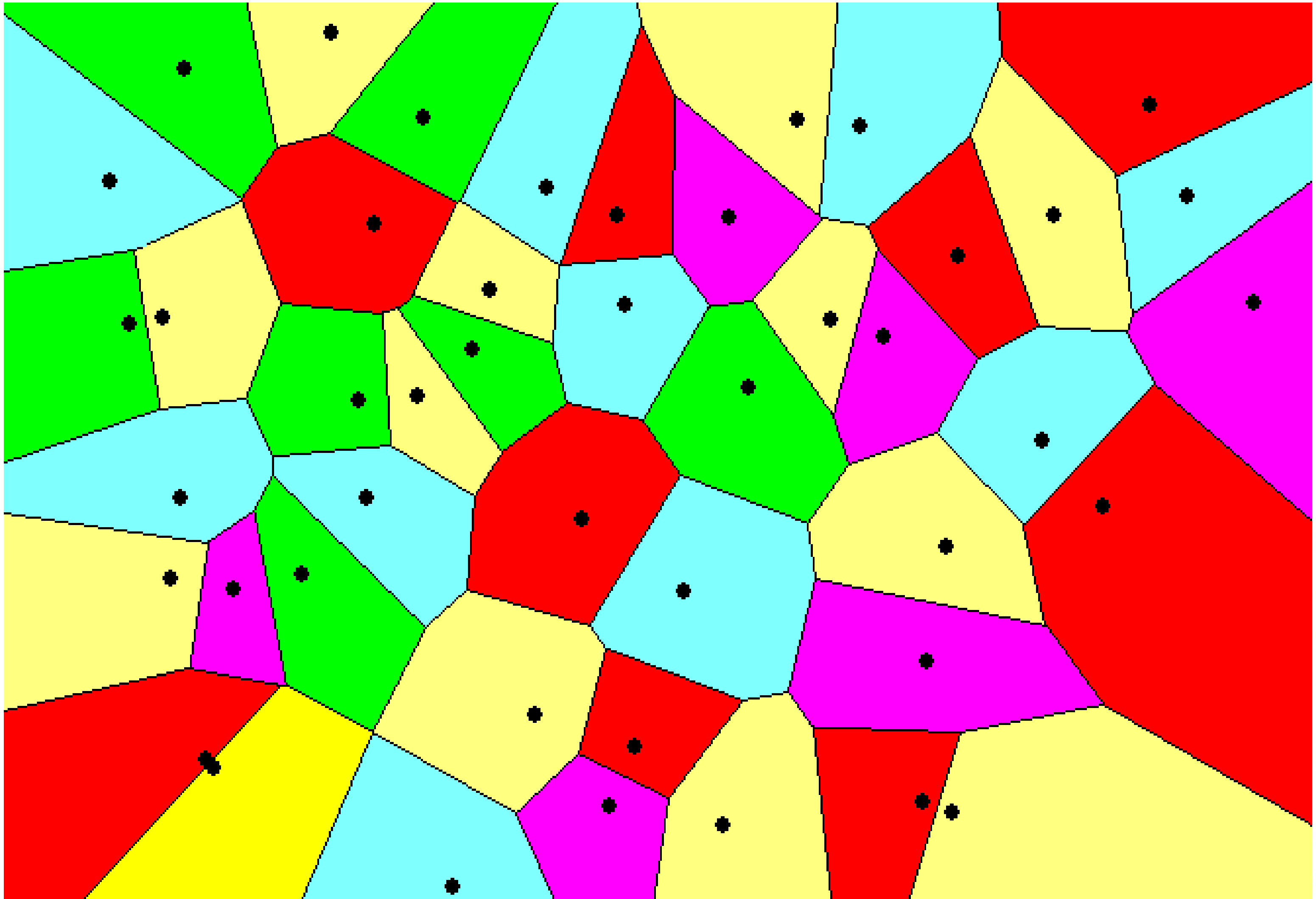
Vertex with O(n) incident edges
From total $|n_e| \le 3n - 6$

16 <= 42

Cell with O(n) vertices
From total $|n_v| \le 2n\text{-}5$

17 <= 29

Voronoi diagram examples

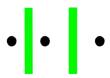


Voronoi diagram (in plane)

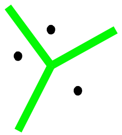
= planar graph

- Subdivides plane into n cells (n = num. of input sites |P|)
- Edge = locus of equidistant pairs of points (cells)
  = part of the bisector of these points
- Vertex = center of the circle defined by ≥ 3 points
  => vertices have degree ≥ 3
- Number of vertices $n_v \le 2n - 5$ => $O(n)$
- Number of edges $n_e \le 3n - 6$ => $O(n)$
  (only $O(n)$ from $O(n^2)$ intersections of bisectors)
- In higher dimensions complexity from $O(n)$ up to $O(n^{\lceil d/2 \rceil})$
- Unbounded cells belong to sites (points) on convex hull

Voronoi diagram O(n) complexity derivation



- For *n* collinear sites: $n_v = 0 \leq 2n - 5$, $n_e = (n-1) \leq 3n - 6$ both hold



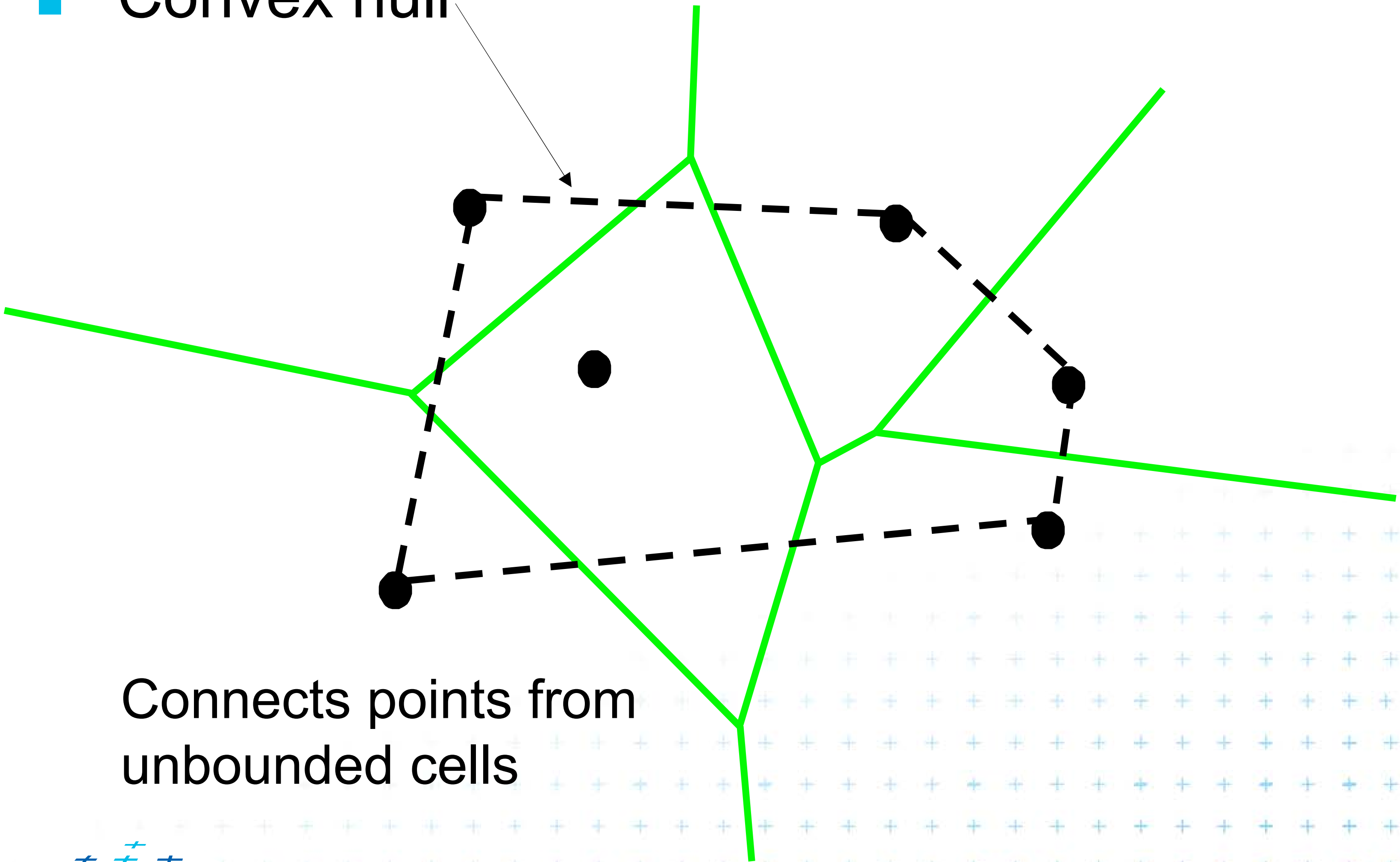
- For *n* non-collinear sites:
  - Add extra VD vertex *v* in infinity $m_v = n_n + 1$
  - Apply Euler's formula: $m_v - m_e + m_f = 2$
  - Obtain $(n_v + 1) - n_e + n = 2$ $\left\{ \begin{array}{l} n_e = n_v + n - 1 \\ n_v = n_e - n + 1 \end{array} \right.$
  - Every VD edge has 2 vertices Sum of vertex degrees $= 2n_e$
  - Every VD vertex has degree ≥ 3 Sum of vertex degrees $= 3m_v = 3(n_v + 1)$
  - Together $2n_e \geq 3(n_v + 1)$

$$2n_e \geq 3(n_v + 1)$$
$$2(n_v + n - 1) \geq 3(n_v + 1)$$
$$2n_v + 2n - 2 \geq 3n_v + 3$$
$$n_v \leq 2n - 5$$

$$2n_e \geq 3(n_v + 1)$$
$$2n_e \geq 3(n_e - n + 1 + 1)$$
$$2n_e \geq 3n_e - 3n + 6$$
$$n_e \leq 3n - 6$$

Voronoi diagram and convex hull

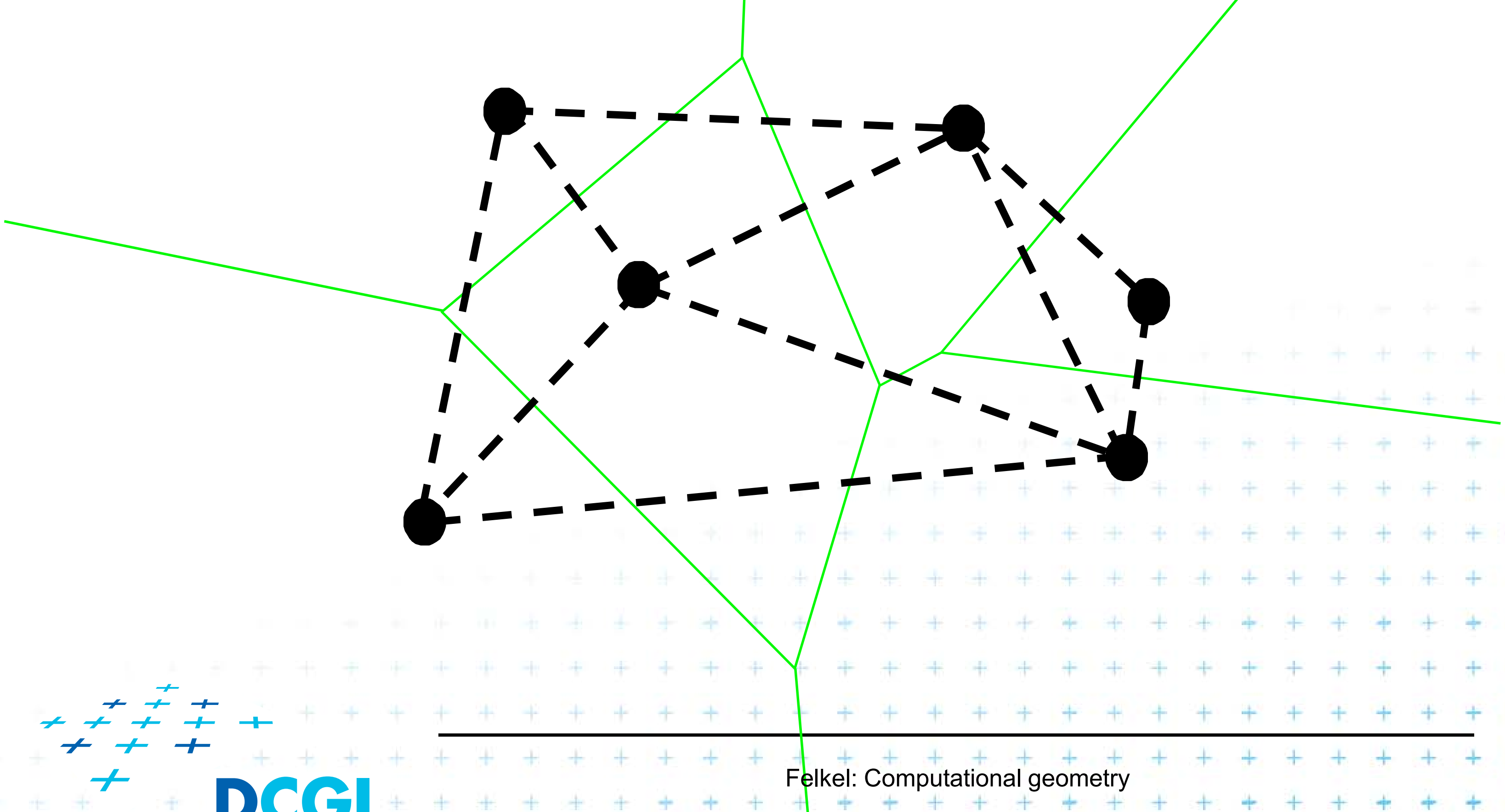
- Convex hull



Connects points from unbounded cells

Delaunay triangulation

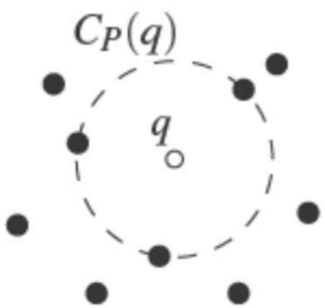
- point set triangulation   (straight line dual to VD)
- maximize the minimal angle (tends to equiangularity)



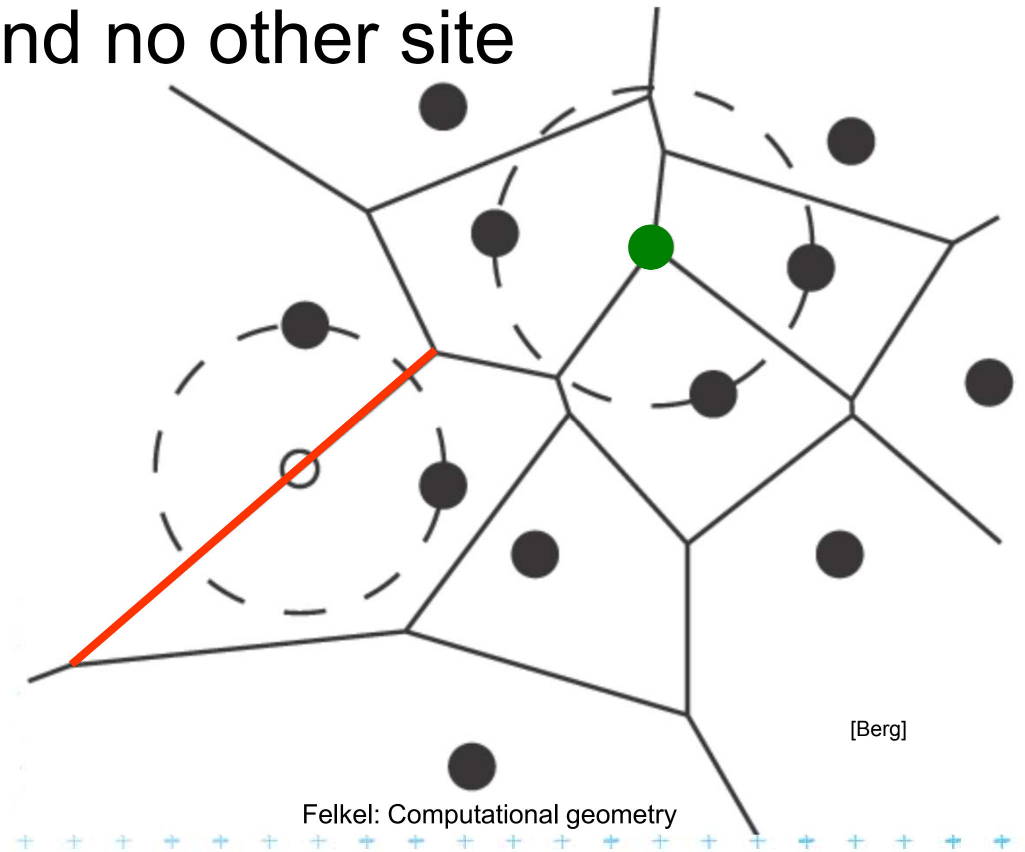
Edges, vertices and largest empty circles

Largest empty circle $C_P(q)$ with center in

1. In VD vertex q: has 3 or more sites on its boundary
2. On VD edge: contains exactly 2 sites on its boundary and no other site



[Berg]

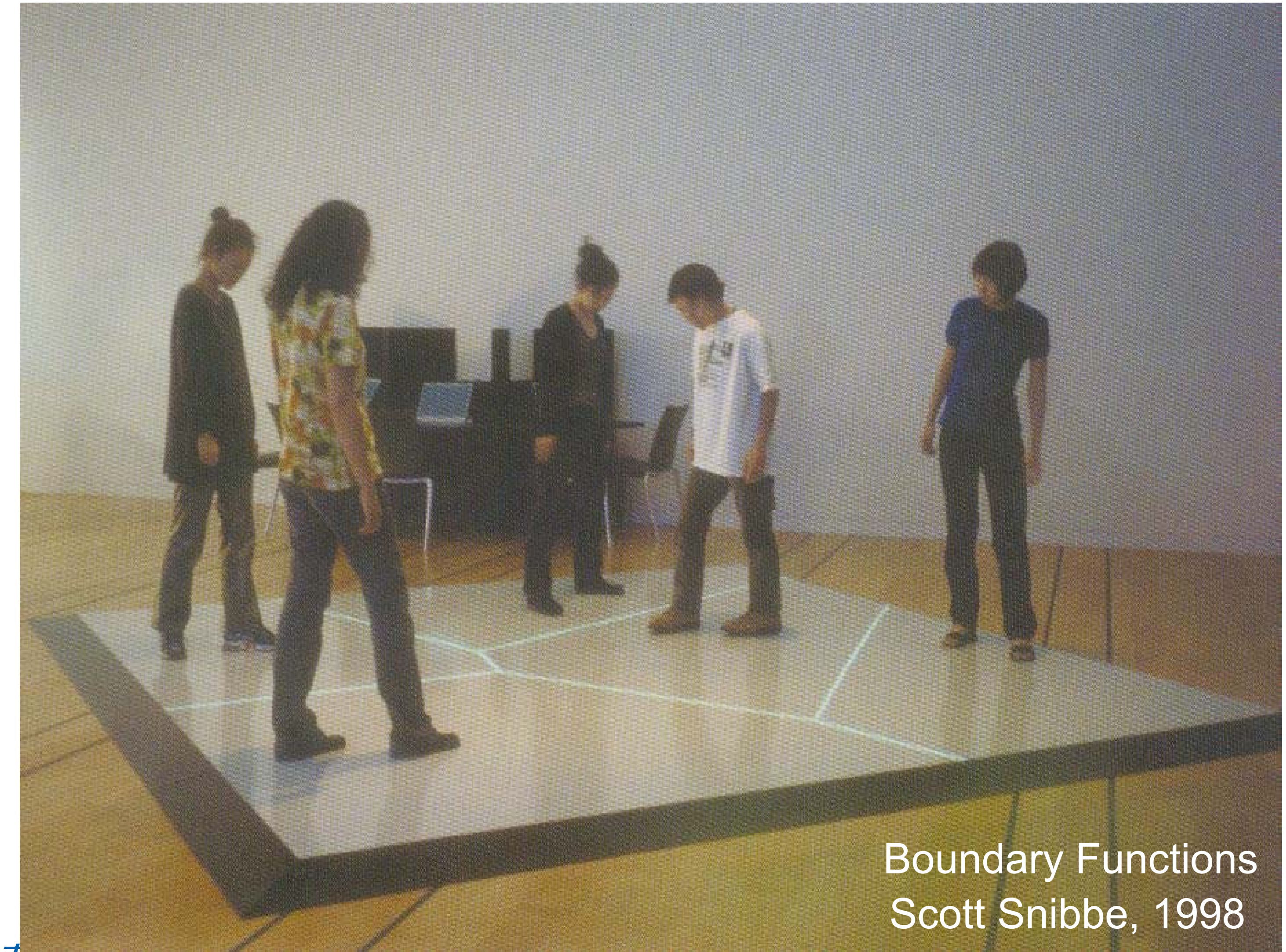


[Berg]

Some applications

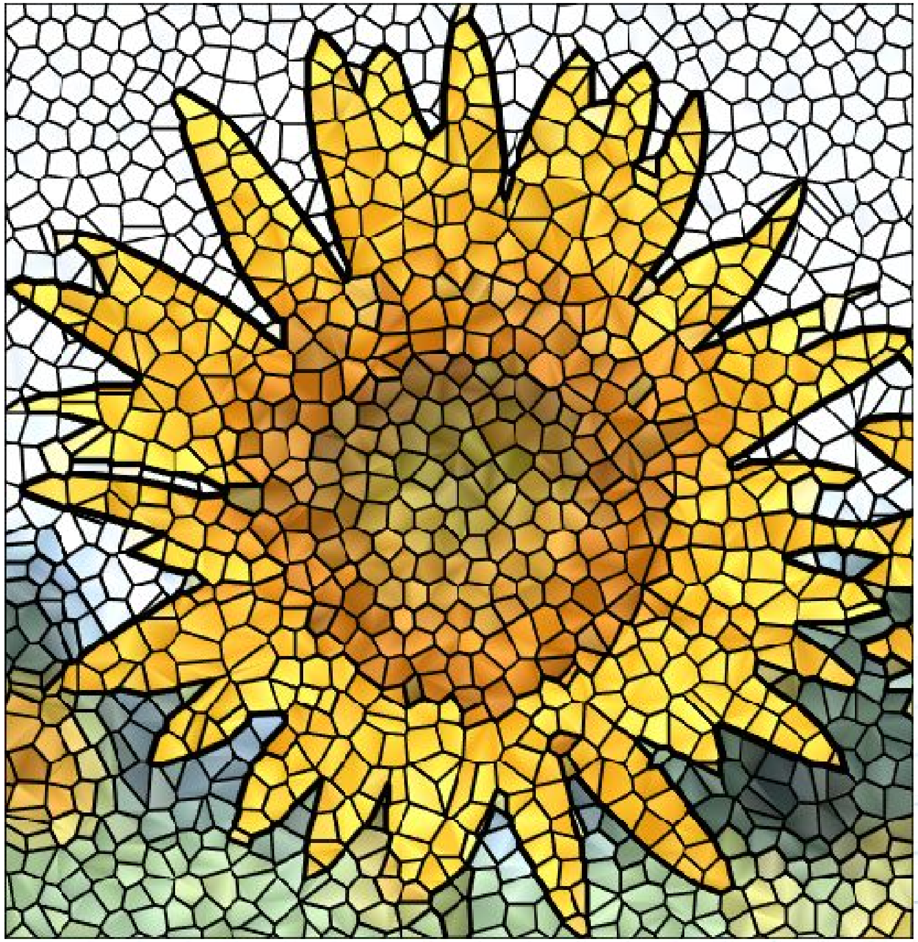
- Nearest neighbor queries in Vor(P) of points P
  - Point q $\in$ P ... search sites across the edges around the cell q
  - Point q $\notin$ P ... point location queries – see Lecture 2 (the cell where point *q* falls)
- Facility location (shop or power plant)
  - Largest empty circle (better in Manhattan metric VD)
- Neighbors and Interpolation
  - Interpolate with the nearest neighbor,
    in 3D: surface reconstruction from points
- Art

Voronoi Art



Boundary Functions
Scott Snibbe, 1998

Voronoi Art



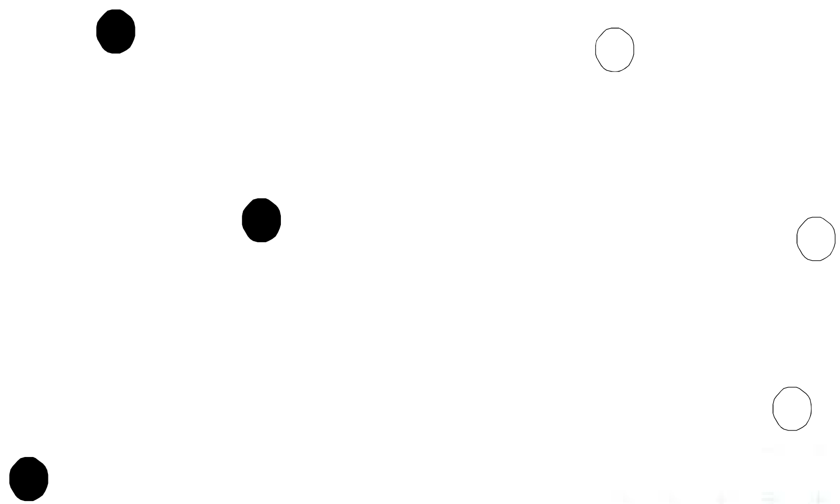
Courtesy [Gold]

Algorithms in 2D

- D&C O(n log n)
- Fortune's Sweep line O(n log n)

Voronoi diagram (VD)

Divide and Conquer method

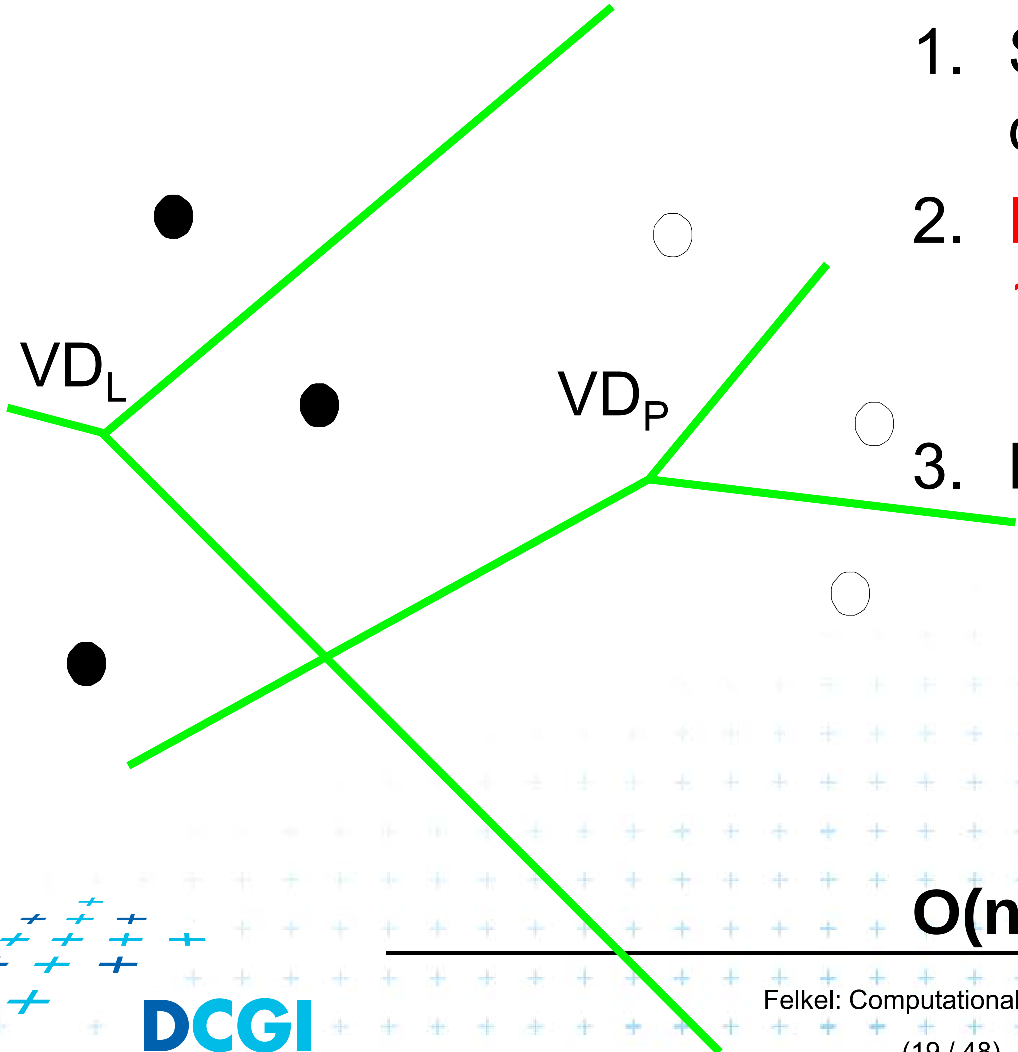


1. Split points based on x-coord into L and R
2. Recursion on L and R
   1-3 points => return
   >3 points => recursion
3. Merge VD_L and VD_R
   - monotone chain
   - trim intersected edges
   - Add new edges from the chain

O(n log n)

Voronoi diagram (VD)

Divide and Conquer method



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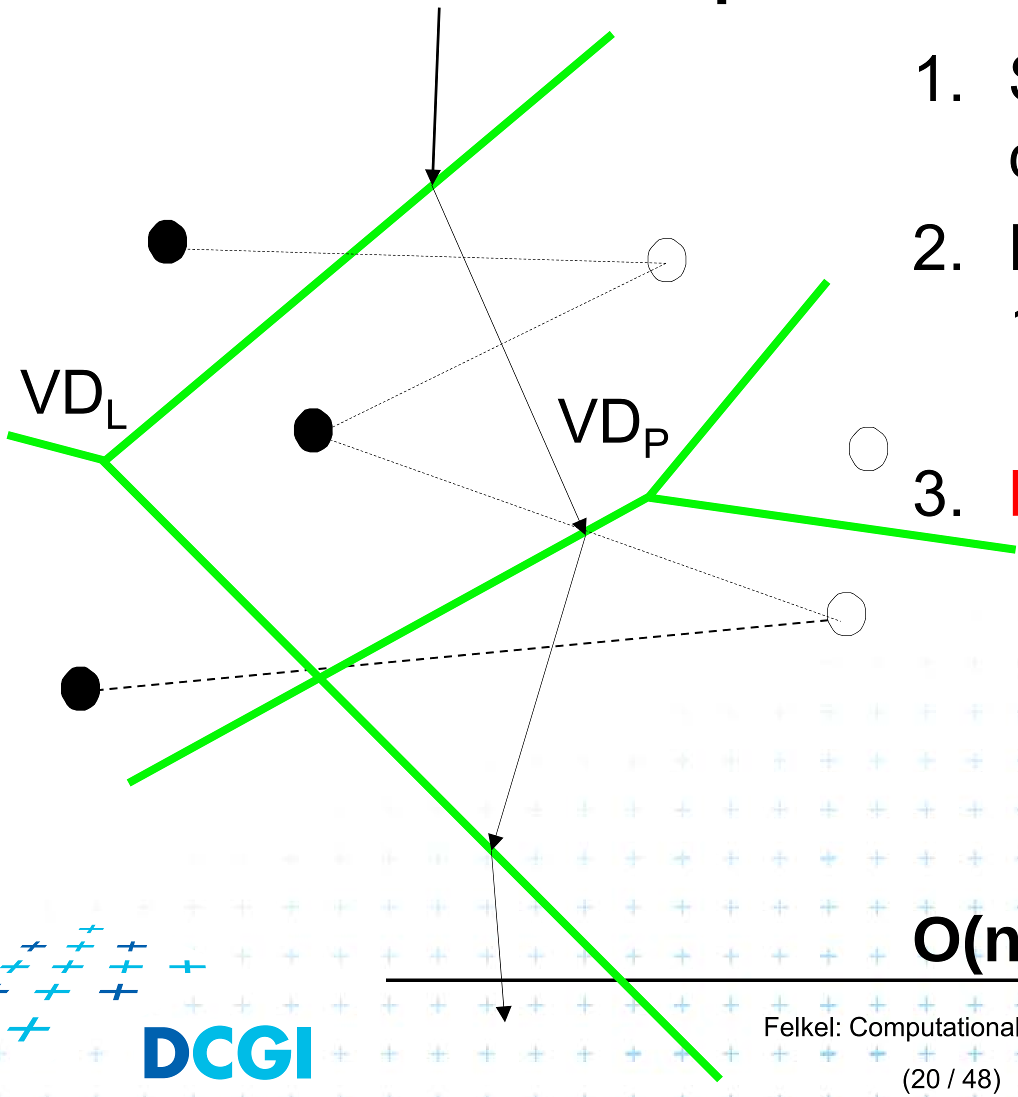
O(n log n)

Voronoi diagram (VD)

Divide and Conquer method



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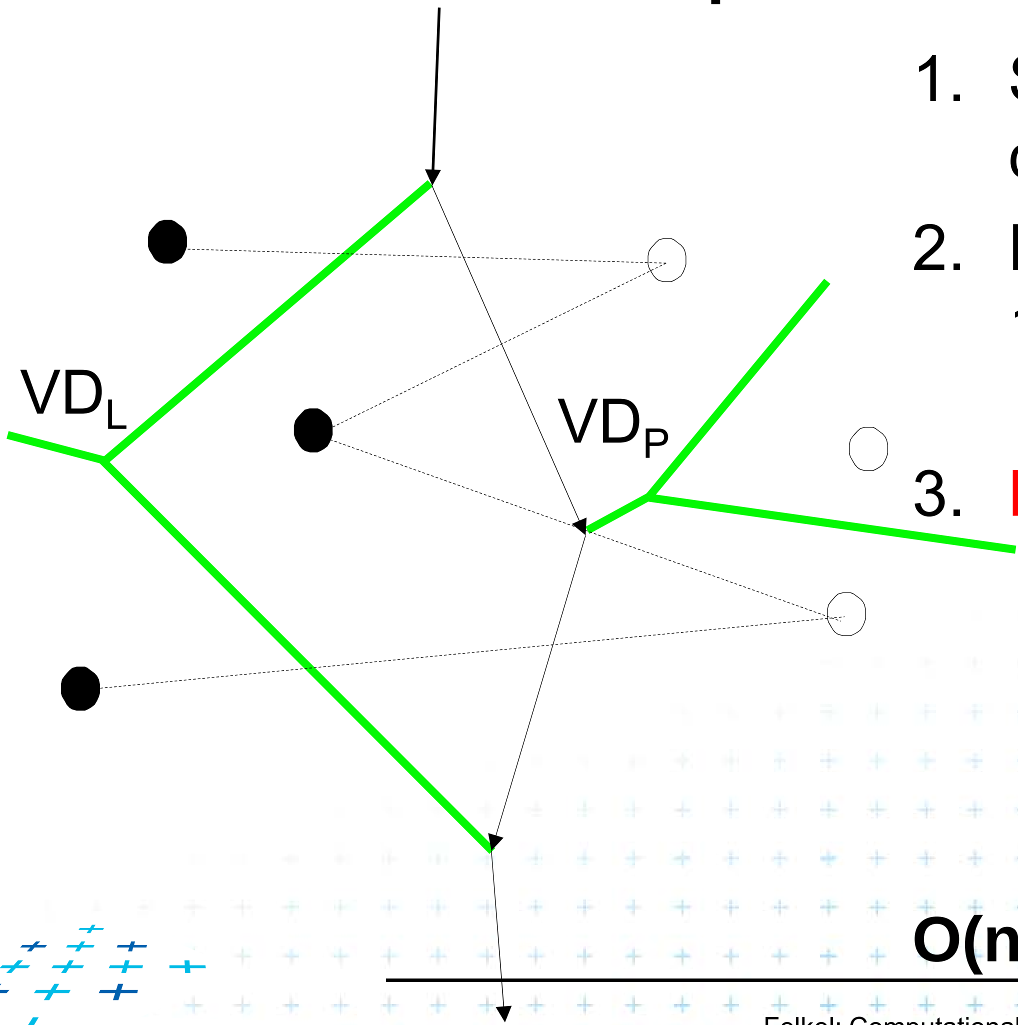
O(n log n)

Voronoi diagram (VD)

Divide and Conquer method



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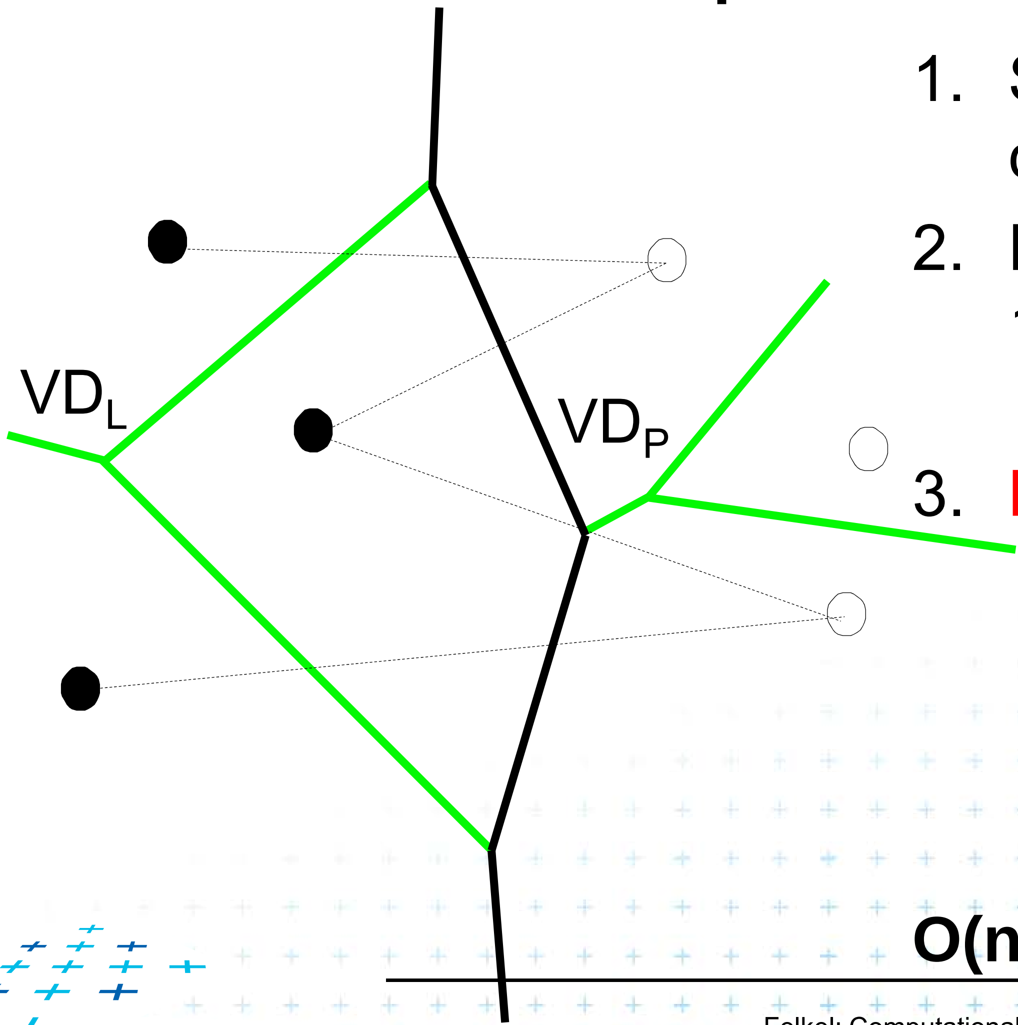
O(n log n)

Voronoi diagram (VD)

Divide and Conquer method



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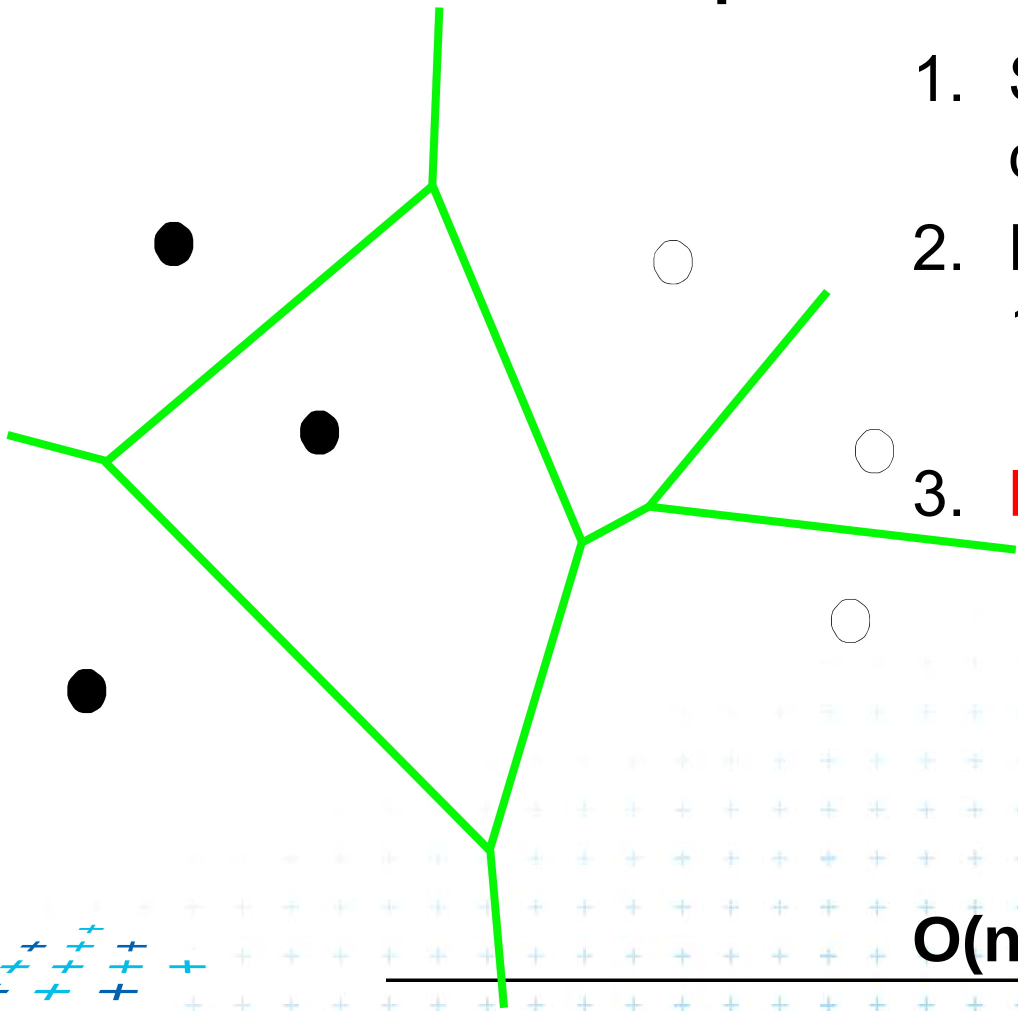
O(n log n)

Voronoi diagram (VD)

Divide and Conquer method



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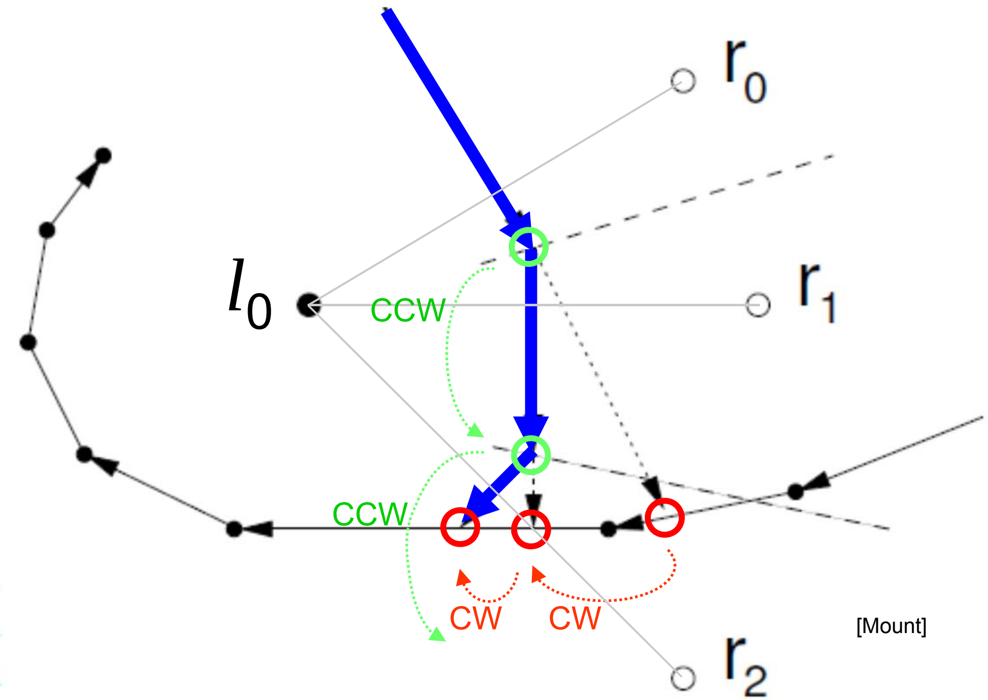
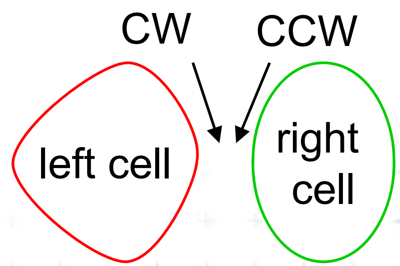
O(n log n)

Monotone chain search in O(n)

- Avoid repeated rescanning of cell edges
- Start in the last tested edge of the cell (each edge tested ~once)
- In the left cell l_i continue CW, in the right cell r_i go CCW
- Image shows CW search on cell l_0 and CCW on cells r_i:

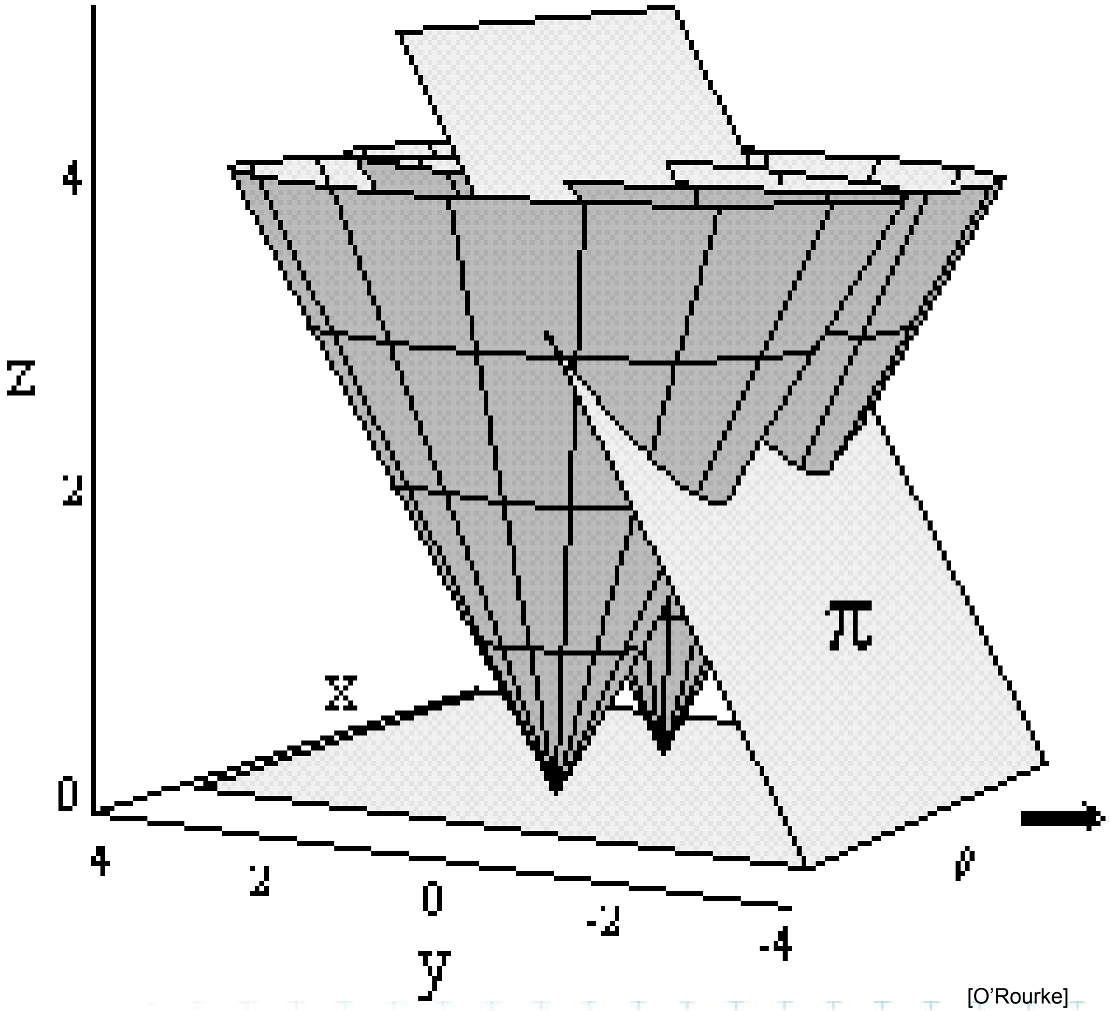




Divide and Conquer method complexity

- Initial sort $O(n \log n)$
- $O(\log n)$ recursion levels
  - O(n) each merge (chain search, trim, add edges to VD)
- Altogether $O(n \log n)$

Fortune's sweep line algorithm – idea in 3D



[O'Rourke]

Cones in sites

Scanning plane π

Both slanted 45°

Projection of the intersection to xy:

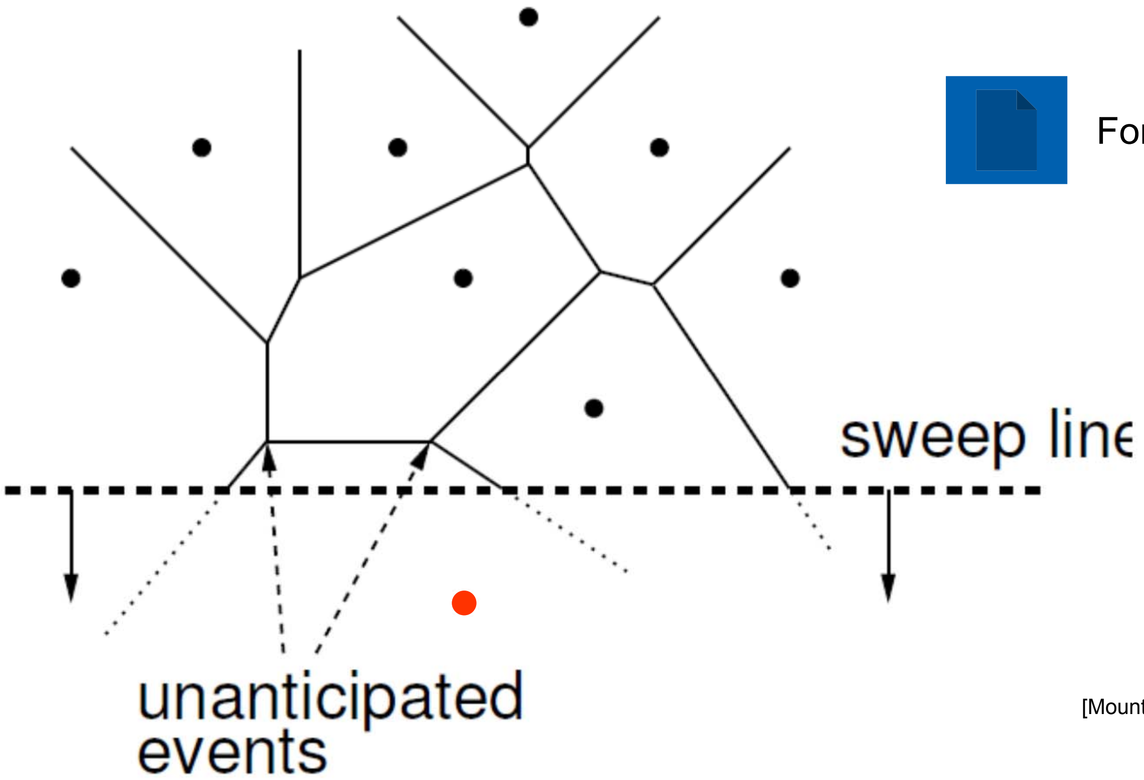
- Cone x plane => parabolic arcs
- Cone x cone => edges of VD

Fortune's sweep line algorithm

DONE

TODO

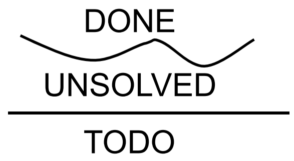
- Differs from "typical" sweep line algorithm
- Unprocessed sites ahead from sweep line may generate Voronoi vertex behind the sweep line



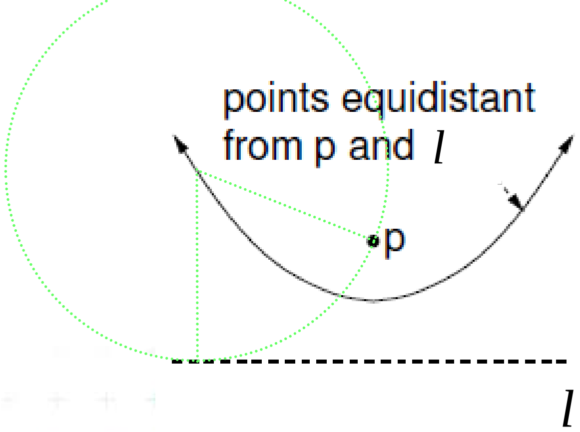
Fortune's applet

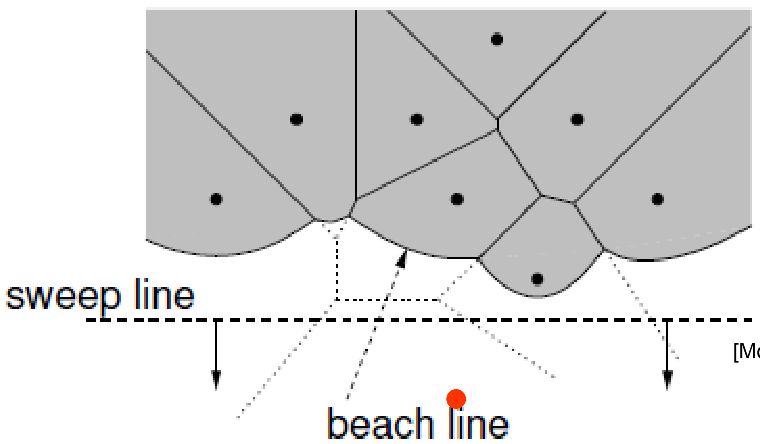
[Mount]

Fortune’s sweep line algorithm idea



- Subdivide the halfplane above the sweep line *l* into 2 regions
  1. Points closer to some site above than to sweep line *l* (solved part)
  2. Points closer to sweep line *l* than any point above (unsolved part – can be changed by sites below *l*)
- Border between these 2 regions is a beach line





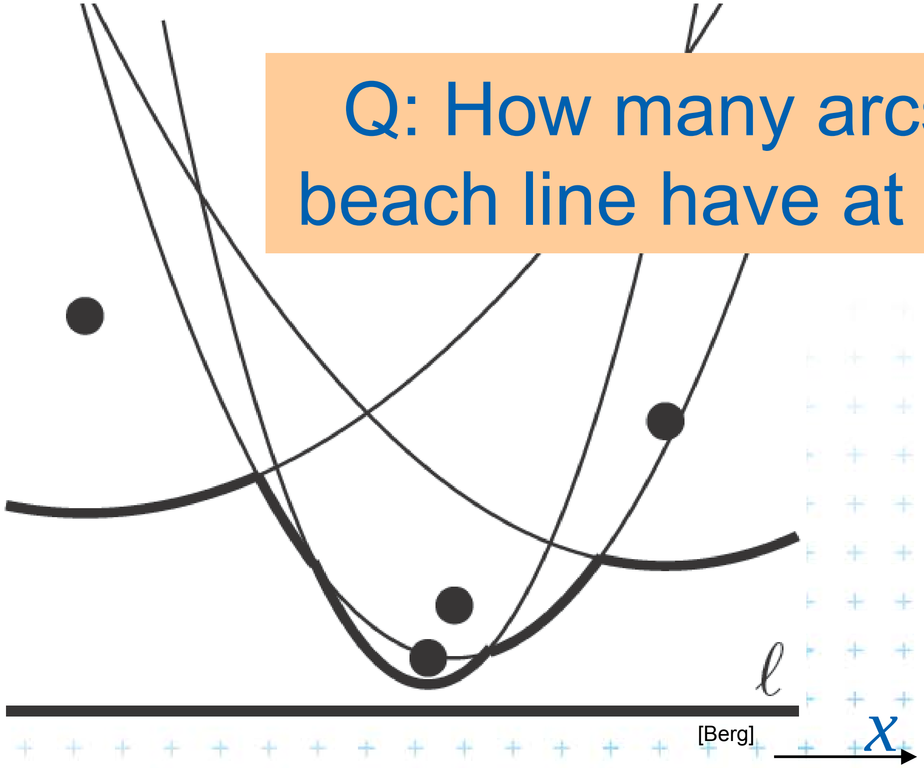
[Mount]

Sweep line and beach line

- Straight sweep line l
  - Separates processed and unprocessed sites (points)
- Beach line (Looks like waves rolling up on a beach)
  - Separates *solved* and *unsolved* regions above sweep line (separates sites above l that can be changed from sites that cannot be changed by sites below l)
  - *x*-monotonic curve made of parabolic arcs
  - Follows the sweep line
  - Prevents us from missing unanticipated events until the sweep line encounters the corresponding site

Beach line

- Every site p_i above *l* defines a complete parabola
- Beach line is the function, that passes through the lowest points of all the parabolas (lower envelope)

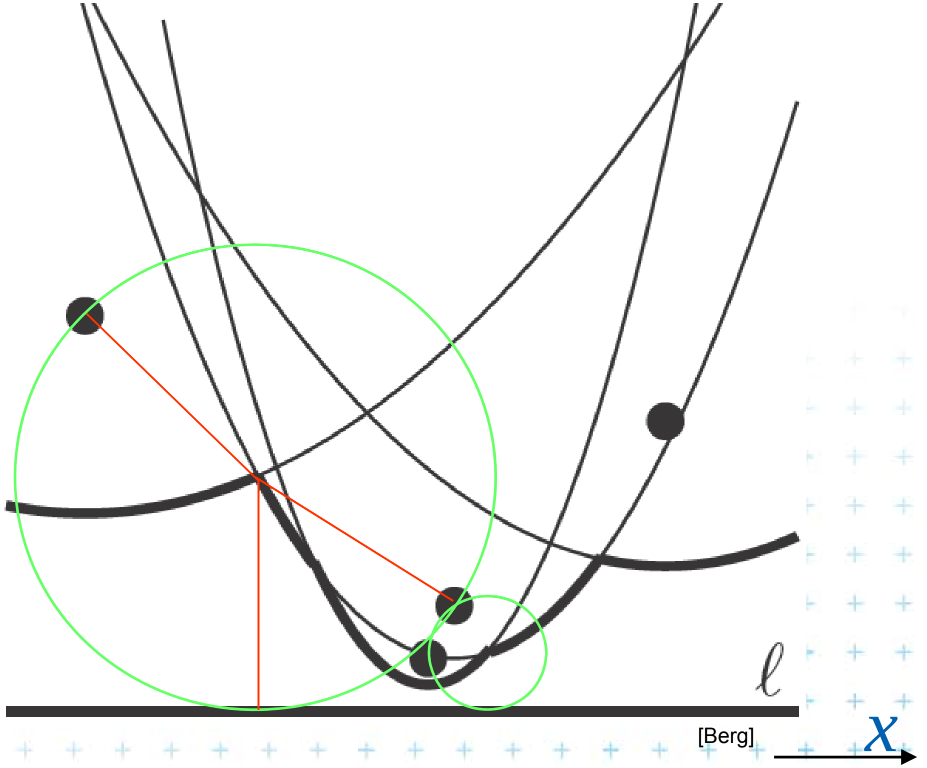


Q: How many arcs may the beach line have at maximum?

Break point (*bod zlomu*)

= Intersection of two arcs on the beach line

- Equidistant to 2 sites and sweep line l
- Lies on Voronoi edge of the final diagram



Notes

Beach line is x-monotone

= every vertical line intersects it in exactly ONE point

Along the beach line

- Parabolic arcs are ordered
- Breakpoints are ordered

Breakpoints

- trace the Voronoi edges
- compute their position on the fly from neighboring arcs

Events

What event types exist?

Events

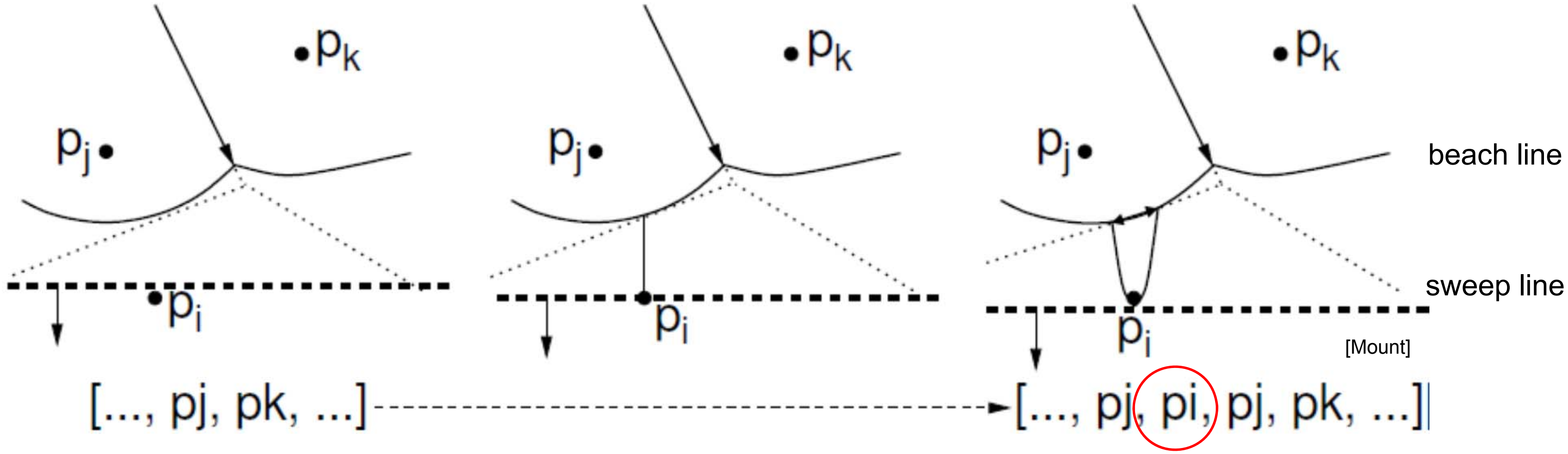
Colors:

- *Beach line*
- *Voronoi diagram -VD*

There are two types of events:

- Site events (SE)
  - When the sweep line passes over a new site p_i,
    - *new arc* is added to the beach line
    - *new edge fragment* added to the VD.
  - All SEs known from the beginning (sites sorted by *y*)
- Voronoi vertex event ([Berg] calls a circle event)
  - When the parabolic arc shrinks to zero and disappears, *new Voronoi vertex* is created.
  - Created dynamically by the algorithm for triples or more neighbors on the beach line (triples changed by both types of events)

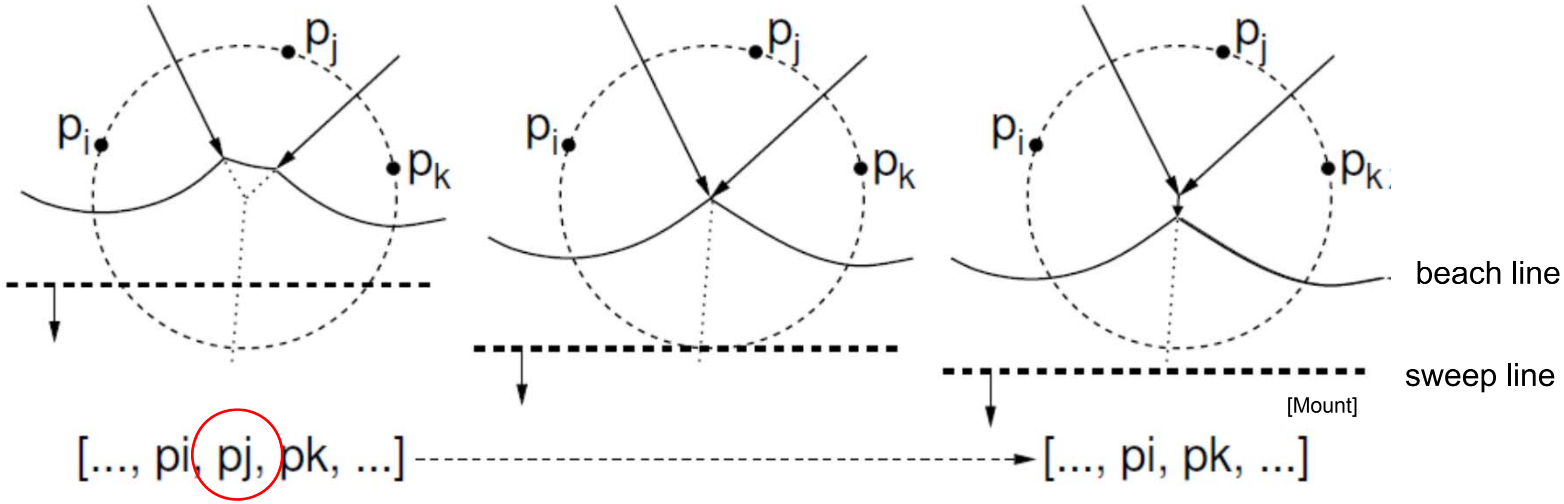
Site event



Generated when the sweep line passes over a site p_i

- New parabolic arc created,
  it starts as a vertical ray from p_i to the beach line
- As the sweep line sweeps on, the arc grows wider
- The entry $\langle \ldots, p_j, \ldots \rangle$ on the sweep line status is replaced by the triple $\langle \ldots, p_j, p_i, p_j, \ldots \rangle$
- Dangling future VD edge created on the bisector (p_i, p_j)

Voronoi vertex event (circle event)



Generated when *l* passes the lowest point of a circle

- Sites p_i , p_j , p_k appear consecutively on the beach line
- Circumcircle lies partially below the sweep line (Voronoi vertex has not yet been generated)
- This circumcircle contains no point below the sweep line (no future point will block the creation of the vertex)
- Vertex & bisector (p_i, p_k) created, (p_i, p_j) & (p_j, p_k) finished
- One parabolic arc removed from the beach line

Data structures

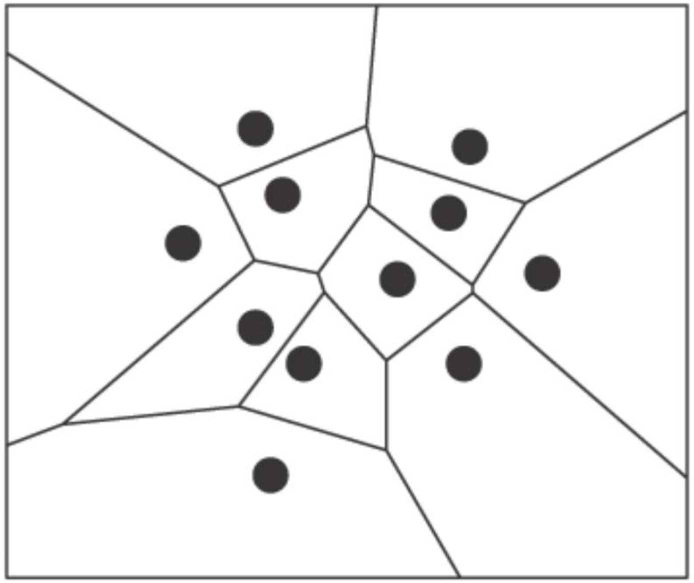
1. (Partial) Voronoi diagram
2. Beach line data structure T
3. Event queue Q

1. VD edges arise during: site event circle event?
2. VD vertices arise during: site event circle event?
3. Site events known from the beginning: yes no?
4. Circle events known from the beginning: yes no?

1. (Partial) Voronoi diagram data structure

Any PSLG data structure, e.g. DCEL (planar stright line graph)

- Stores the VD during the construction
- Contain unbounded edges
  - dangling edges during the construction (managed by the beach line DS) and
  - edges of unbounded cells at the end
    => create a bounding box



[Berg]

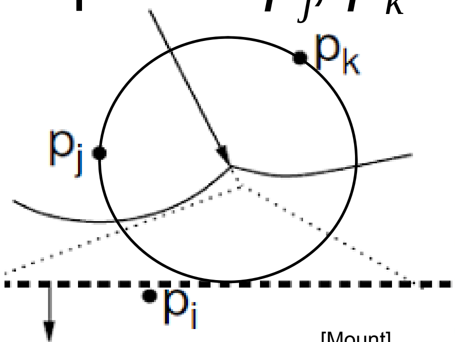
2. Beach line tree data structure T – status

- Used to locate the arc directly above a new site
- E.g. Binary tree *T*

  p_i – possibly multiple times

  – Leaves - ordered arcs along the beach line (x-monotone)
    - *T* stores only the sites p_i in leaves, *T* does not store the parabolas

  – Inner tree nodes - breakpoints as ordered pairs <p_j, p_k>
    - p_j, p_k are neighboring sites
    - Breakpoint position computed on the fly from p_j, p_k and y-coord of the sweep line

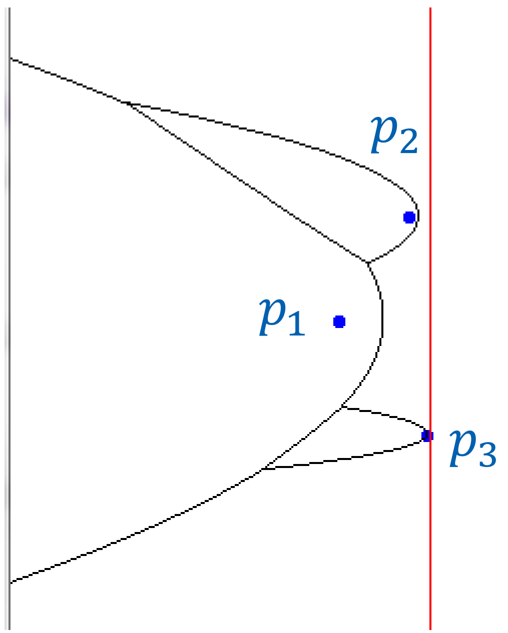


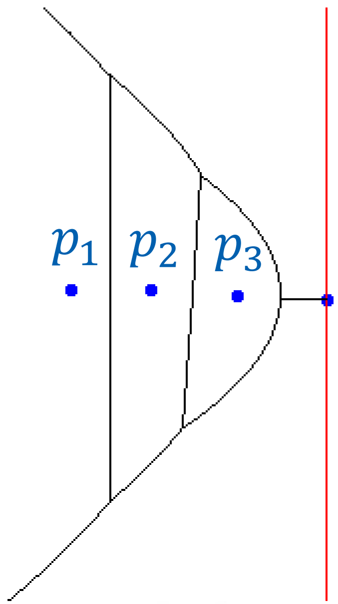
[Mount]

  – Pointers to other two DS
    - In leaves – pointer to event queue, point to node when arc disappears via Voronoi vertex event – if it exists
    - In inner nodes - pointer to (dangling) half-edge in DCEL of VD, that is being traced out by the break point

Max 2n -1 arcs on the beach line

New site splits just one arc



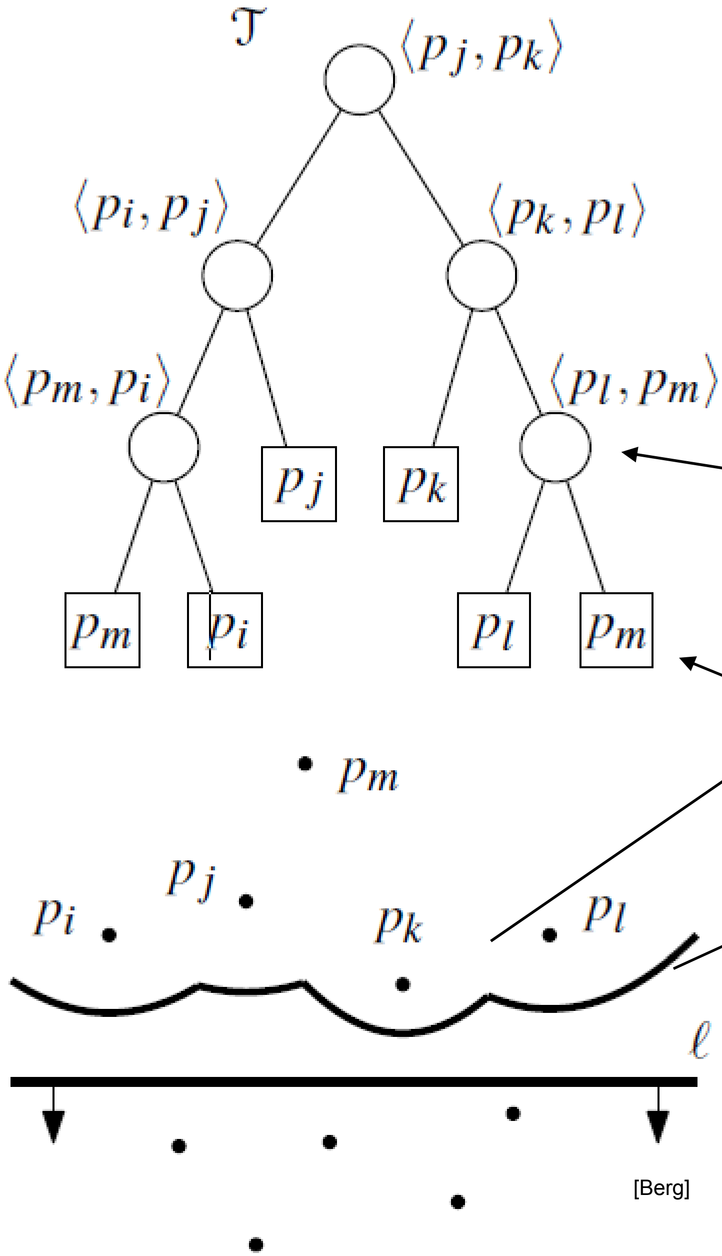


p_1	+1	p_1	+1
$p_1 p_2 p_1$	+2	$p_1 p_2 p_1$	+2
$p_1 p_3 p_1 p_2 p_1$	+2	$p_1 p_2 p_3 p_2 p_1$	+2

→ Leaves in T

2. Beach line tree T

x-coord computed on the fly for a given position of the beach line l



Break points
= inner nodes in T

Arcs = Leaves in T

3. Event queue Q

- Priority queue, ordered by y-coordinate
- For site event
  - stores the site itself
  - known from the beginning
- For Voronoi vertex event (circle event)
  - stores the lowest point of the circle
  - stores also pointer to the leaf in tree T (represents the parabolic arc that will disappear)
  - created by both events, when triples of points become neighbors (possible max three triples for a site)
  - p_i, p_j, p_k, p_l, p_m insert of p_k can create up to 3 triples and delete up to 2 triples (p_i, p_j, p_l) and (p_j, p_l, p_m)

Fortune's algorithm

FortuneVoronoi(*P*)

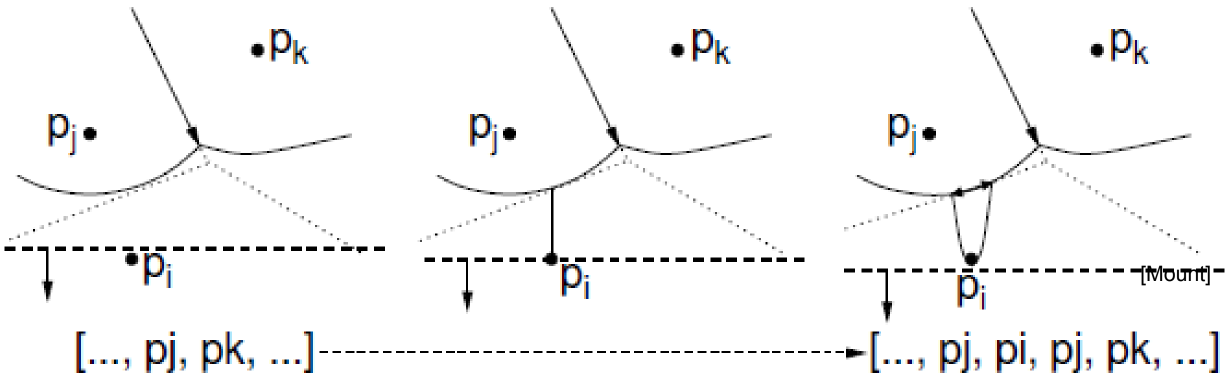
Input: A set of point sites $P = \{p_1, p_2, \ldots, p_n\}$ in the plane

Output: Voronoi diagram Vor(*P*) inside a bounding box in a DCEL struct.

1. Init event queue Q with all *site events*
2. **while**(Q not empty) **do**
3. consider the event with largest *y*-coordinate in Q (next in the queue)
4. **if**(event is a *site event* at site p_i)
5. **then** HandleSiteEvent(p_i)
6. **else** HandleVoroVertexEvent(p_i), where p_i is the lowest point of the circle causing the event
7. remove the event from Q
8. Create a bbox and attach half-infinite edges in *T* to it in DCEL.
9. Traverse the halfedges in DCEL and add cell records and pointers to and from them

Handle site event

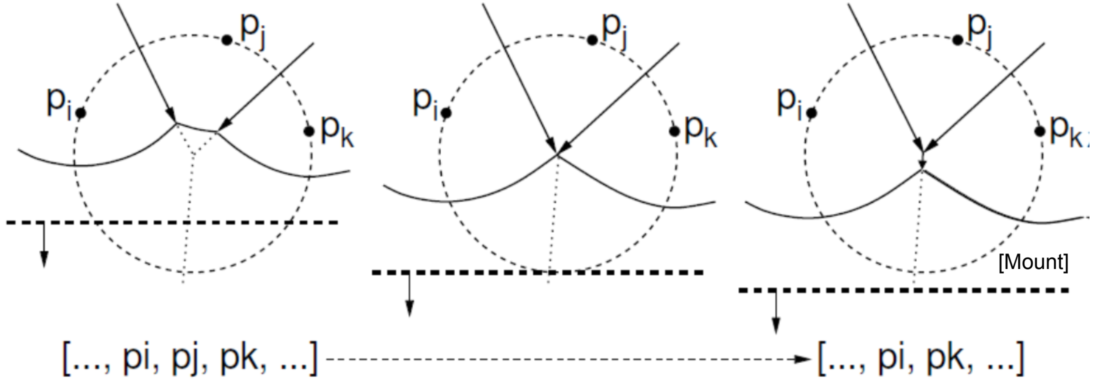
HandleSiteEvent(p_i)
Input: event site p_i
Output: updated DCEL



1. Search in T for arc α vertically above p_i. Let p_j be the corresponding site
2. Apply insert-and-split operation, inserting a new entry of p_i to the beach line T (new arc), thus replacing $\langle \ldots, p_j, \ldots \rangle$ with $\langle \ldots, p_j, p_i, p_j, \ldots \rangle$
3. Create a new (dangling) edge in the Voronoi diagram, which lies on the bisector between p_i and p_j
4. Neighbors on the beach line changed -> check the neighboring triples of arcs and *insert or delete Voronoi vertex events* (insert only if the circle intersects the sweep line and it is not present yet).
   Note: Newly created triple p_j , p_i , p_j cannot generate a circle event because it only involves two distinct sites.

Handle Voronoi vertex (circle) event

HandleVoroVertexEvent(p_j)
Input: event site p_j
Output: updated DCEL



Let p_i , p_j , p_k be the sites that generated this event (from left to right).

1. Delete the entry p_j from the beach line (thus eliminating its arc α), i.e.: Replace a triple $\langle \ldots, p_i , p_j , p_k , \ldots \rangle$ with $\langle \ldots, p_i , p_k , \ldots \rangle$ in *T.*
2. Create a new vertex in the Voronoi diagram (at circumcenter of $\langle p_i , p_j , p_k \rangle$) and join the two Voronoi edges for the bisectors $\langle p_i , p_j \rangle$ and $\langle p_j , p_k \rangle$ to this vertex (dangling edges – created in step 3 above).
3. Create a new (dangling) edge for the bisector between $\langle p_j , p_k \rangle$
4. Delete any Voronoi vertex events (max. three) from Q that arose from triples involving the arc α of p_j and generate (two) new events corresponding to consecutive triples involving p_i, and p_k.

DCGI

Beach line modification

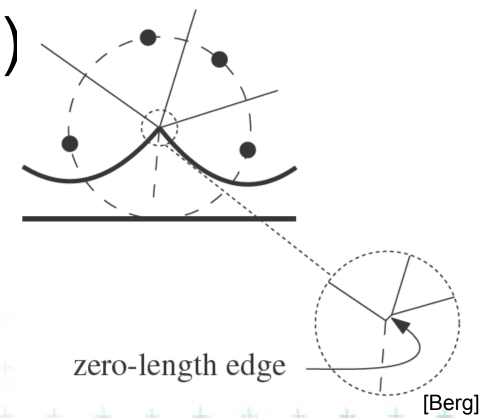
Q: Beach line contains: abcdef

After deleting of d, which triples vanish and which triples are added to the beach line?

Handling degeneracies

Algorithm handles degeneracies correctly

- 2 or more events with the same y
  - if x coords are different, process them in any order
  - if x coords are the same (cocircular sites)
    process them in any order,
    it creates duplicated vertices with
    zero-length edges,
    remove them in post processing step



[Berg]

- degeneracies while handling an event
  - Site below a beach line breakpoint
  - Creates circle event on the same position
    remove zero-length edges in post processing step

[Berg]

References

[Berg] Mark de Berg, Otfried Cheong, Marc van Kreveld, Mark Overmars: Computational Geometry: *Algorithms and Applications*, Springer-Verlag, 3rd rev. ed. 2008. 386 pages, 370 fig. ISBN: 978-3-540-77973-5, Chapter 7, http://www.cs.uu.nl/geobook/

[Mount] Mount, D.: *Computational Geometry Lecture Notes for Fall 2016*, University of Maryland, Lectures 11 and 16. http://www.cs.umd.edu/class/fall2016/cmsc754/Lects/cmsc754-fall16-lects.pdf

[Preparata] Preperata, F.P., Shamos, M.I.: *Computational Geometry. An Introduction.* Berlin, Springer-Verlag,1985. Chapter 5

[VoroGlide] VoroGlide applet: http://www.pi6.fernuni-hagen.de/GeomLab/VoroGlide/

[Fortune] Fortune's algorithm applet: http://www.personal.kent.edu/~rmuhamma/Compgeometry/MyCG/Voronoi/Fortune/fortune.htm

[Muhama] http://www.personal.kent.edu/~rmuhamma/Compgeometry/compgeom.html

http://www.personal.kent.edu/~rmuhamma/Compgeometry/MyCG/Voronoi/DivCongVor/divCongVor.htm