DEPARTMENT OF COMPUTER GRAPHICS AND INTERACTION

GEOMETRIC SEARCHING PART 2: RANGE SEARCH

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https://cw.felk.cvut.cz/doku.php/courses/a4m39vg/start

Based on [Berg] and [Mount]

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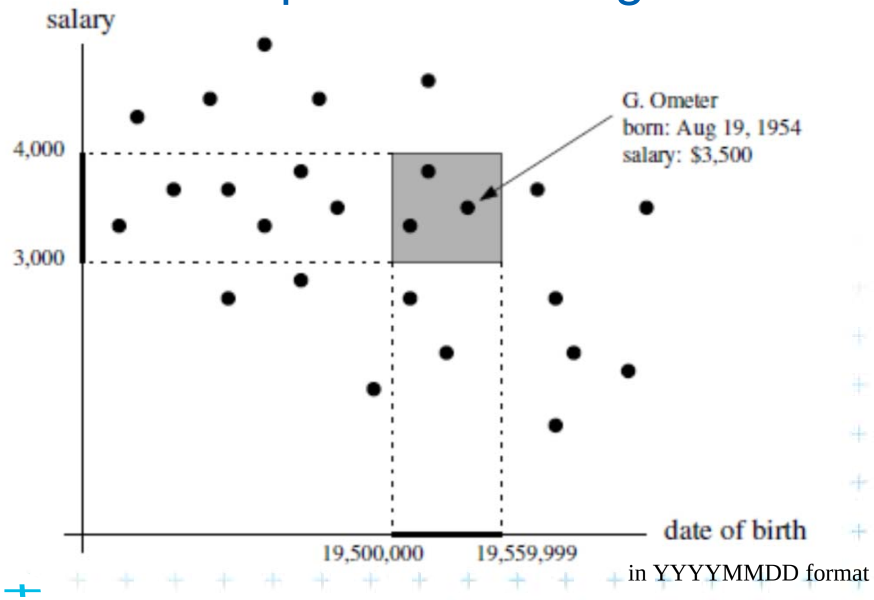
Range search

- Orthogonal range searching
- Canonical subsets
- 1D range tree
- 2D-nD Range tree
  - With fractional cascading (Layered tree)
- Kd-tree

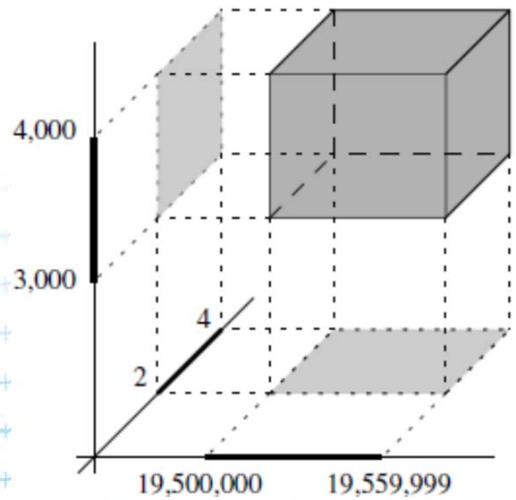
Orthogonal range searching

– Given a set of points P, find the points in the region Q
  - Search space: a set of points P (somehow represented)
  - Query: intervals Q (axis parallel rectangle)
  - Answer: points contained in Q
– Example: Databases (records->points)
  - Find the people with given range of salary, date of birth, kids, …

2D: axis parallel rectangle



3D: axis parallel box



[Berg]

Orthogonal range searching

Query region = axis parallel rectangle

- nDimensional search can be decomposed into set of 1D searches (separable)

Other range searching variants

- Search space S: set of
  - line segments,
  - rectangles, …
- Query region Q: any other searching region
  - disc,
  - polygon,
  - halfspace, …
- Answer: subset of S laying in Q

- We concentrate on points in orthogonal ranges

How to represent the search space?

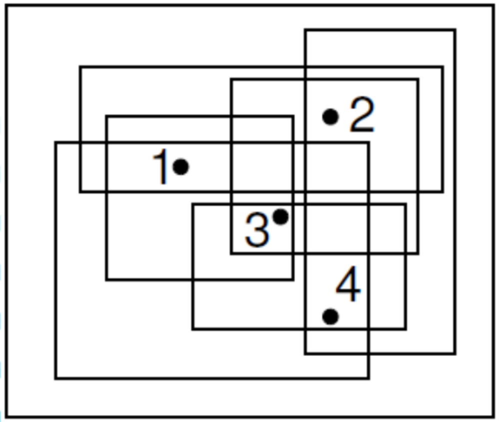
Basic idea:

- Not all possible combination can be in the output (not the whole power set potenční množina)
- => Represent only the "selectable" things (a well selected subset –> one of the canonical subsets)

Example?

Subsets selectable by given range class

- The number of subsets that can be selected by simple ranges Q is limited
- It is usually much smaller than the power set of P
  - Power set = set of all possible subsets (potenční množina)
  - Power set of P where P = {1,2,3,4} is {{ }, {1},{2},{3},{4}, {1,2},{1,3},{1,4}, {2,3},…,{2,3,4}, {1,2,3,4} } … $O(2^n)$
  - Simple rectangular queries are limited
    - Q defined by max 4 points along 4 sides => $O(n^4)$ of $O(2^n)$ power set, but
    - left border must be smaller than the right border
    - not all sets can be formed by □ query Q => $<O(n^4)$ e.g. sets {1,4} and {1,2,4} cannot be formed



[Mount]

Canonical subsets S_i

Search space $S = (P, Q)$ represented as a collection of canonical subsets $\{S_1, S_2, \dots, S_k\}$, each $S_i \subseteq S$,

- S_i may overlap each other (elements can be multiple times there)
- Any set can be represented as disjoint union disjunktní sjednocení of canonical subsets S_i each element knows from which subset it came
- Elements of disjoint union are ordered pairs (x, i) (every element x with index i of the subset S_i)

S_i may be selected in many ways

- from *n* singletons $\{p_i\}$ … $O(n)$
- to power set of P … $O(2^n)$

- Good DS balances between total number of canonical subsets and number of CS needed to answer the query

Disjoint union example

$$A = \{1,2,3\}$$
$$B = \{1,3,5\}$$

$$A \cup B = \{1,2,3,5\}$$
$$A \sqcup B = \{1_a, 1_b, 2_a, 3_a, 3_b, 5_b\}$$

$$|A \cup B| \leq |A| + |B|$$
$$|A \sqcup B| = |A| + |B|$$

1D range queries (interval queries)

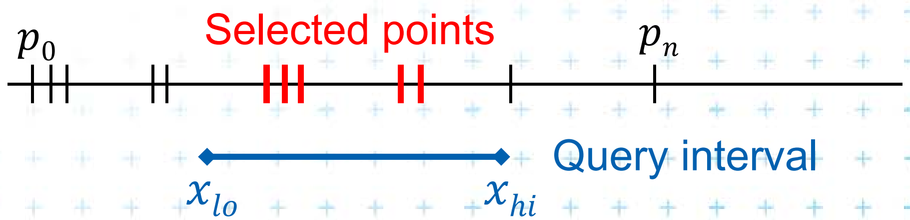
- Query: Interval $[x_{lo}, x_{hi}]$
- Search space: Points $P = \{p_1, p_2, \dots, p_n\}$ on the line

a) Binary search in an ordered array

- Simple, but
- not generalize to any higher dimensions

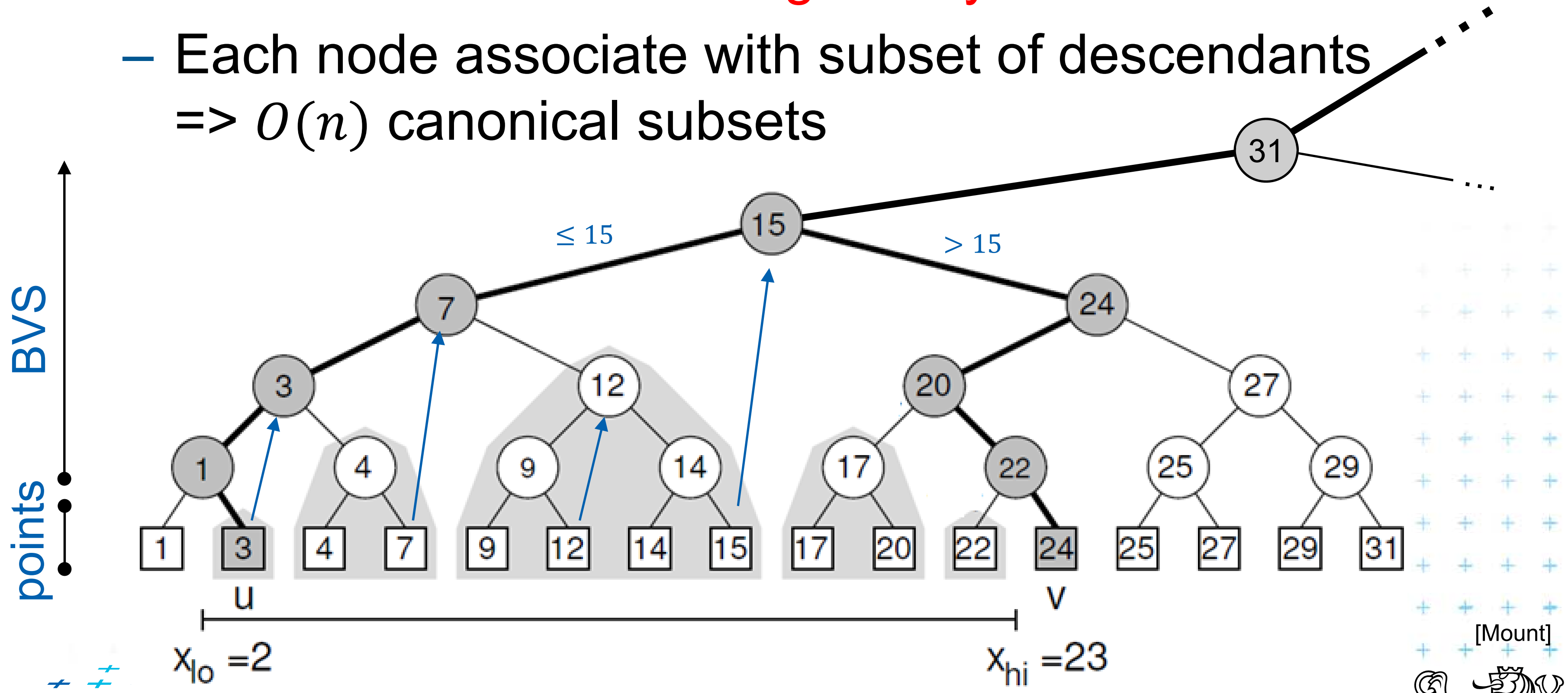
b) Balanced binary search tree

- 1D range tree
- maintains canonical subsets
- generalize to higher dimensions



1D range tree definition

- Balanced binary search tree (with repeated keys)
  - leaves – sorted points
  - inner node label – the largest key in its left child
  - Each node associate with subset of descendants => $O(n)$ canonical subsets



Canonical subsets and <2,23> search

Canonical subsets for this subtree are — #

{ {1}, {3}, …, {31}, — 16

{1, 3}, {4, 7}, …, {29, 31} — 8

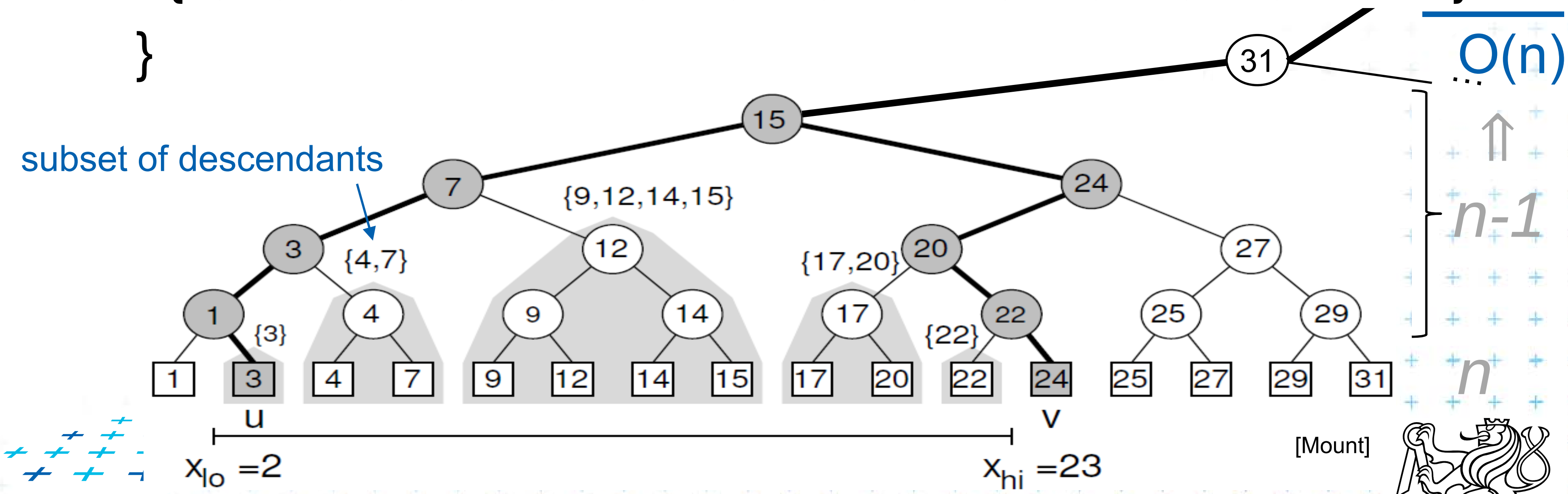
{1, 3, 4, 7}, {9, 12, 14, 15}, …, {25, 27, 29, 31} — 4

{1, 3, 4, 7, 9, 12, 14, 15}, {17, 20, 22, 24, 25, 27, 29, 31} — 2

{1, 3, 4, 7, 9, 12, 14, 15, 17, 20, 22, 24, 25, 27, 29, 31} — 1

}

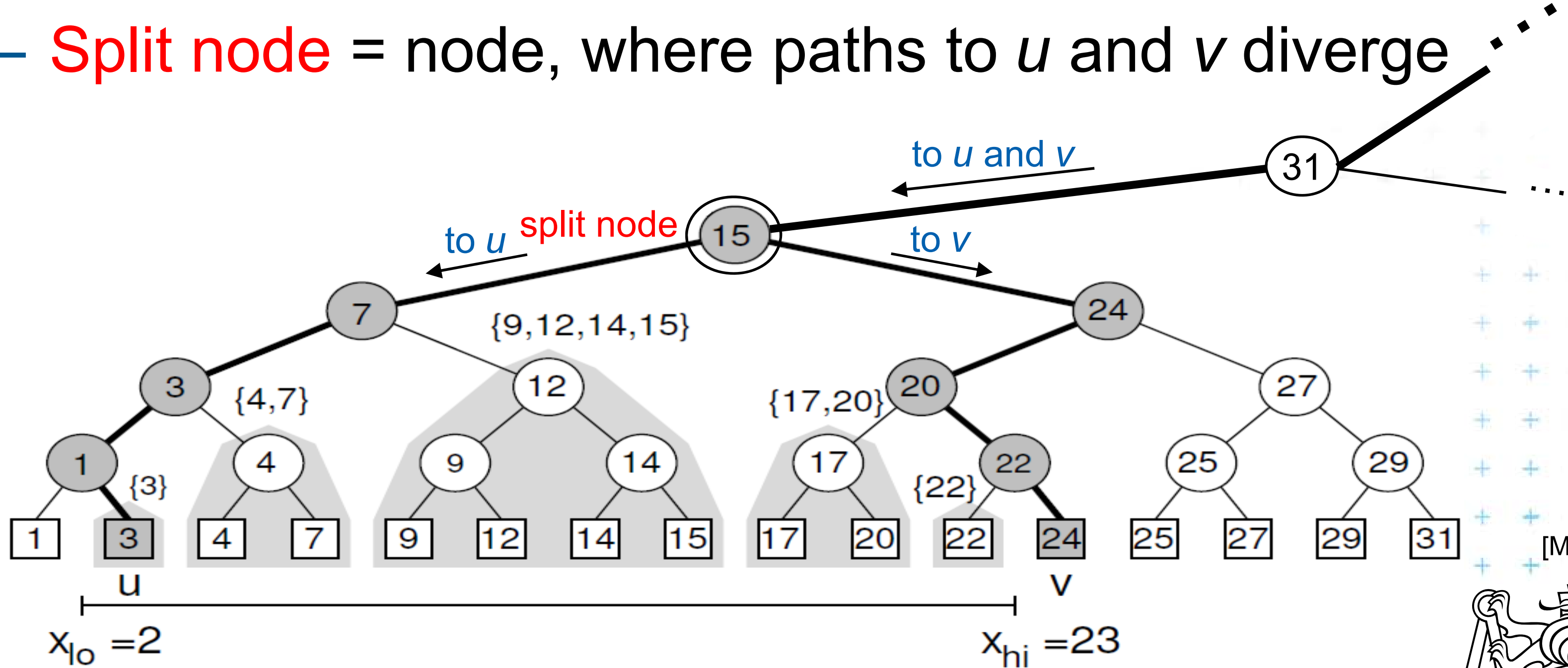
O(n)



[Mount]

1D range tree search interval <2,23>

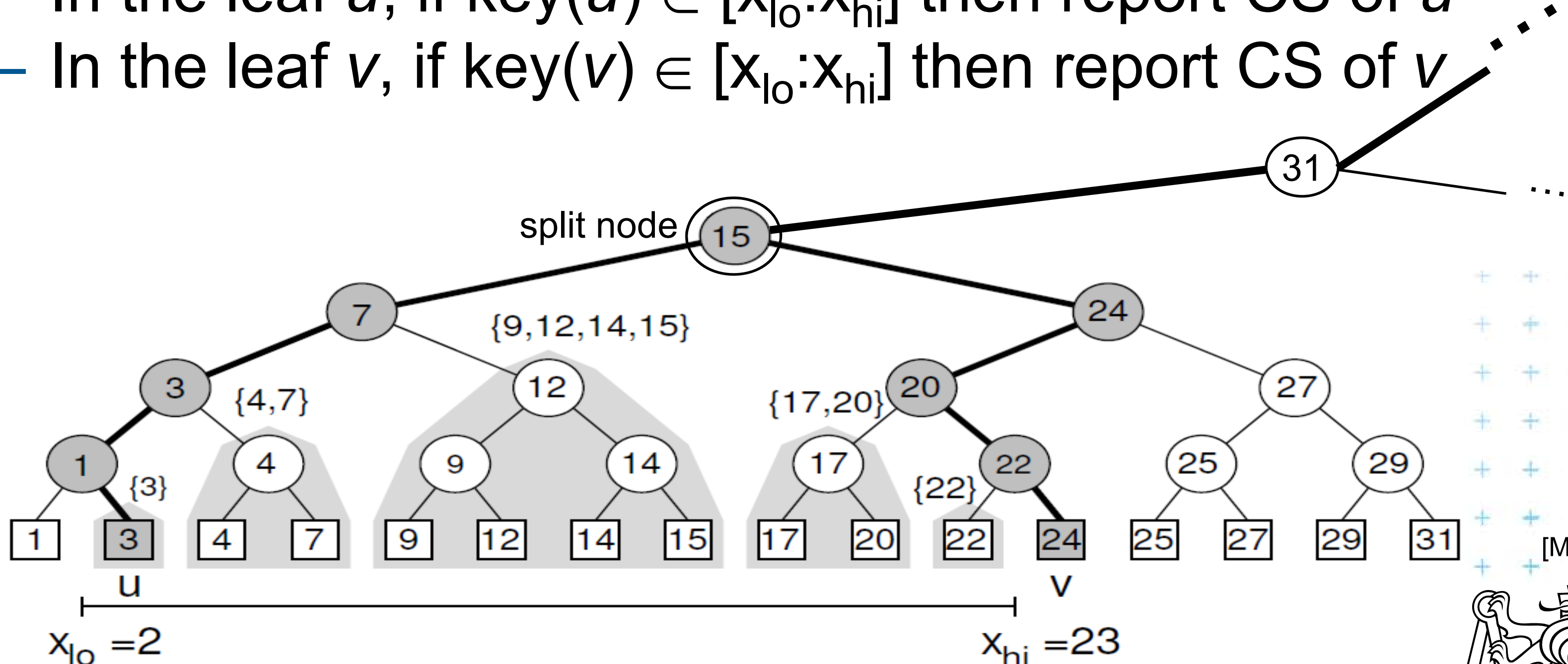
- Canonical subsets for any range ~~query~~ found in O(log *n*)
  - Search x_{lo}: Find leftmost leaf *u* with key(*u*) $\geq x_{lo}$ 2 -> 3
  - Search x_{hi}: Find leftmost leaf *v* with key(*v*) $\geq x_{hi}$ 23 -> 24
  - Points between *u* and *v* lie within the range => report canon. subsets of maximal subtrees between *u* and *v*
  - Split node = node, where paths to *u* and *v* diverge



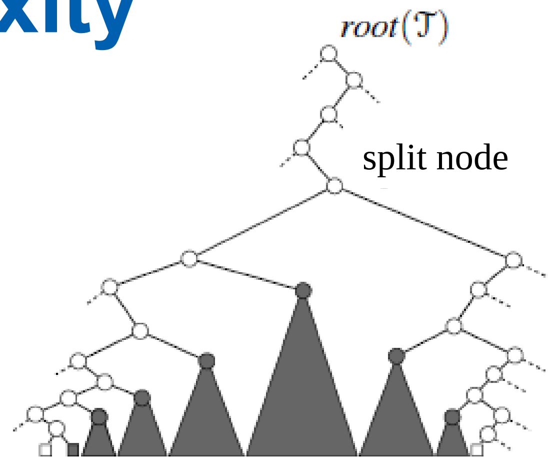
[Mount]

1D range tree search

- Reporting the subtrees (below the split node)
  - On the path to *u* whenever the *path goes left*, report the canonical subset (CS) associated to right child
  - On the path to *v* whenever the *path goes right*, report the canonical subset associated to left child
  - In the leaf *u*, if key(*u*) $\in [x_{lo}:x_{hi}]$ then report CS of *u*
  - In the leaf *v*, if key(*v*) $\in [x_{lo}:x_{hi}]$ then report CS of *v*



1D range tree search complexity



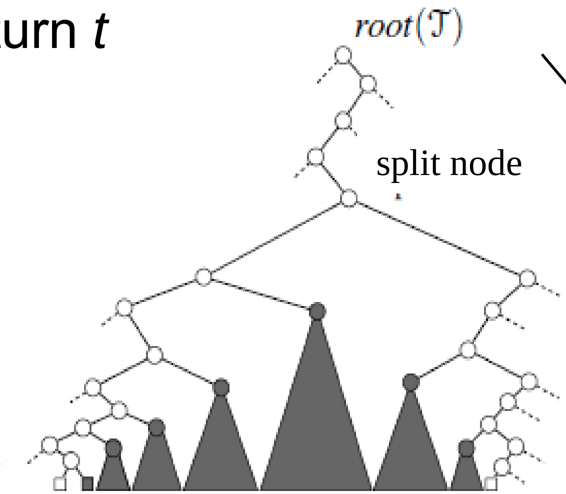
[Berg]

- Path lengths O(log n)
  => O(log n) canonical subsets (subtrees)
- Range counting queries
  – Return just the number of points in given range
  – Sum the total numbers of leaves stored in maximum subtree roots … O(log *n*) time
- Range reporting queries
  – Return all *k* points in given range
  – Traverse the canonical subtrees … O(log *n* + *k*) time
- O(*n*) storage, O(*n* log *n*) preprocessing – sort P

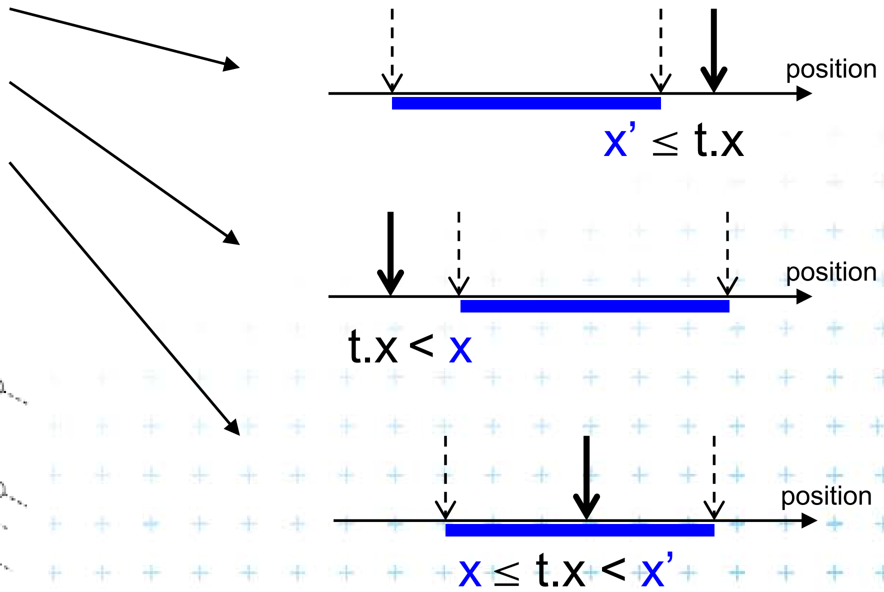
Find split node

FindSplitNode(*T*, [x:x'])
Input: Tree *T* and Query range [x:x'], x ≤ x'
Output: The node, where the paths to x and x' split
or the leaf, where both paths end

1. *t = root(T)*
2. while(*t* is not a leaf **and (**x' ≤ t.x **or** t.x < x**)**) // *t* out of the range [x:x']
3. if(*x'* ≤ t.x) *t = t.left*
4. else *t = t.right*
5. return *t*



[Berg]



STOP

1D range search

(2D on slide 22)

1dRangeQuery(*t*, [x:x’])

Input: 1d range tree *t* and Query range $[x: x']$

Output: All points in *t* lying in the range

t_{split} = FindSplitNode(*t, x, x’*) // find point t with $t_x \in [x: x']$

if(t_{split} is leaf) // e.g. Searching [16:17] or [16:16.5] both stops in the leaf 17 in the previous example

    check if the point in t_{split} must be reported // $t_x \in [x: x']$

else // follow the path to x, reporting points in subtrees right of the path

    t = t_{split}.left

    while(t is not a leaf)

        if($x \leq$ t.x)

            ReportSubtree(*t.right*) // any kind of tree traversal

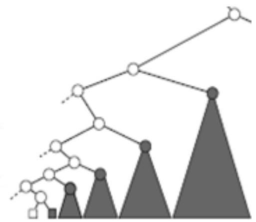
            t = t.left

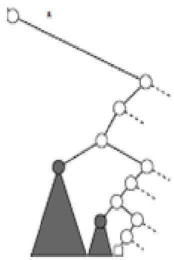
        else *t = t.right*

    check if the point in leaf *t* must be reported

    // Symmetrically follow the path to x' reporting points left of the path

    t = t_{split}.right …



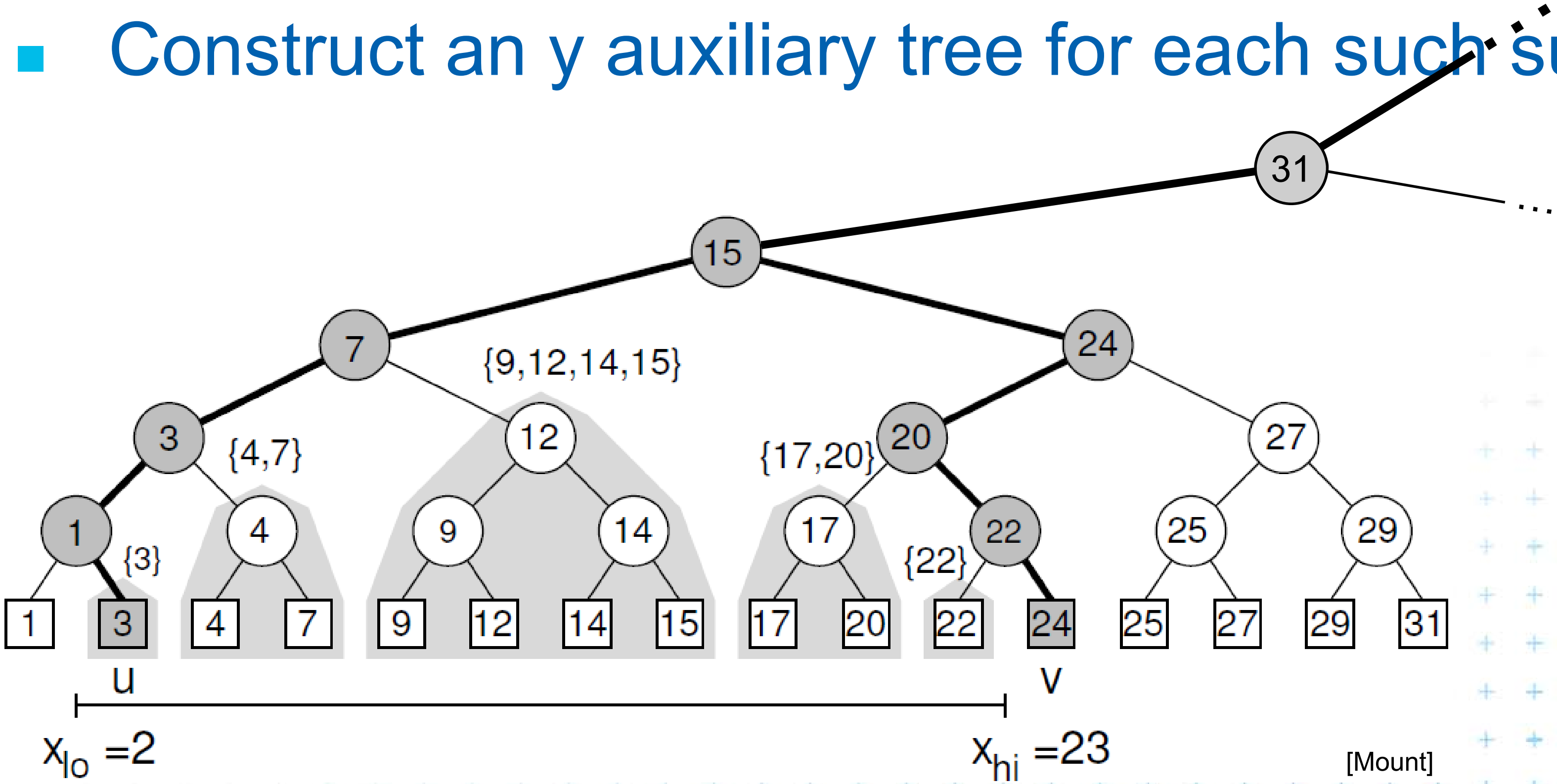


Multidimensional range searching

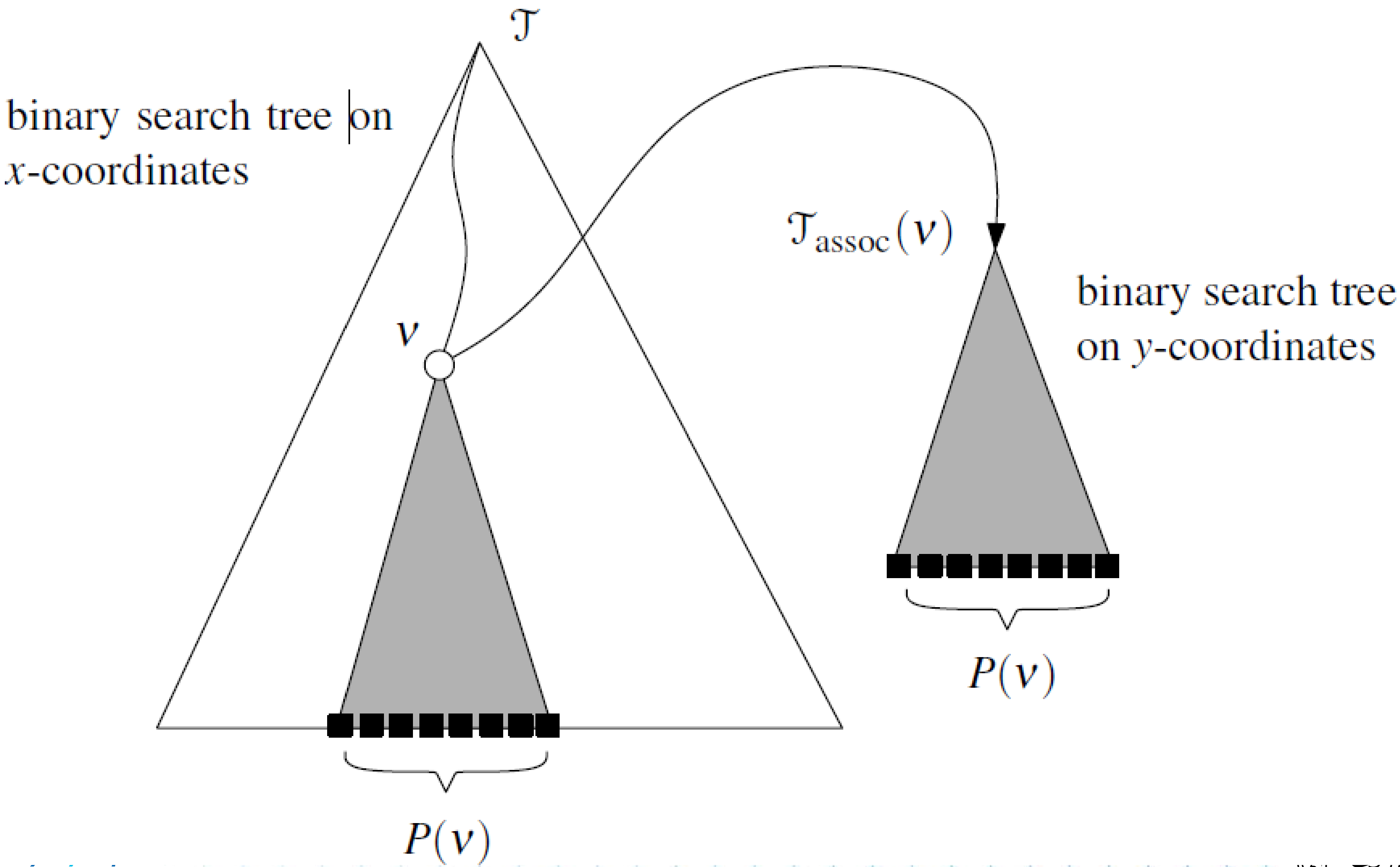
- Equal principle – find the largest subtrees contained within the range
- Separate one *n*-dimensional search into *n* 1-dimensional searches
- Different tree organization
  – Orthogonal (Multilevel) range search tree
    e.g. nd range tree
  – Kd tree

From 1D to 2D range tree

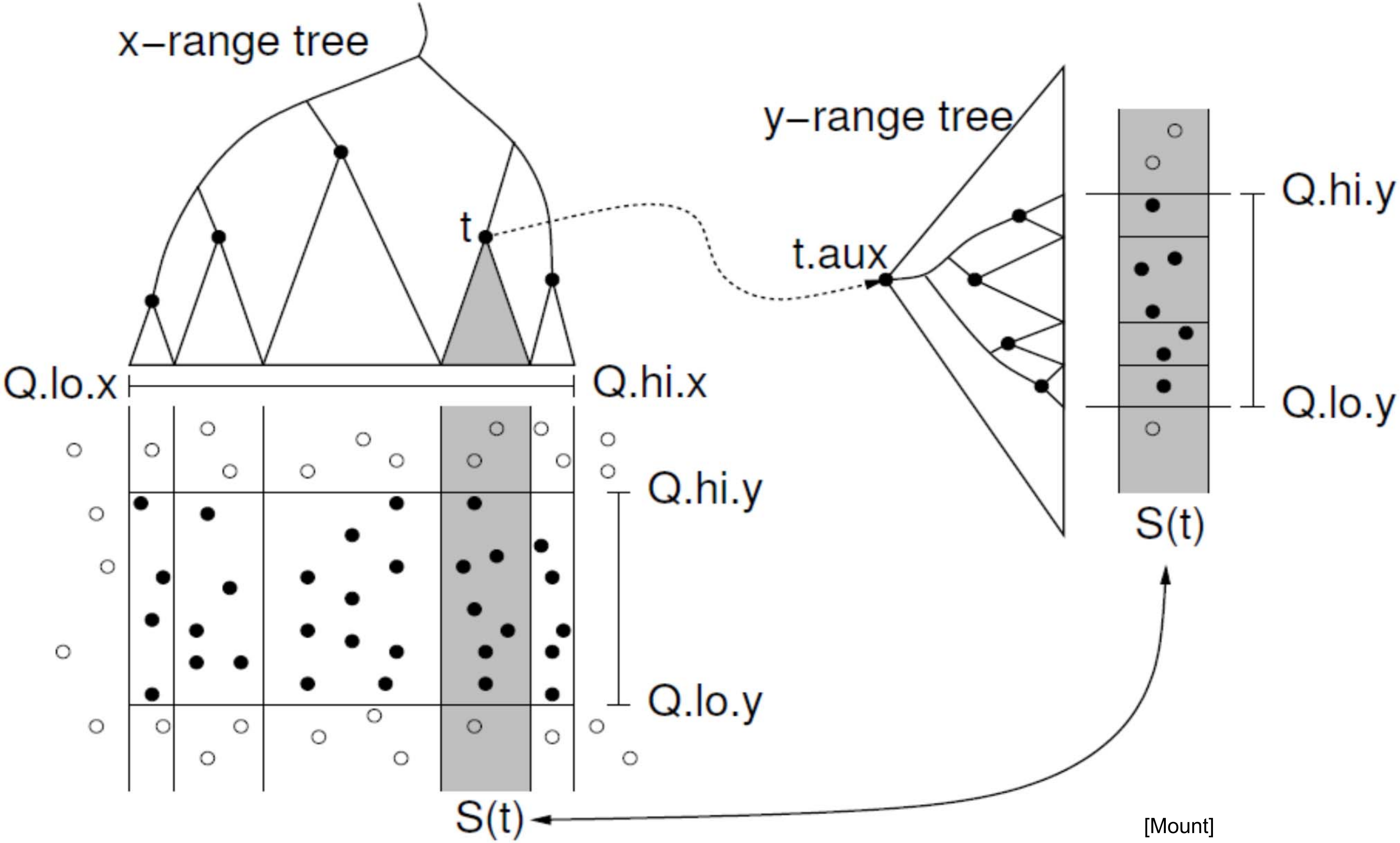
- Search points from [Q.x_{lo}, Q.x_{hi}] [Q.y_{lo}, Q.y_{hi}]
- 1d range tree: log n canonical subsets based on x
- Construct an y auxiliary tree for each such subset



y-auxiliary tree for each canonical subset



2D range tree



[Mount]

2D range search

(1D on slide 17)

2dRangeQuery(*t*, [x:x'] × [y:y'])

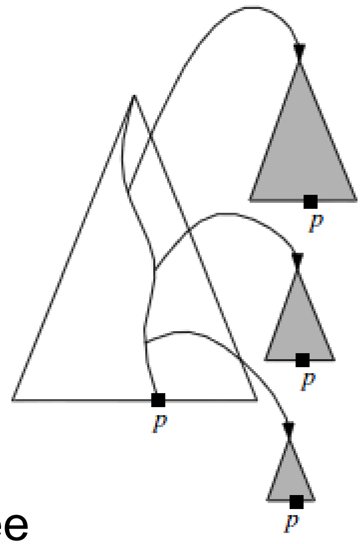
Input: 2d range tree t and Query ranges $[x: x'] \times [y: y']$

Output: All points in t laying in both ranges

```
t_split = FindSplitNode( t, x, x' )           // find point t with t_x ∈ [x: x']
if( t_split is leaf )
    check if the point in t_split must be reported      // t_x ∈ [x: x'], t_y ∈ [y: y']
else // follow the path to x, calling 1dRangeQuery on y
    t = t_split.left     // path to the left
    while( t is not a leaf )
        if( x ≤ t.x)
            1dRangeQuerry( t_assoc( t.right ), [y:y'] ) // check associated subtree
            t = t.left
        else t = t.right
    check if the point in leaf t must be reported      … t_x ≤ x', t_y ∈ [y: y']
    // Similarly for the path to x'   … // path to the right
    t = t_split.right …
```

2D range tree

- Search $O(\log^2 n + k)$ … $\log n$ in x, $\log n$ in y
- Space $O(n \log n)$
  - $O(n)$ the tree for x-coords
  - $O(n \log n)$ trees for y-coords
    - Point p is stored in all canonical subsets along the path from root to the leaf with p,
    - once for each x-tree level along the path
    - each canonical subset is stored in one auxiliary y-tree
    - $\log n$ levels of x-tree => $O(n \log n)$ space for y-trees



[Berg]

- Construction - $O(n \log n)$
  - Sort points (by x and by y). Bottom up construction

Canonical subsets

- Canonical subsets for this subtree are #

{ {1}, {3}, …, {31}, 16

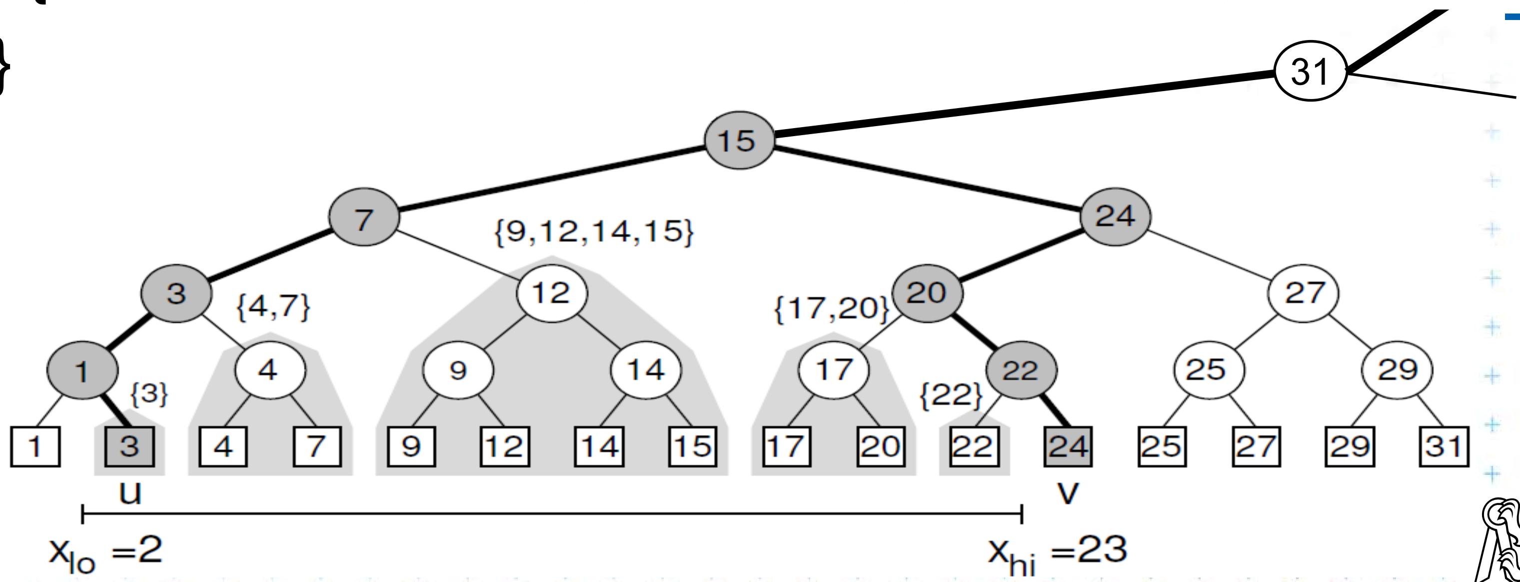
{1, 3}, {4, 7}, …, {29, 31} 8

{1, 3, 4, 7}, {9, 12, 14, 15}, …, {25, 27, 29, 31} 4

{1, 3, 4, 7, 9, 12, 14, 15}, {17, 20, 22, 24, 25, 27, 29, 31} 2

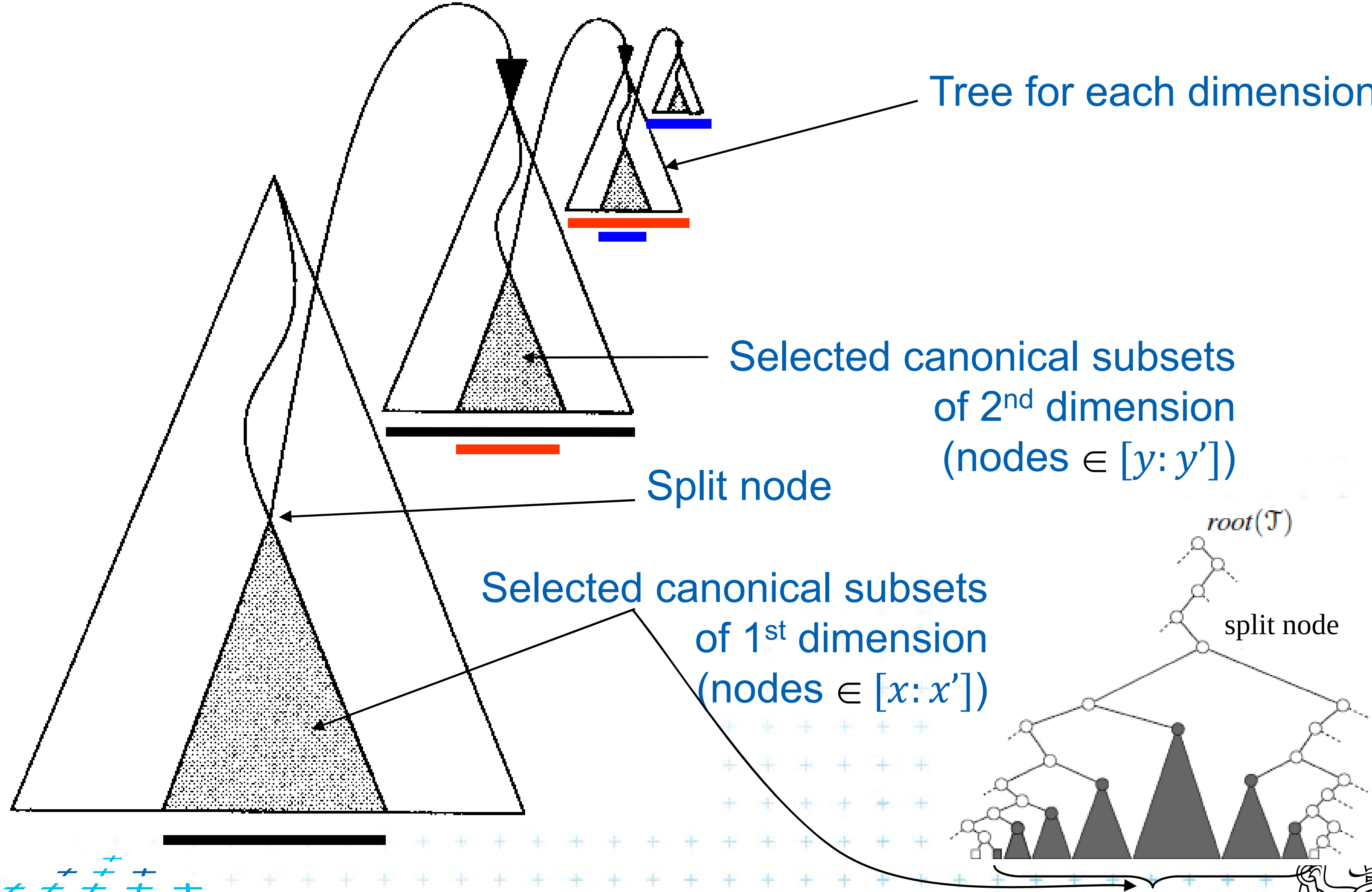
{1, 3, 4, 7, 9, 12, 14, 15, 17, 20, 22, 24, 25, 27, 29, 31} 1

} O(n)



[Mount]

nD range tree (multilevel search tree)



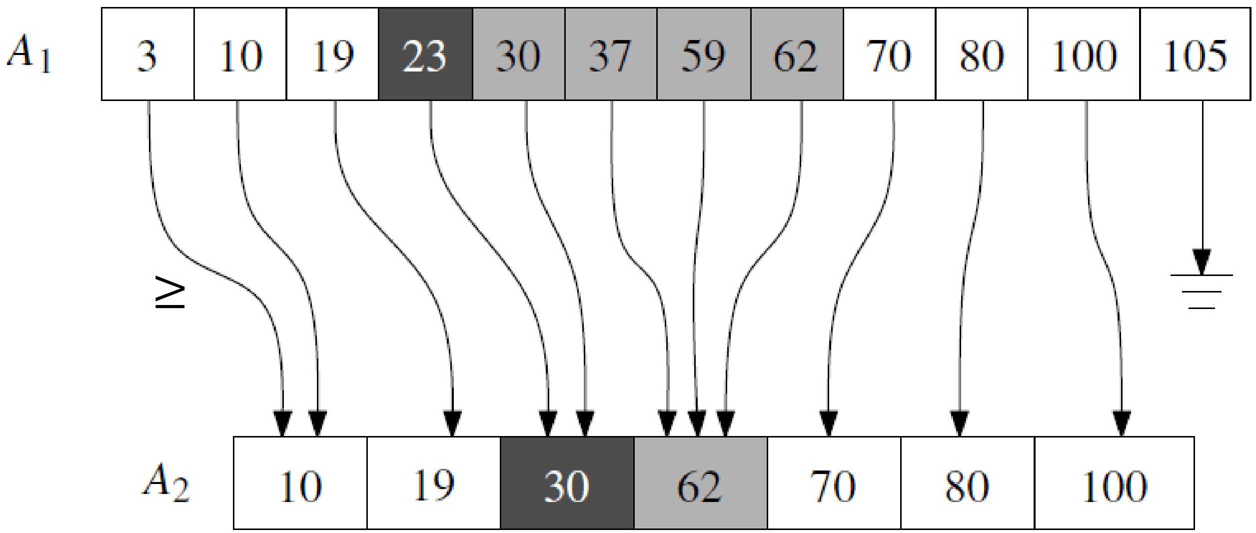
[Berg]

Fractional cascading - principle

- Two sets S_1, S_2 stored in sorted arrays A_1, A_2
- $S_1 \supseteq S_2$ (S_2 is subset of S_1)
- Report objects in both arrays whose keys in $[y : y']$
- Naïve approach – search twice independently
  - $O(\log n_1 + k_1)$ – search in A_1 + report k_1 elements
  - $O(\log n_2 + k_2)$ – search in A_2 + report k_2 elements
- Fractional cascading – adds pointers from A_1 to A_2
  - $O(\log n_1 + k_1)$ – search in A_1 + report k_1 elements
  - $O(1 + k_2)$ – jump to A_2 + report k_2 elements
  - Saves the $O(\log n_2)$ – search

Fractional cascading – principle for arrays

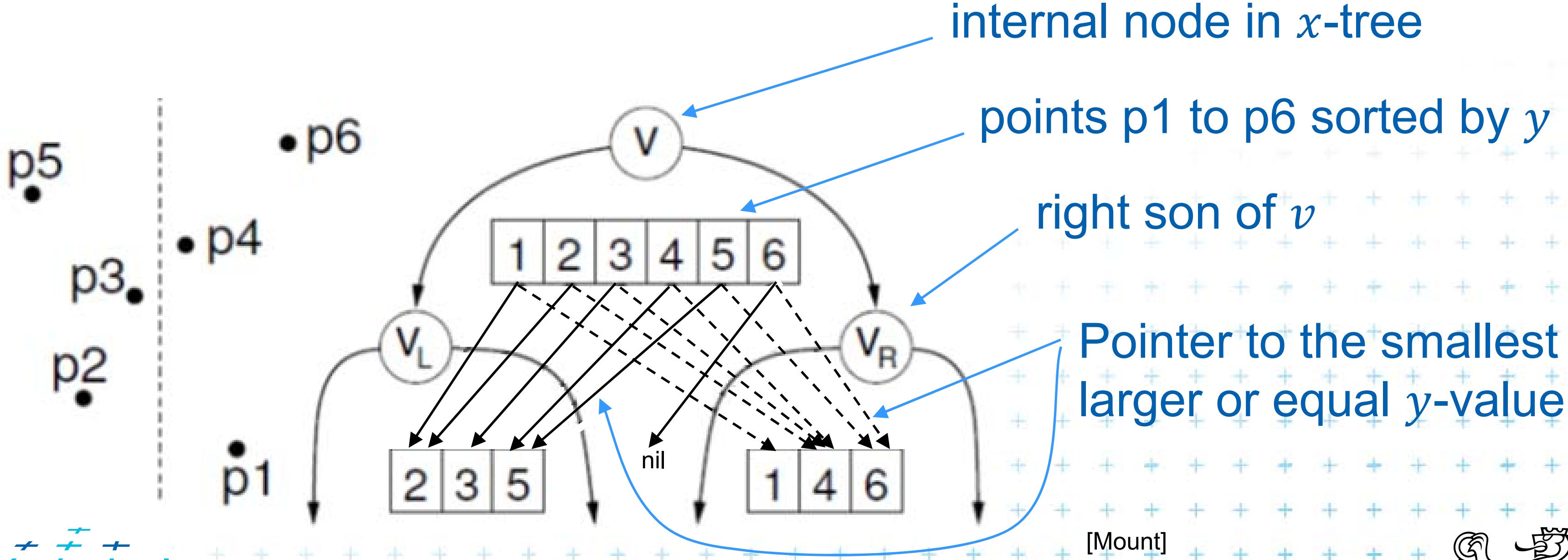
- Add pointers from A_1 to A_2
  - From element in array A_1 with a key y_i point to the element in A_2 with the smallest key *larger or equal* to y_i
- Example query with the range [20 : 65]



[Berg]

Fractional cascading in the 2D range tree

- How to save one $\log n$ during last dim. search?
  - Store canonical subsets in arrays sorted by y
  - Pointers to subsets for both child nodes v_L and v_R
  - $O(1)$ search in lower levels => in two dimensional search $O(\log^2 n)$ time -> $O(\log n)$



[Mount]

Orthogonal range tree - summary

- Orthogonal range queries in plane
  - Counting queries $O(\log^2 n)$ time,
    or with fractional cascading $O(\log n)$ time
  - Reporting queries plus $O(k)$ time, for k reported points
  - Space $O(n \log n)$
  - Construction $O(n \log n)$
- Orthogonal range queries in d-dimensions, $d \geq 2$
  - Counting queries $O(\log^d n)$ time,
    or with fractional cascading $O(\log^{d-1} n)$ time
  - Reporting queries plus $O(k)$ time, for k reported points
  - Space $O(n \log^{d-1} n)$
  - Construction $O(n \log^{d-1} n)$ time

Kd-tree

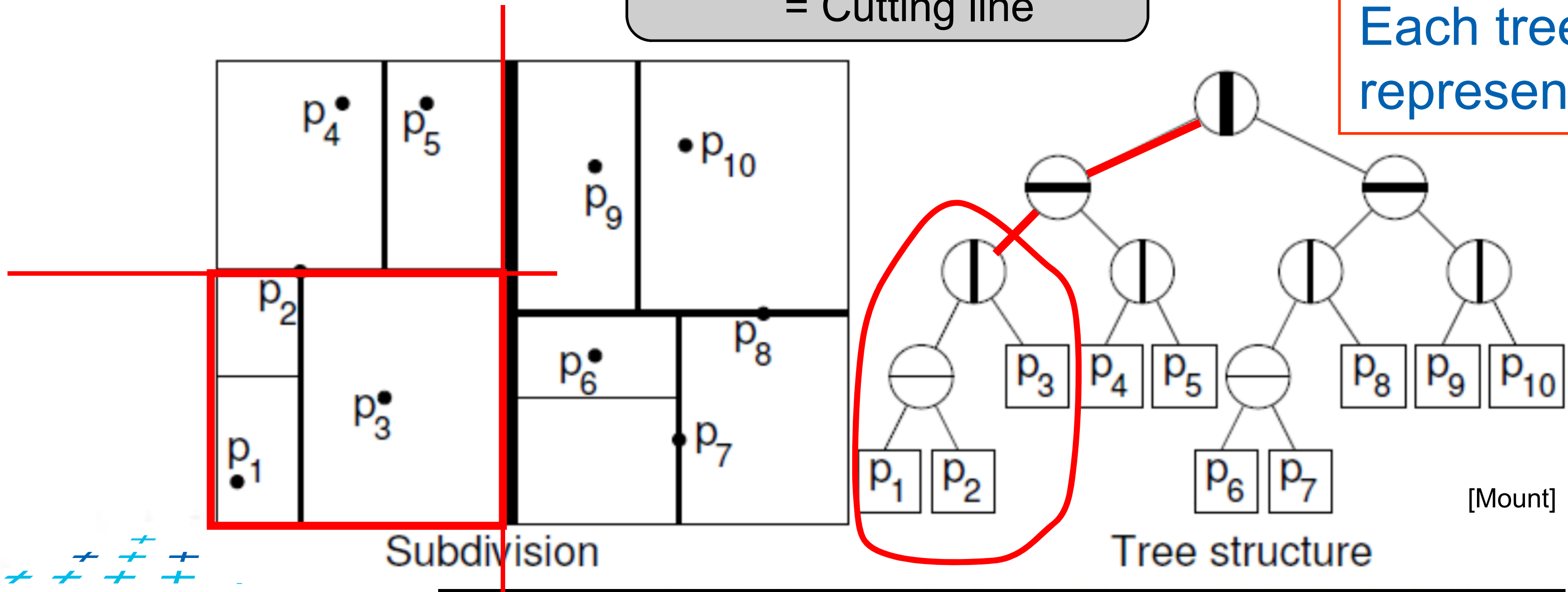
- Easy to implement
- Good for different searching problems (counting queries, nearest neighbor,…) see DPG
- Designed by Jon Bentley as k-dimensional tree (2-dimensional kd-tree was a $2d$ tree, …)
- Not the asymptotically best for orthogonal range search (=> range tree is better)
- Types of queries
  - Reporting – points in range
  - Counting – number of points in range

Kd-tree principle

- Subdivide space according to different dimension (x-coord, then y-coord, …)
- This subdivides space into rectangular cells => hierarchical decomposition of space
- In node *t* store: cutDim, cutVal, (size (for counting queries))

= Cutting line

Each tree node represents a region



Subdivision

Tree structure

Where is a mistake in the figure?

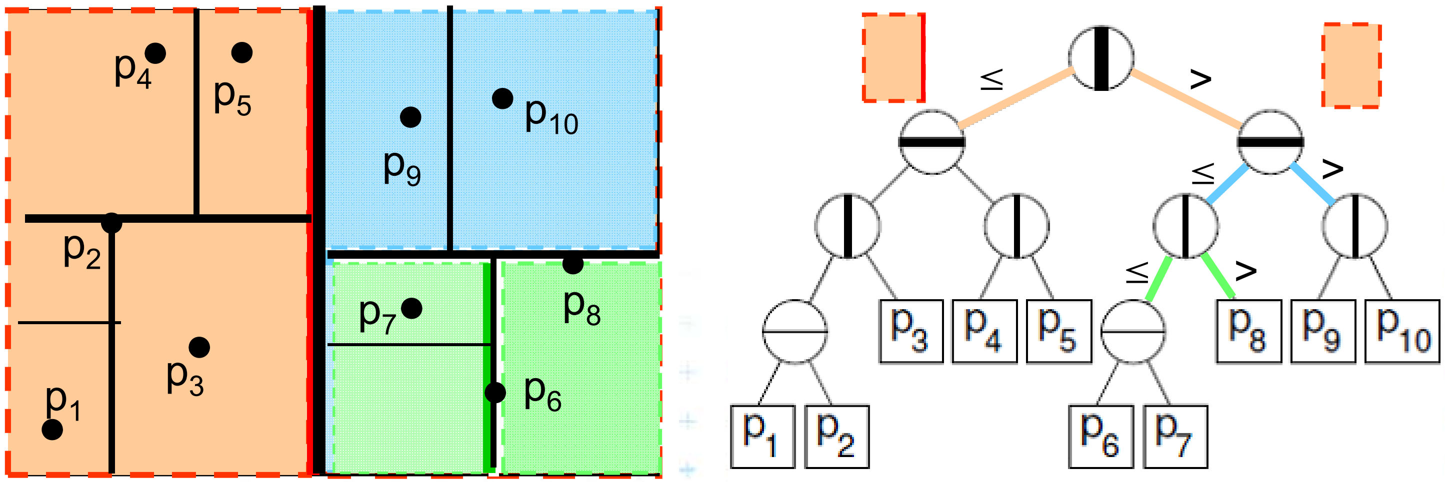
Kd-tree principle

- Which dimension to cut? (cutDim)
  - Cycle through dimensions (round robin)
    - Save storage – cutDim is implicit ~ depth in the tree
    - May produce elongated cells (if uneven data distribution)
  - Greatest spread (the largest difference of coordinates)
    - Adaptive
    - Called “Optimal kd-tree”

- Where to cut? (cutVal)
  - Median, or midpoint between upper and lower median -> $O(n)$
  - Presort coords of points in each dimension $(x, y, \dots)$ for $O(1)$ median – resp. $O(d)$ for all d dimensions

Kd-tree principle

- What about points on the cell boundary?
  - Boundary belongs to the left child
  - Left: $p_{cutDim} \leq cutVal$
  - Right: $p_{cutDim} > cutVal$



Subdivision

Tree structure

[Mount]

Kd-tree construction in 2-dimensions

BuildKdTree(*P, depth*)
Input: A set of points *P* and current *depth.*
Output: The root of a kD tree storing P.

1. **If** (*P* contains only one point) [or small set of (10 to 20) points]
2. **then return** a leaf storing this point
3. **else if** (*depth* is even)
4. **then** split *P* with a vertical line *l* through median *x* into two subsets P_1 and P_2 (left and right from median)
5. **else** split *P* with a horiz. line *l* through median y into two subsets P_1 and P_2 (below and above the median)
6. t_{left} = BuildKdTree(P_1, *depth*+1)
7. t_{right} = BuildKdTree(P_2, *depth*+1)
8. create node *t* storing *l*, t_{left} and t_{right} children // l = cutDim, cutVal
9. **return** *t*

Split according to (*depth%max_dim)* dimension

If median found in $O(1)$ and array split in $O(n)$
$T(n) = 2\,T(n/2) + n \Rightarrow O(n \log n)$ construction

Kd-tree test variants

Test interval-interval

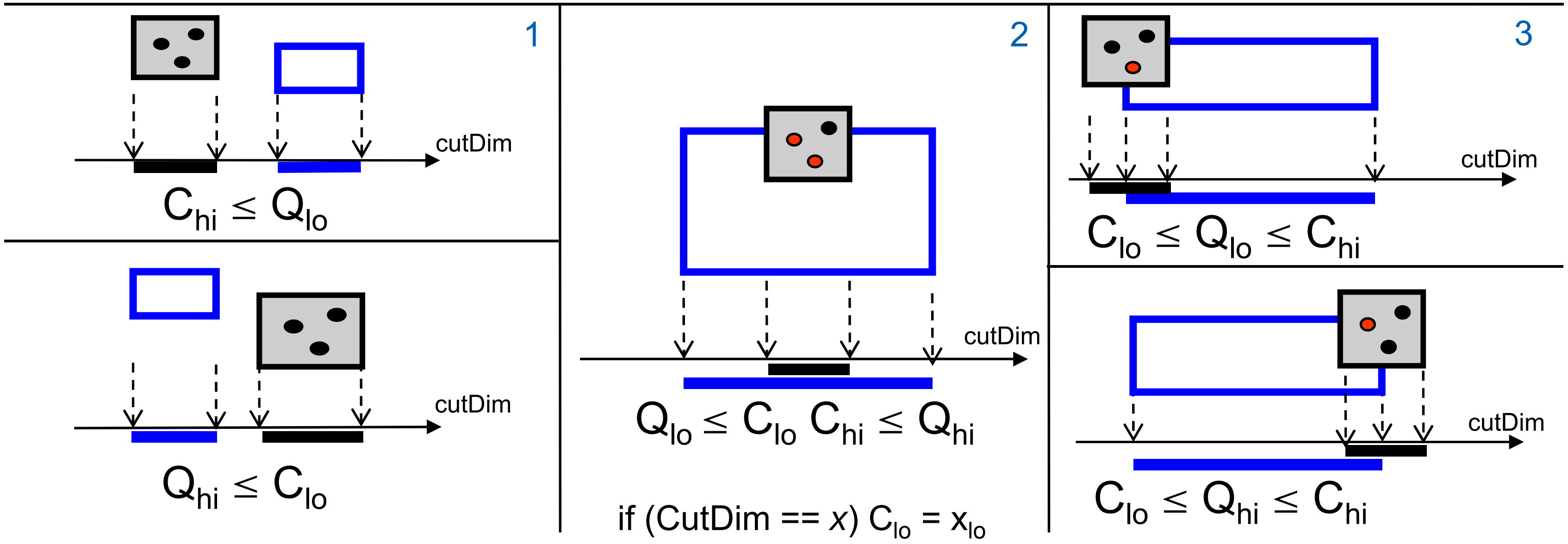
a) Compare rectang. array Q with rectangular cells C

– Rectangle C:[x_{lo}, x_{hi}, y_{lo}, y_{hi}] – computed on the fly

– Test of kD node cell C against query Q (in one cutDim)

1. if cell is disjoint with Q … $C \cap Q = \emptyset$ … stop
2. If cell C completely inside Q … $C \subseteq Q$ … stop and report cell points
3. else cell C overlaps Q … recurse on both children

– Recursion stops on the largest subtree (in or out)



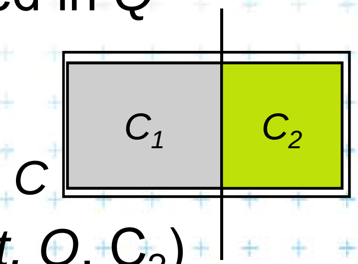
Kd-tree rangeCount (with rectangular cells)

int rangeCount(*t*, *Q*, *C*)

Input: The root *t* of kD tree, query range *Q and t's* cell C.

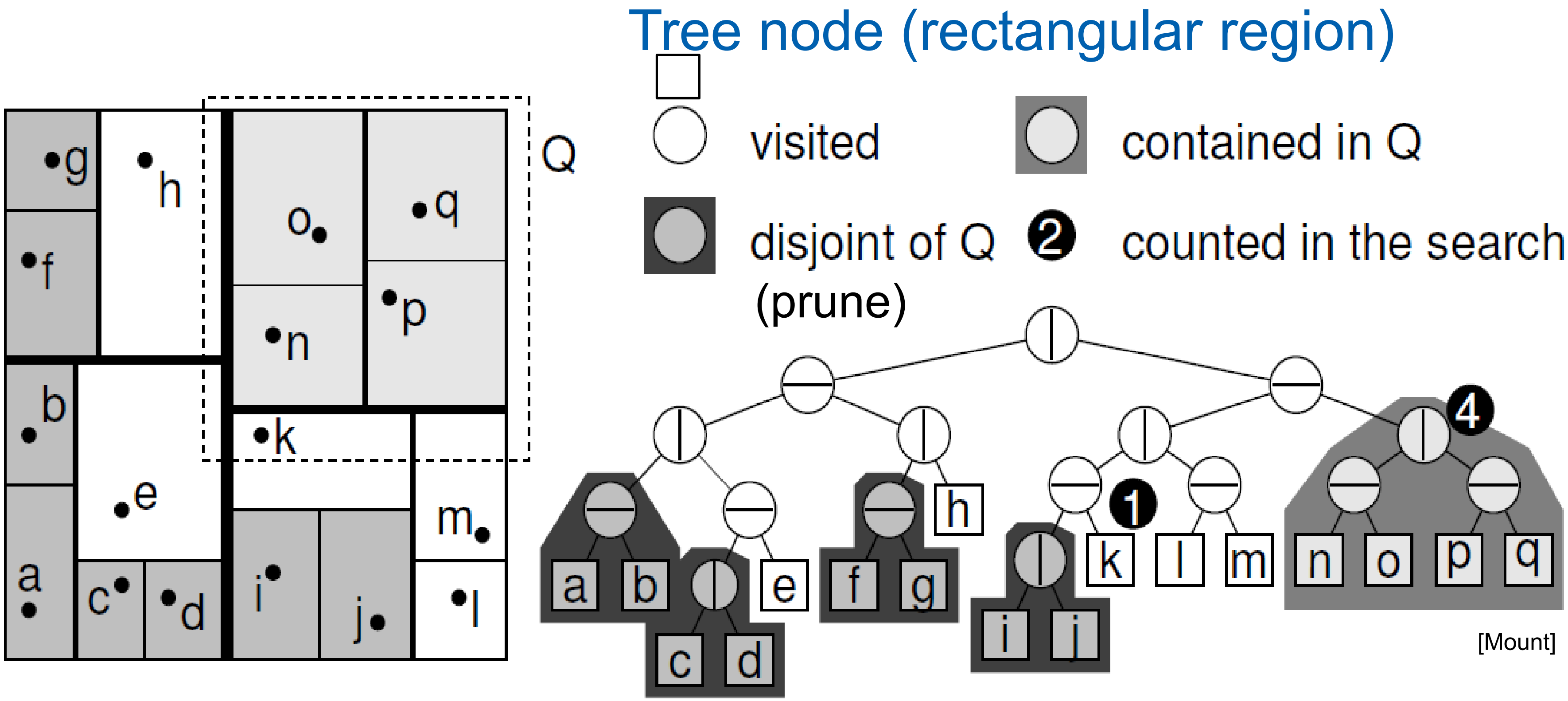
Output: Number of points at leaves below *t* that lie in the range.

if (*t* is a leaf)
    if (*t.point* lies in *Q) return 1*   // or loop this test for all points in leaf
    else return 0   // visited, not counted
else *//* **(***t* is not a leaf)
    if ($C \cap Q = \emptyset$ **)** *return 0*   … *disjoint*
    else if ($C \subseteq Q$ **)** *return t.size*   … *C* is fully contained in *Q*
    else
        split C along *t's* cutting value and dimension,
        creating two rectangles C_1 and C_2.
        return rangeCount(*t.left*, *Q*, C_1) + rangeCount(*t.right*, *Q*, C_2)



// (pictograms refer to the next slide)

Kd-tree rangeCount example



kd–tree subdivision

Nodes visited in range search

Kd-tree test variants

Test point-interval

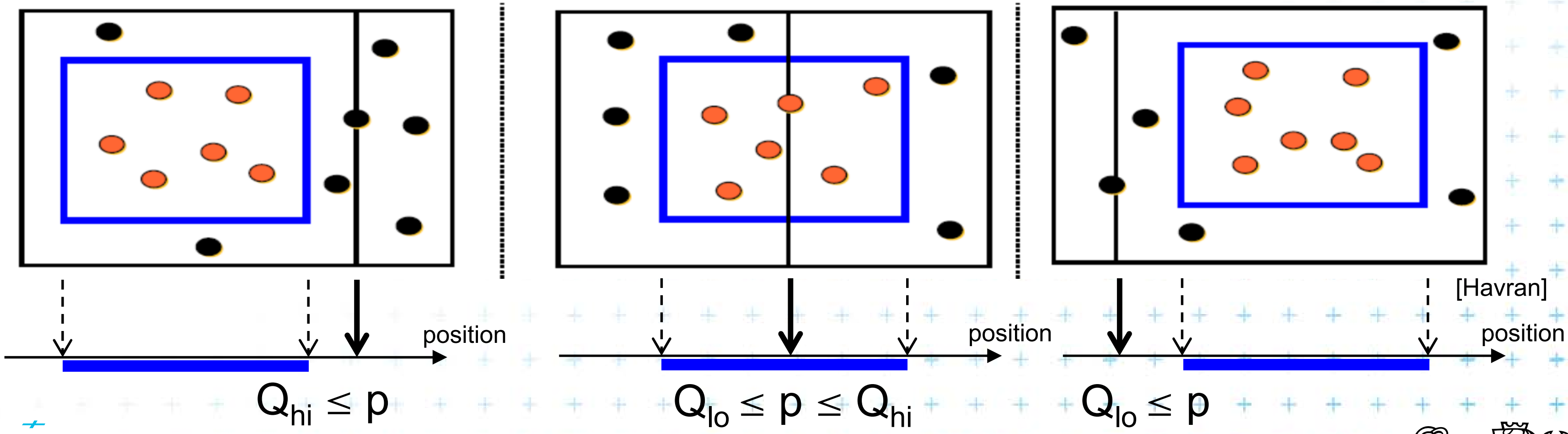
b) Compare Q with cutting lines

– Line = Splitting value *p* in one of the dimensions

– Test of single position given by dimension against Q

1. Line *p* is right from Q … recurse on left child only (prune right child)
2. Line *p* intersects Q … recurse on both children
3. Line *p* is left from Q … recurse on right child only (prune left ch.)

– Recursion stops in leaves - traverses the whole tree



Kd-tree rangeSearch (with cutting lines)

int rangeSearch(*t*, *Q*)

Input: The root *t* of (a subtree of a) kD tree and query range *Q*.

Output: Points at leaves below *t* that lie in the range.

```
if (t is a leaf)
    if (t.point lies in Q) report t.point   // or loop test for all points in leaf
    else return
else  (t is not a leaf)
    if (Q_hi ≤ t.cutVal) rangeSearch(t.left, Q)   // go left only
    if (Q_lo > t.cutVal) rangeSearch(t.right, Q) // go right only
    else
        rangeSearch(t.left, Q)                   // go to both
        rangeSearch(t.right, Q)
```

Kd-tree - summary

- Orthogonal range queries in the plane (in balanced 2d-tree)
  - Counting queries $O(\sqrt{n})$ time
  - Reporting queries $O(\sqrt{n} + k)$ time, where k = No. of reported points
  - Space $O(n)$
  - Preprocessing: Construction $O(n \log n)$ time (Proof: if presorted points to arrays in dimensions. Median in $O(1)$ and split in $O(n)$ per level, $\log n$ levels of the tree)
- For d≥2:
  - Construction $O(dn \log n)$, space $O(dn)$, Search $O(dn^{(1-1/d)} + k)$

Proof $\sqrt{n}$

Každé sudé patro se testuje osa x.

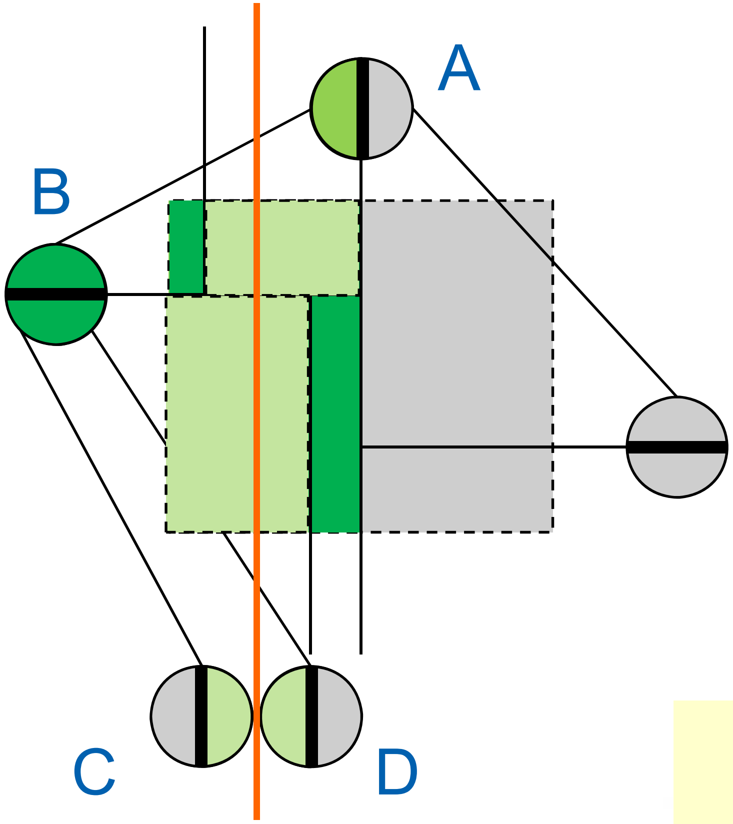
- V patře 0 je jeden uzel a jde se do obou synů (v patře 1 se jde taky do obou)
- v patře 2 jsou 4 uzly, z nich jsou ale 2 bud úplně mimo, nebo úplně in => stab jen 2
- v 4. patře stab 4 z 8, …
- v i-tém patře stab 2^i uzlů

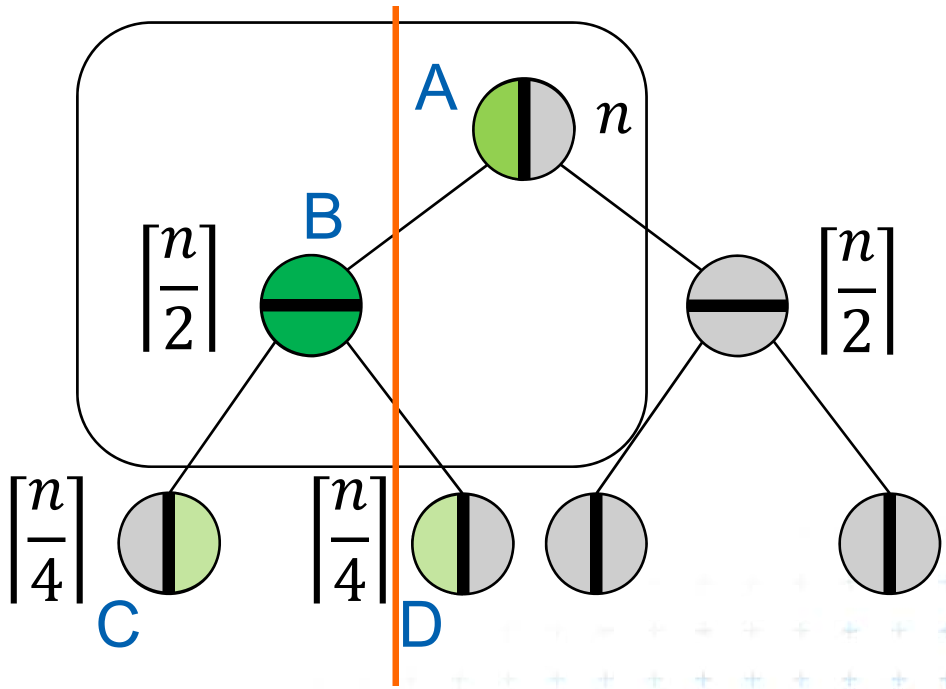
Výška stromu je $\log n$

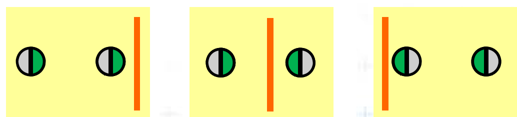
Proto tedy sčítám sudé členy z $0..\log n$ z 2^i. Je to exponenciála, proto dominuje poslední člen

$$2^{\log n/2} = 2^{\log(\sqrt{n})} = \sqrt{n}$$

Proof $O(\sqrt{n})$ search complexity – line







$$Q(n) = \begin{cases} O(1) & if\ n = 1 \\ 2 + 2Q\left(\frac{n}{4}\right) & if\ n > 1 \end{cases}$$

Proof $O(\sqrt{n})$ search complexity – line

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The master method

$$T(n) = f(n) + aT\left(\frac{n}{b}\right)$$

$$f(n) = 2$$
$$a = 2$$
$$b = 4$$
$$n^{\log_b a} = n^{\log_4 2} = \sqrt{n}$$

$$\text{if } n^{\log_b a} > f(n) \Rightarrow \ T(n) = \Theta(n^{\log_b a})$$
$$\text{if } \sqrt{n} > 2 \Rightarrow \ T(n) = \Theta(\sqrt{n})$$

Proof $O(\sqrt{n})$ search complexity – quad

$Q(n) = O(\sqrt{n})$ for single line

$= $ vertical query line intersects $O(\sqrt{n})$ regions

The same for rectangular (2D) query range

$$Q(n) = O(\sqrt{n})$$

In higher dimensions

- Binary tree with n leaves $\Rightarrow O(n)$ storage
- Construction $O(n \log n\,) -$ assuming d be constant
- Query visits the nodes of intersected regions
- Time bounded by $O(n^{1-1/d)} + k)$

Orthogonal range tree (RT)

- DS highly tuned for orthogonal range queries
- Reporting query times in the plane

2d tree	versus	2d range tree (+frac.c.)
$O(\sqrt{n} + k)$ time	>	$O(\log n + k)$ time
$O(n)$ space	<	$O(n \log n)$ space

n = number of points

k = number of reported points

References

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- [Havran] Vlastimil Havran, Materiály k předmětu Datové struktury pro počítačovou grafiku, přednáška č. 6, Proximity search and its Applications 1, CTU FEL, 2007