GEOMETRIC SEARCHING PART 1: POINT LOCATION in 2D

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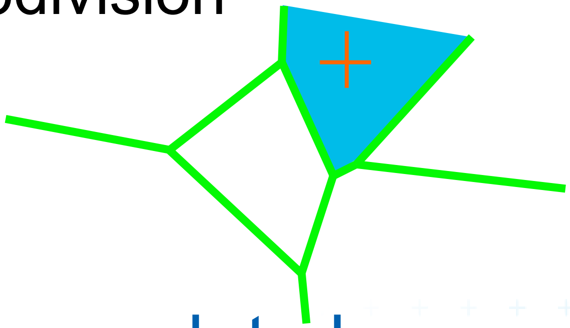
https://cw.felk.cvut.cz/doku.php/courses/a4m39vg/start

Based on [Berg] and [Mount]

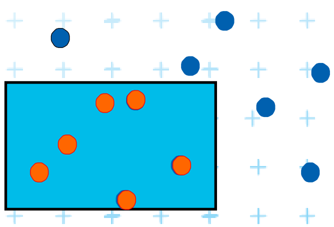
Version from 04.10.2023

Geometric searching problems

1. Point location (static) – Where am I?
   – (Find the name of the state, pointed by mouse cursor)
   – Search space S: a planar (spatial) subdivision
   – Query: point Q
   – Answer: region containing Q



2. Orthogonal range searching – Query a data base (Find points, located in d-dimensional axis-parallel box)
   – Search space S: a set of points
   – Query: set of orthogonal intervals q
   – Answer: subset of points in the box
   – (Was studied in DPG)

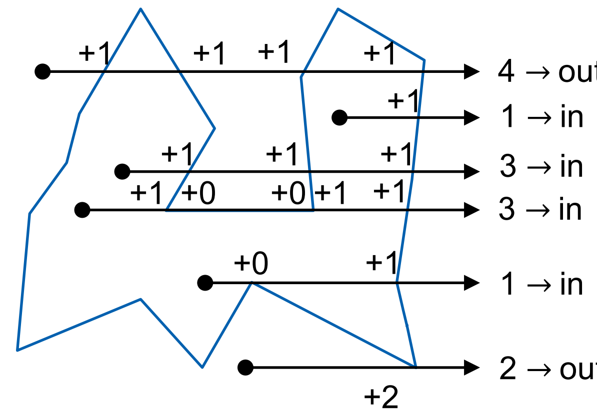


Part 1: Point location

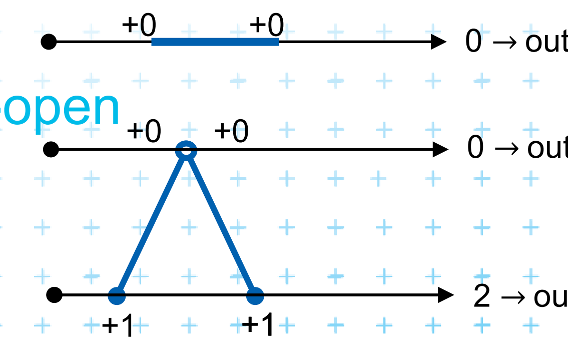
- Point location in polygon
- Planar subdivision
- DCEL data structure
- Point location in planar subdivision
  - slabs
  - monotone sequence
  - trapezoidal map

Point location in polygon by ray crossing

1. Ray crossing - O(n)
   – Compute number *t* of ray intersections with polygon edges (e.g., ray X+ after point moved to origin)
   – If odd(*t*) then inside
     else out

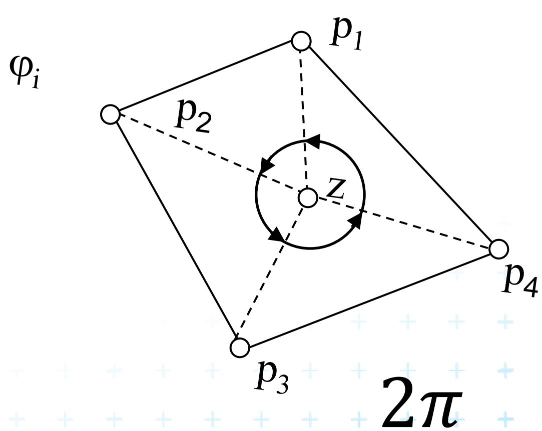


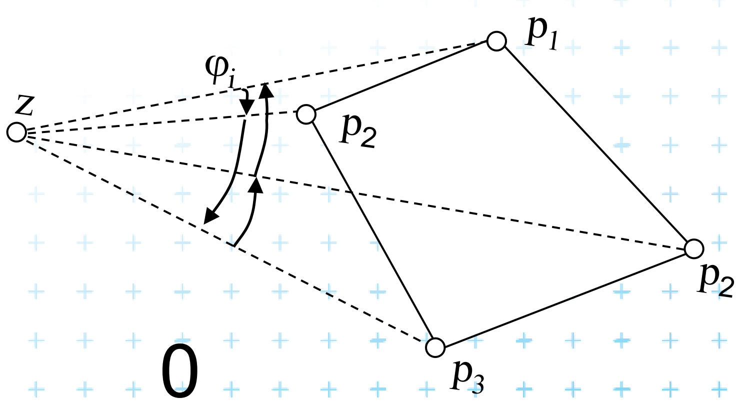
   – Singular cases must be handled!
     • Do not count horizontal line segments
     • Take non-horizontal segments as half-open (upper point not part of the segment)



Point location in polygon

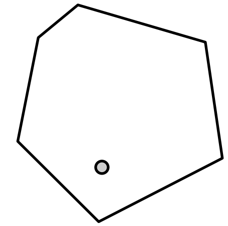
2. Winding number - $O(n)$
   (number of turns around the point)
   - Sum oriented angles $\varphi_i = \sphericalangle(p_i, z, p_{i+1})$
   - If $(\sum\varphi_i = 2\pi)$ then inside (1 turn)
   - If $(\sum\varphi_i = 0)$ then outside (no turn)
   - About 20-times slower than ray crossing



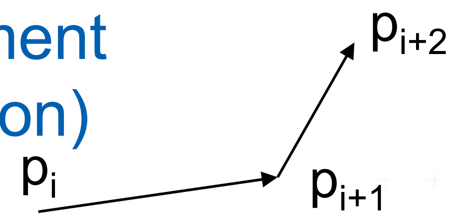


Point location in convex polygon

3. Position relative to all edges
   - For convex polygons
   - If (left from all edges) then inside



- Position of point in relation to the line segment (Determination of convex polygon orientation)



Convex polygon, non-collinear points

$p_i = [x_i, y_i, 1], \;\; p_{i+1} = [x_{i+1}, y_{i+1}, 1], \;\; p_{i+2} = [x_{i+2}, y_{i+2}, 1]$

$$\begin{vmatrix} x_i & y_i & 1 \\ x_{i+1} & y_{i+1} & 1 \\ x_{i+2} & y_{i+2} & 1 \end{vmatrix}$$

> 0 => point left from edge (for CCW polygon)

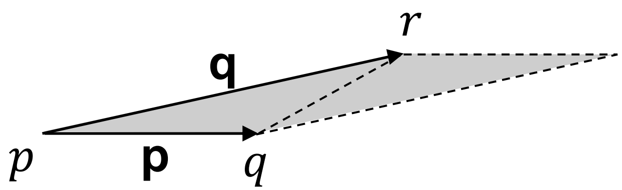
< 0 => point right from edge (for CW polygon)

Area of Triangle

$$T = \frac{1}{2}|\mathbf{p} \times \mathbf{q}|$$

$$\mathbf{p} = q - p$$

$$\mathbf{q} = r - p$$



$$2T = \mathbf{p}_x\mathbf{q}_y - \mathbf{p}_y\mathbf{q}_x$$

using vector product $\mathbf{p} \times \mathbf{q}$

$$2T = \begin{vmatrix} p_x & p_y & 1 \\ q_x & q_y & 1 \\ r_x & r_y & 1 \end{vmatrix}$$

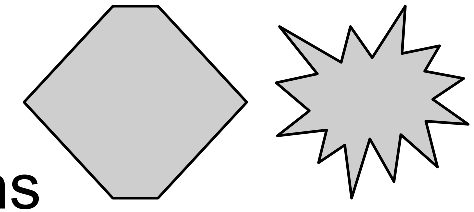
using coordinates of points

Orientation is computed as $\mathrm{sign}(2\mathrm{T}) =$

$$= \mathrm{sign}(p_x q_y + q_x r_y + r_x p_y - p_x r_y - q_x p_y - r_x q_y)$$

$$= \mathrm{sign}\left((q_x - p_x)(r_y - p_y) - (q_y - p_y)(r_x - p_x)\right)$$ for pivot p

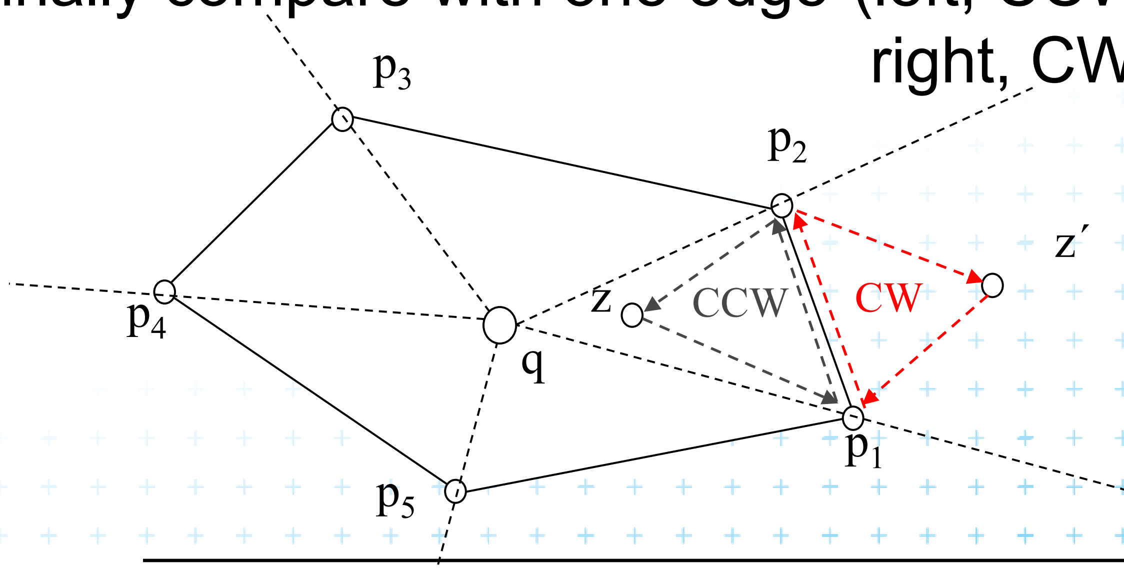
Point location in polygon



4. Binary search in angles

Works for convex and star-shaped polygons

1. Choose any point *q* inside / in the polygon core
2. *q* forms wedges with polygon edges
3. Binary search of wedge výseč based on angle
4. Finally compare with one edge (left, CCW => in, right, CW => out)

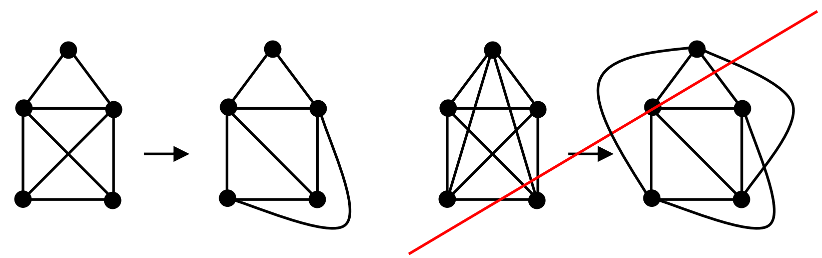


Planar graph

Planar graph

U=set of nodes, H=set of arcs

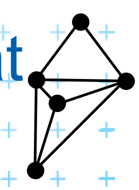
= Graph G = (U,H) is planar, if it can be embedded into plane without crossings



Planar embedding of planar graph G = (U,H)

= mapping of each *node in U to vertex* in the plane and each *arc in H into simple curve (edge)* between the two images of extreme nodes of the arc, so that **no** two **images of arc intersect** except at their endpoints

Every planar graph can be embedded in such a way that arcs map to straight line segments [Fáry 1948]

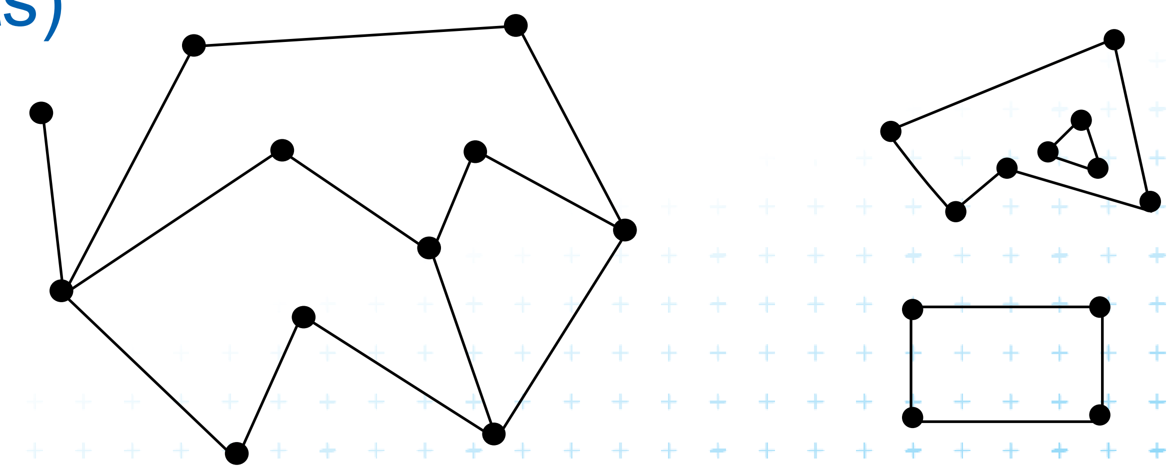


=> Planar Straight Line Graph

Planar subdivision

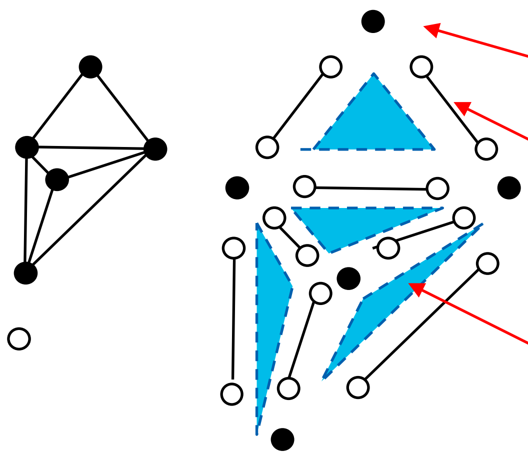
= Partition of the plane determined by straight line planar embedding of a planar graph.
Also called PSLG – Planar Straight Line Graph

- (embedding of a planar graph in the plane such that its arcs are mapped into straight line segments)



connected disconnected

Planar subdivision



Vertex = embedding of graph node

Edge = embedding of graph arc
(open – without vertices)

Face = maximal connected subset of a plane that doesn't contain points on edges nor vertices (open polygonal region whose boundary is formed by edges and vertices from the subdivision)

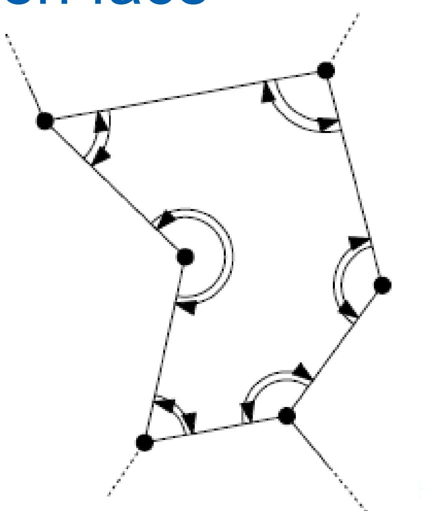
Complexity (size) of a subdivision = sum of number of vertices +
+ number of edges +
+ number of faces it consists of

Euler's formula: $|V| - |E| + |F| \geq 2$
$|V| = n,\ |E| \leq 3v - 6,\ |F| \leq 2v - 4$
} $O(n)$ data structure

DCEL = Double Connected Edge List [Eastman 1982]

- A structure for storage of planar subdivision
- Operations like:

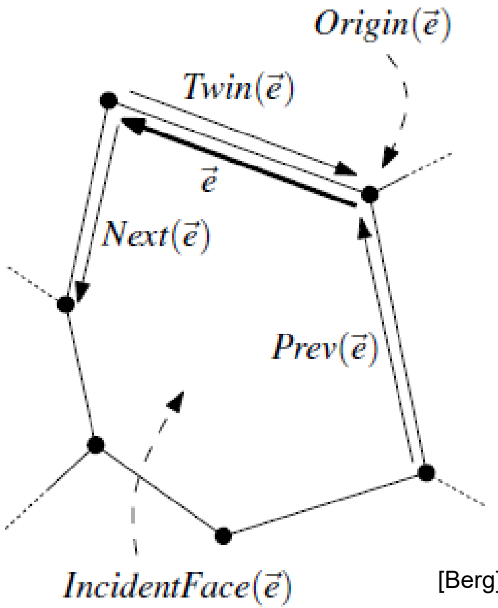
Walk around boundary of a given face



[Berg]

Pointers to next and prev edge

Get incident face

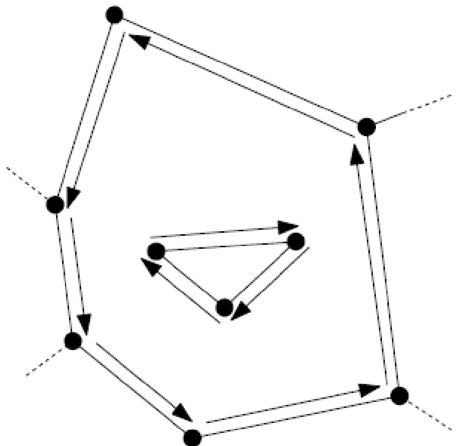


[Berg]

Half-edge, op. Twin(e), unique Next(e), Prev(e)

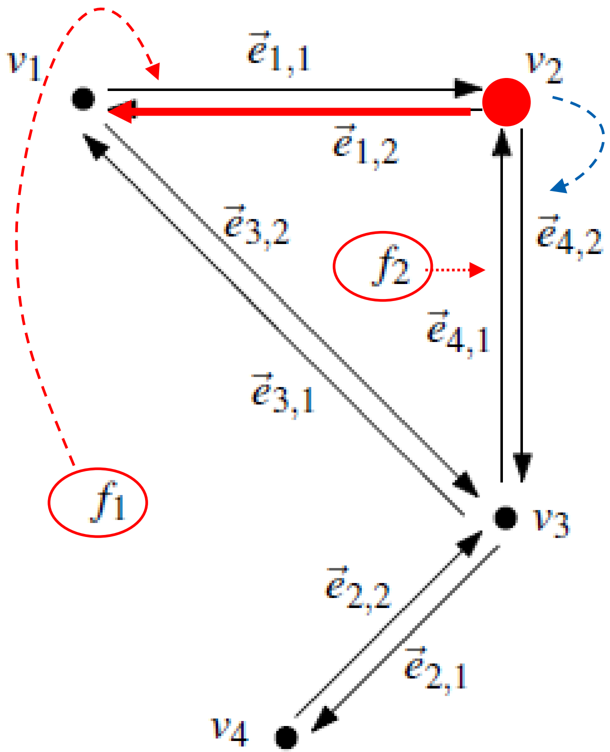
DCEL = Double Connected Edge List

- Vertex record v
  - Coordinates(v) and pointer to one IncidentEdge(v)
- Face record f
  - OuterComponent(f) pointer (boundary)
  - List of holes – InnerComponent(f)
- Half-edge record e
  - Origin(e), Twin(e), IncidentFace(e)
  - Next(e), Prev(e)
  - [Dest(e) = Origin(Twin(e))]
- Possible attribute data for each



[Berg]

DCEL = Double Connected Edge List



G

Vertex	Coordinates	IncidentEdge
v_1	$(0,4)$	$\vec{e}_{1,1}$
v_2	$(2,4)$	$\vec{e}_{4,2}$
v_3	$(2,2)$	$\vec{e}_{2,1}$
v_4	$(1,1)$	$\vec{e}_{2,2}$

One of edges

List of holes

T

Face	OuterComponent	InnerComponents
f_1	**nil**	$\vec{e}_{1,1}$
f_2	$\vec{e}_{4,1}$	**nil**

T

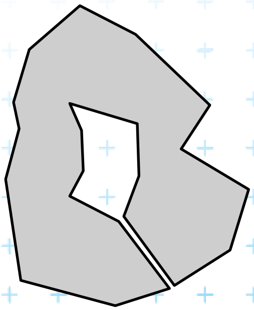
Half-edge	Origin	Twin	IncidentFace	Next	Prev
$\vec{e}_{1,1}$	v_1	$\vec{e}_{1,2}$	f_1	$\vec{e}_{4,2}$	$\vec{e}_{3,1}$
$\vec{e}_{1,2}$	v_2	$\vec{e}_{1,1}$	f_2	$\vec{e}_{3,2}$	$\vec{e}_{4,1}$
$\vec{e}_{2,1}$	v_3	$\vec{e}_{2,2}$	f_1	$\vec{e}_{2,2}$	$\vec{e}_{4,2}$
$\vec{e}_{2,2}$	v_4	$\vec{e}_{2,1}$	f_1	$\vec{e}_{3,1}$	$\vec{e}_{2,1}$
$\vec{e}_{3,1}$	v_3	$\vec{e}_{3,2}$	f_1	$\vec{e}_{1,1}$	$\vec{e}_{2,2}$
$\vec{e}_{3,2}$	v_1	$\vec{e}_{3,1}$	f_2	$\vec{e}_{4,1}$	$\vec{e}_{1,2}$
$\vec{e}_{4,1}$	v_3	$\vec{e}_{4,2}$	f_2	$\vec{e}_{1,2}$	$\vec{e}_{3,2}$
$\vec{e}_{4,2}$	v_2	$\vec{e}_{4,1}$	f_1	$\vec{e}_{2,1}$	$\vec{e}_{1,1}$

[Berg]

G – geometry
T – topology

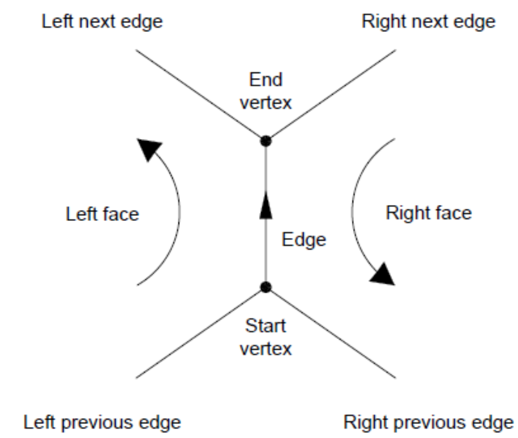
DCEL simplifications

- If no operations with vertices and no attributes
  - No vertex table (no separate vertex records)
  - Store vertex coords in half-edge origin (in the half-edge table)
- If no need for faces (e.g. river network)
  - No face record and no IncidentFace() field (in the half-edge table)
- If only connected subdivision allowed
  - Join holes with rest by dummy edges
  - Visit all half-edges by simple graph traversal
  - No InnerComponent() list for faces

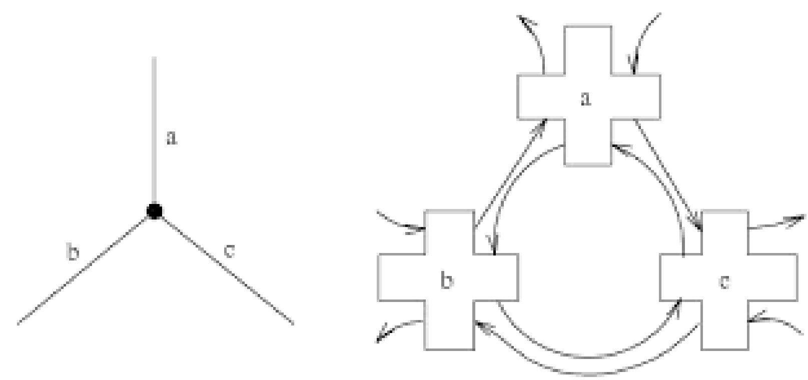


Other structures for representing PSLG

https://i.stack.imgur.com/yadGo.png

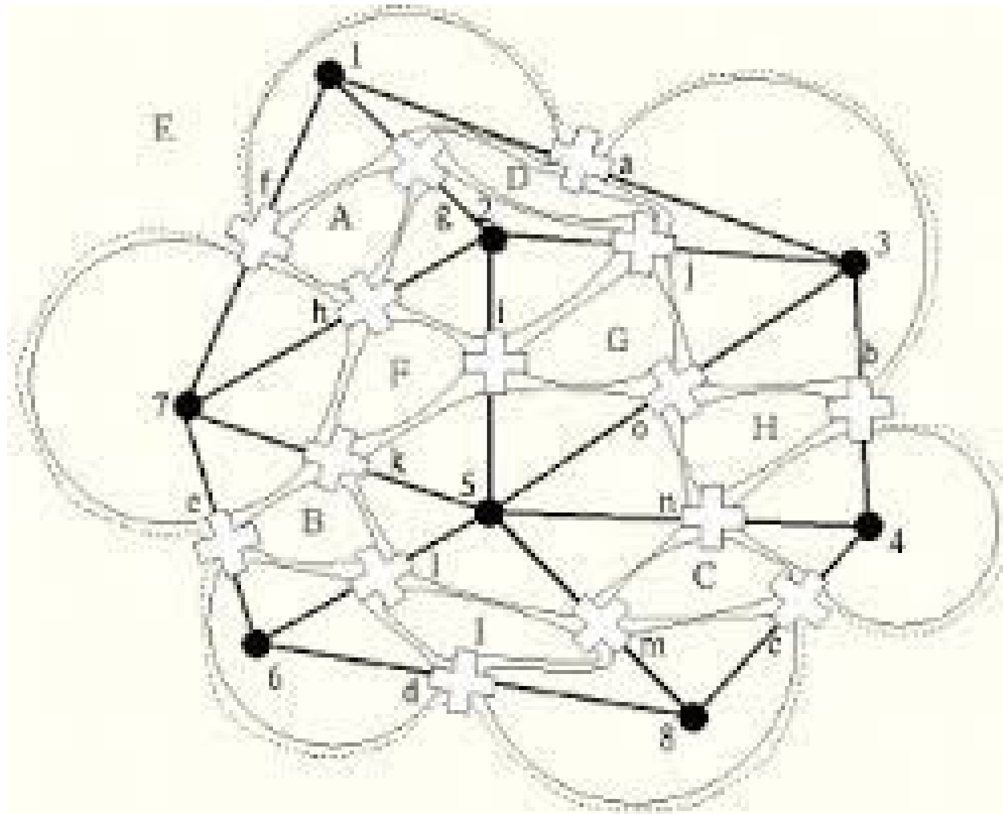


- **Winged edge** [Baumgart 1975]
  - The oldest, complicated manipulation
  - Randomly stored edge direction around faces
- **Quad edge** [Guibas & Stolfi 1985]
  - Stores PSLG and its dual
  - Pointers to edges
    - Around vertex
    - Around face



  - E.g., for Voronoi diagrams & Delaunay triangulations

Quad edge



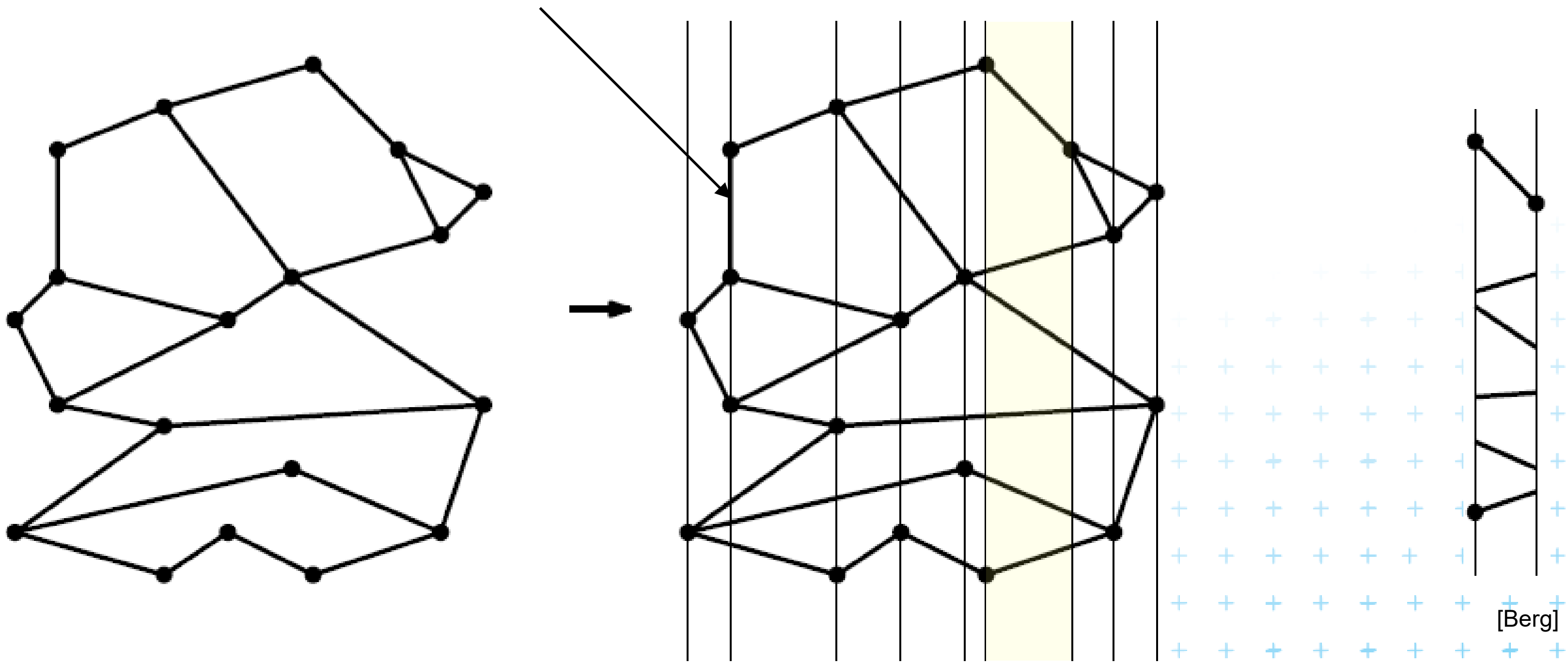
Point location in planar subdivision

- Using special search structures
  an optimal algorithm can be made with
  - O(n) preprocessing,
  - O(n) memory and
  - O(log n) query time.
- Simpler methods
  1. Slabs — O(log n) query, $O(n^2)$ memory
  2. monotone chain tree — $O(\log^2 n)$ query, $O(n^2)$ memory
  3. trapezoidal map — O(log n) query expected time
     O(n) expected memory

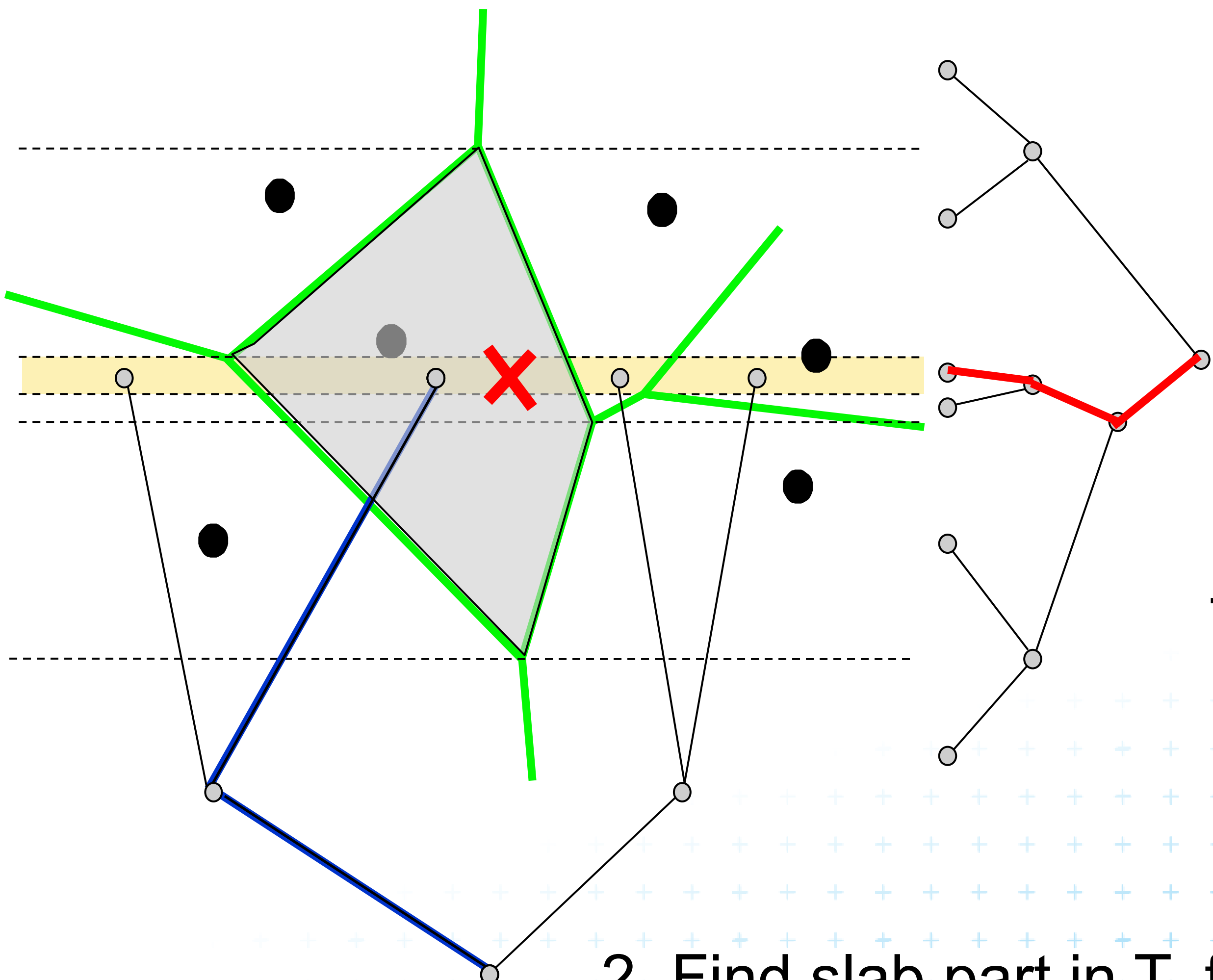
1. Vertical (horizontal) slabs

[Dobkin and Lipton, 1976]

- Draw vertical or horizontal lines through vertices
- It partitions the plane into vertical slabs
  - Avoid points with same x coordinate (to be solved later)



Horizontal slabs example



1. Find slab in T_y for y

T_x and T_y are arrays

2. Find slab part in T_x for x

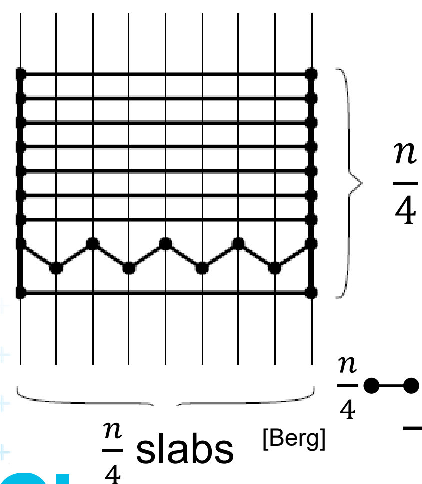
Horizontal slabs complexity

- Query time $O(\log n)$

  $O(\log n)$ time in slab array T_y (size max 2n endpoints)

  + $O(\log n)$ time in slab array T_x (slab crossed max by n edges)

- Memory $O(n^2)$
  - Slabs: Array with y-coordinates of vertices … $O(n)$
  - For each slab $O(n)$ edges intersecting the slab



$O(n^2)$ construction

$O(\log n)$ query

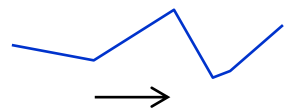
$O(n^2)$ memory

$\frac{n}{4} + 2\frac{n}{4} + \frac{n}{4} = O(n)$ edges $\quad \frac{n}{4} * \frac{n}{4} = O(n^2)$ faces

2. Monotone chain tree

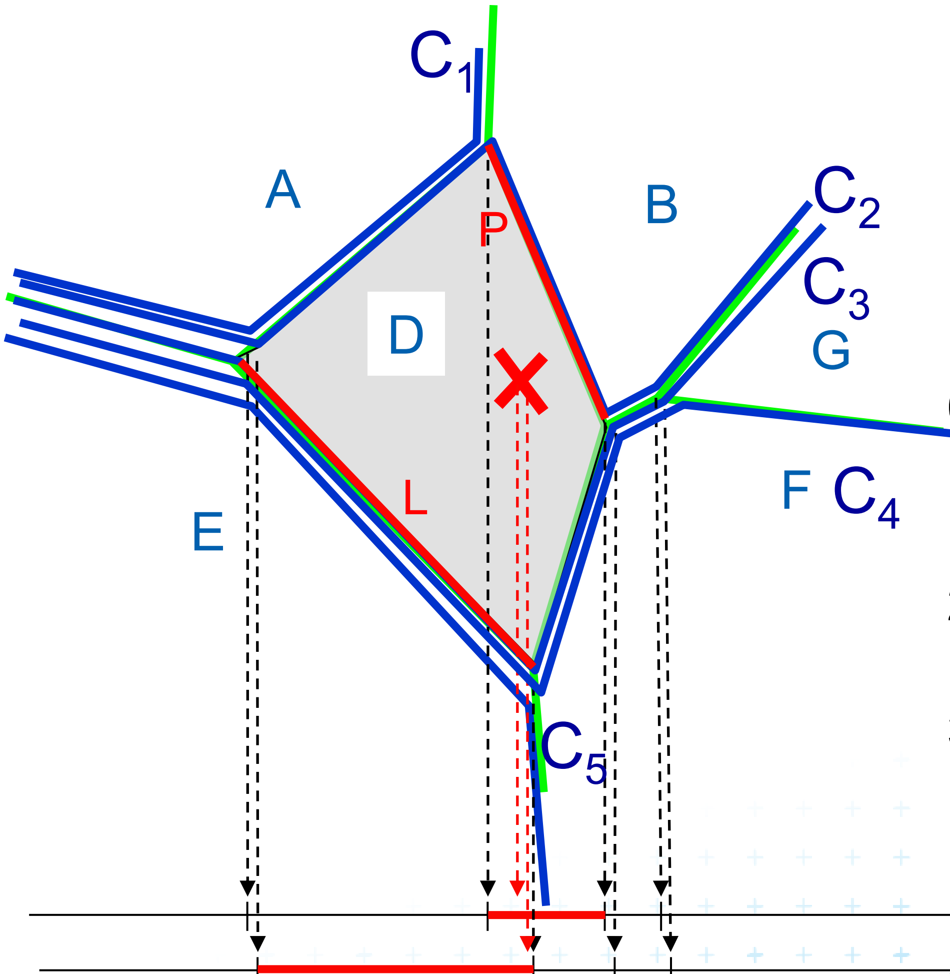
[Lee and Preparata, 1977]

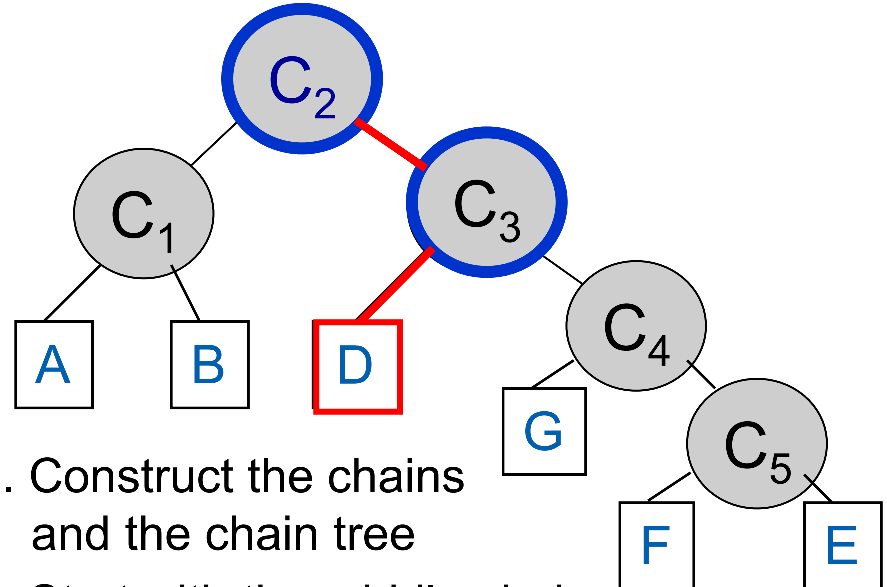
- Construct monotone planar subdivision
  - The edges are all monotone in the same direction



- Each separator chain
  - is monotone (can be projected to line and searched)
  - splits the plane into two parts – allows binary search
- Algorithm
  - Preprocess: Find the separators (e.g., horizontal)
  - Search:
    - Binary search among separators (Y) … $O(\log n)$ times
      - Binary search along the separator (X) … $O(\log n)$
    - $O(\log^2 n)$ query
  - Not optimal, but simple
  - Can be made optimal, but the algorithm and data structures are complicated
    - $O(n^2)$ memory

Monotone chain tree example

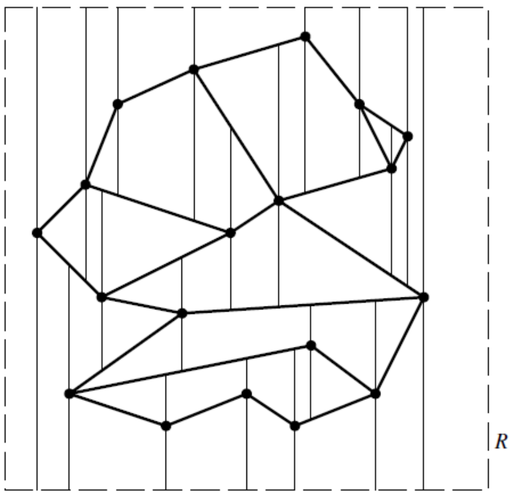




0. Construct the chains and the chain tree
1. Start with the middle chain
2. Find projection of *x* in the projection of the chain – determine the segment
3. Identify position of x in relation to the segment – Left or Right (This is the position of *x* relatively to the whole chain)
4. Continue in L or R chain -> goto 2. or stop if in the leaf

3. Trapezoidal map (TM) search

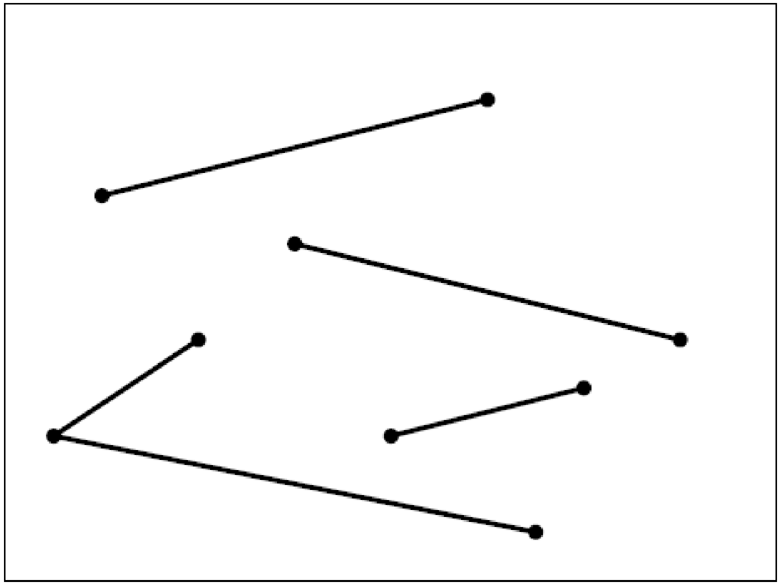
- The simplest and most practical known optimal algorithm
- Randomized algorithm with $O(n)$ expected storage and $O(\log n)$ expected query time
- Expectation depends on the random order of segments during construction, not on the position of the segments
- TM is refinement of original subdivision
- Converts complex shapes into simple ones
- Weaker assumption on input:
  - Input individual segments, not polygons
  - $S = \{s_1, s_2, \dots, s_n\}$
  - S_i subset of first i segments
  - Answer: segment below the pointed trapezoid (Δ)



[Berg]

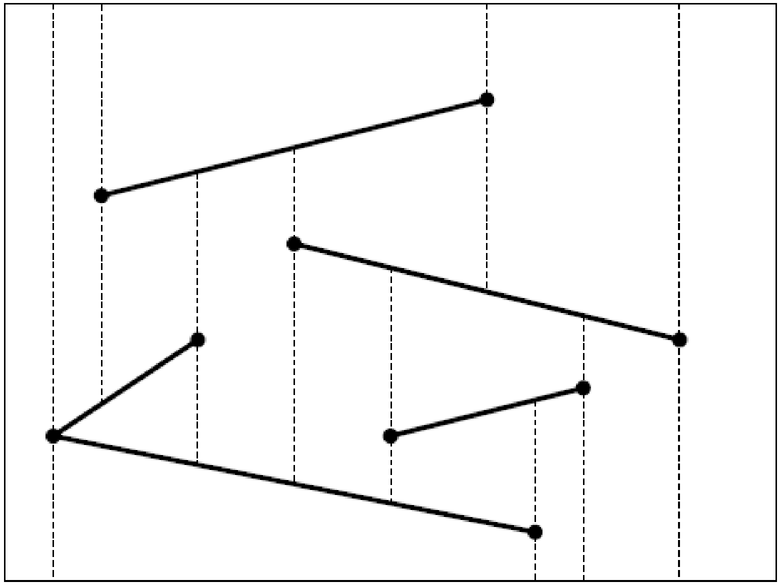
Trapezoidal map of line segments in general position

Input: individual segments *S*



Constru-
ction

Trapezoidal map *T*



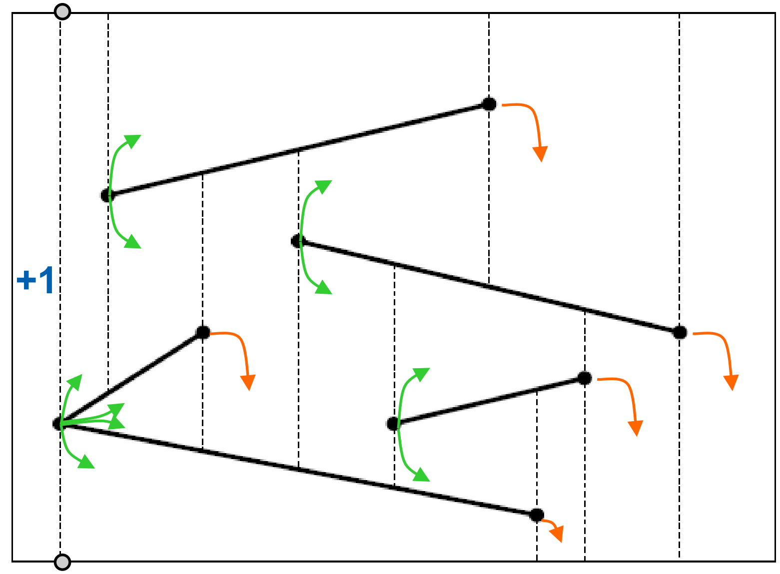
[Mount]

- They do not intersect, except in endpoints
- No vertical segments
- No 2 distinct endpoints with the same x-coordinate

- Bounding rectangle
- 4 Bullets up and down
- Stop on input segment or on bounding rectangle

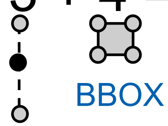
Trapezoidal map of line segments in general position

- Faces are trapezoids ∆ with vertical sides
- Given n segments, TM has
  - at most 6n+4 vertices
  - at most 3n+1 trapezoids



[Mount]

- Proof:
  - each endpoint 2 bullets -> 1+2 points
  - 2n endpoints * 3 + 4 = 6n+4 vertices

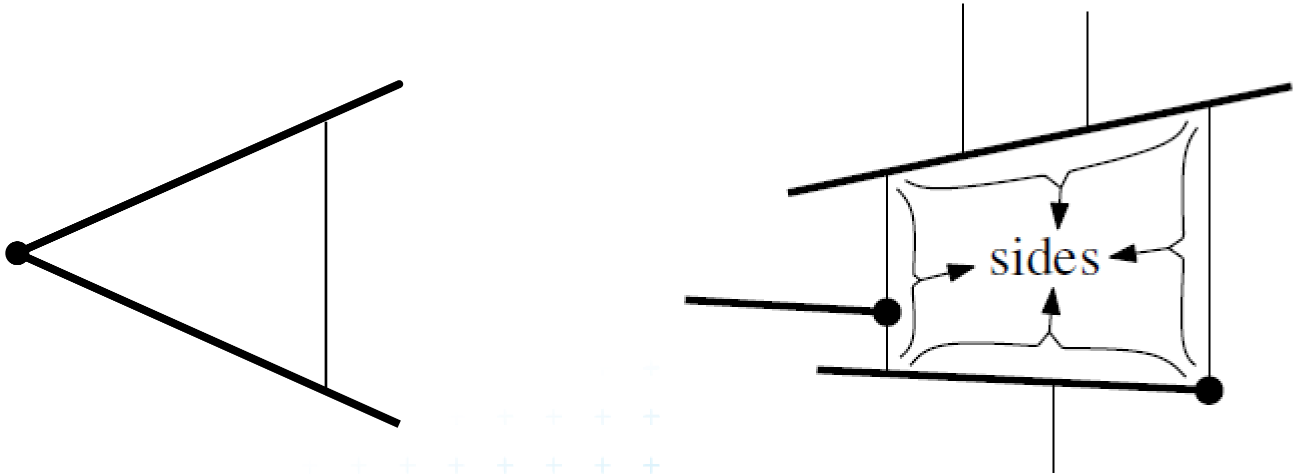


  - start point –> max 2 trapezoids ∆
  - end point –> 1 trapezoid ∆
  - 3 * (n segments) + 1 left ∆ => max 3n+1 ∆

Trapezoidal map of line segments in general position

Each face has

- one or two vertical sides (trapezoid or triangle) and
- exactly two non-vertical sides



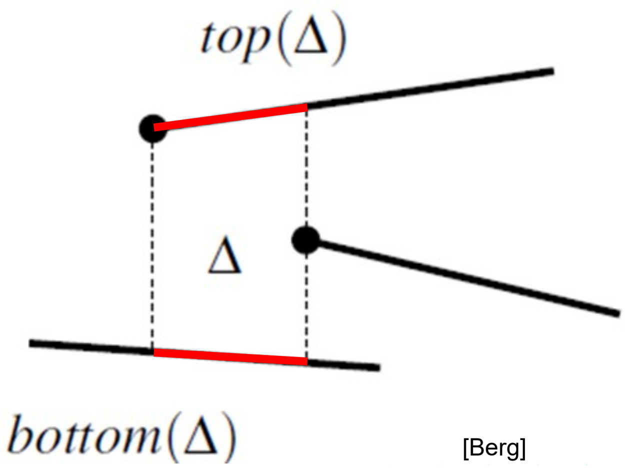
One vertical side   Two vertical sides

[Berg]

Two non-vertical sides

Non-vertical side or

- is contained in one of the segments of set *S*
- or in the horizontal edge of bounding rectangle *R*



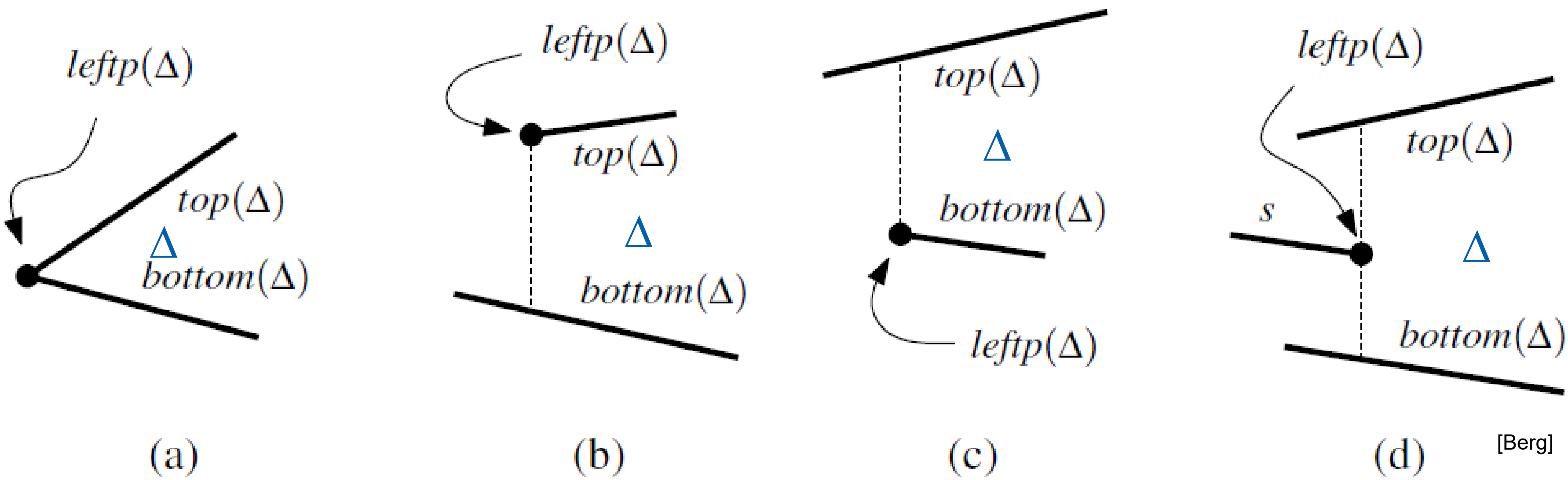
[Berg]

segments:

top(Δ) - bounds from above

bottom(Δ) - bounds from below

Vertical sides – left vertical side of Δ



Left vertical side is defined by the segment end-point p=*leftp*(Δ)

(a) common left point p itself

(b) by the lower vert. extension of left point p ending at bottom()

(c) by the upper vert. extension of left point p ending at top()

(d) by both vert. extensions of the right point p

(e) the left edge of the bounding rectangle R (leftmost Δ only)

Vertical sides - summary

Vertical edges are defined by segment endpoints

- *leftp*(Δ) = the end point defining the left edge of Δ
- *rightp*(Δ) = the end point defining the right edge of Δ

leftp(Δ) is

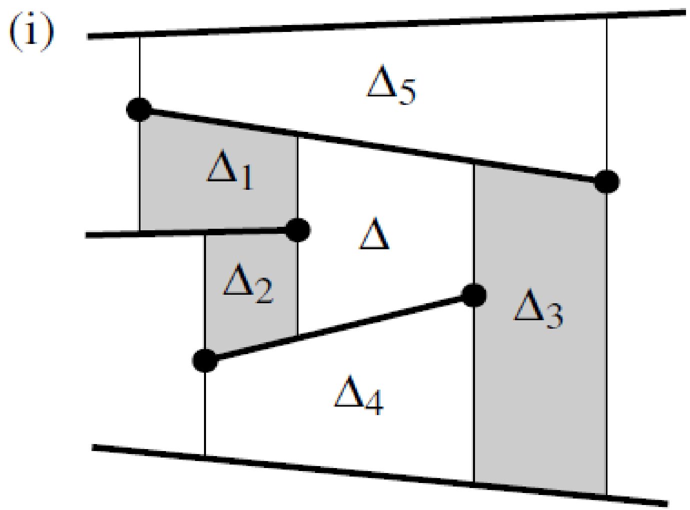
- the left endpoint of *top*() or *bottom*() or both (b, c, a)
- the right point of a third segment (d)
- the lower left corner of the bounding rectangle R (e)

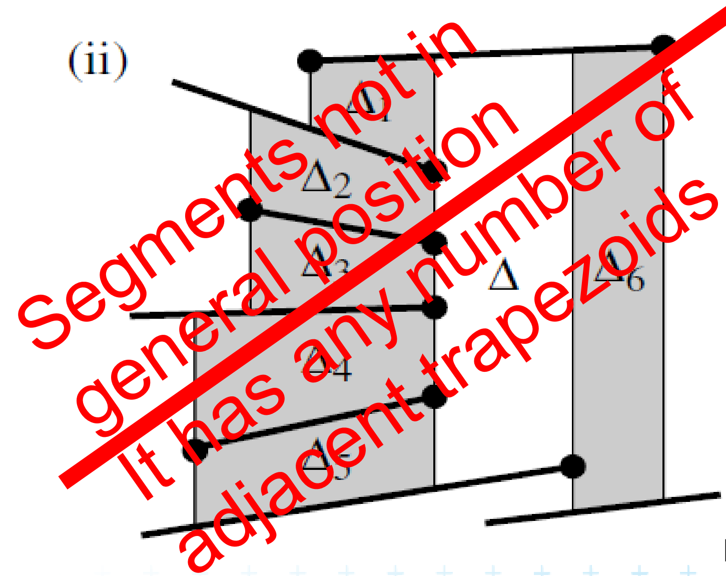
Trapezoid Δ

- Trapezoid Δ is uniquely defined by
  - the segments *top*(Δ), *bottom*(Δ)
  - And by the endpoints *leftp*(Δ), *rightp*(Δ)

Adjacency of trapezoids segments in general position

- Trapezoids Δ and Δ' are adjacent, if they meet along a vertical edge





[Berg]

- Δ_1= upper left neighbor of Δ (common *top*(Δ) edge)
- Δ_2 = lower left neighbor of Δ (common *bottom*(Δ))
- Δ_3 is a right neighbor of Δ (common *top*(Δ) or *bottom*(Δ))

Representation of the trapezoidal map *T*

Special trapezoidal map structure $T(S)$ stores:

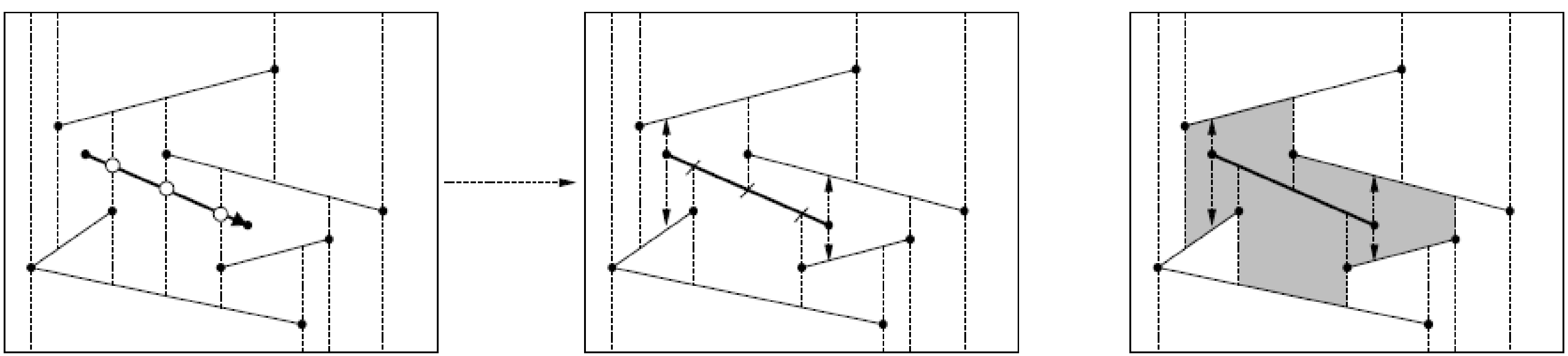
- Records for all line segments and end points
- Records for each trapezoid $\Delta \in T(S)$
  - Definition of Δ - pointers to segments *top*(Δ), *bottom*(Δ),
    - pointers to points *leftp*(Δ), *rightp*(Δ)
  - Pointers to its max four neighboring trapezoids
  - Pointer to the leaf A in the search structure *D* (see below)

- Does not store the geometry explicitly!
- Geometry of trapezoids is computed in O(1)

Construction of trapezoidal map

- Randomized incremental algorithm
  1. Create the initial bounding rectangle ($T_0 = 1\Delta$) … $O(n)$
  2. Randomize the order of segments in S
  3. for i = 1 to n do
  4. Add segment S_i to trapezoidal map T_i
  5. locate left endpoint of S_i in T_{i-1} ⇒ start trapezoid
  6. find intersected trapezoids
  7. shoot 4 bullets from endpoints of S_i ⇒ create new trapezoids
  8. trim intersected vertical bullet paths

[Mount]



Locate left endpoint and determine intersections — Shoot new bullet paths and trim intersecting rays — Newly created trapezoids

Trapezoidal map point location

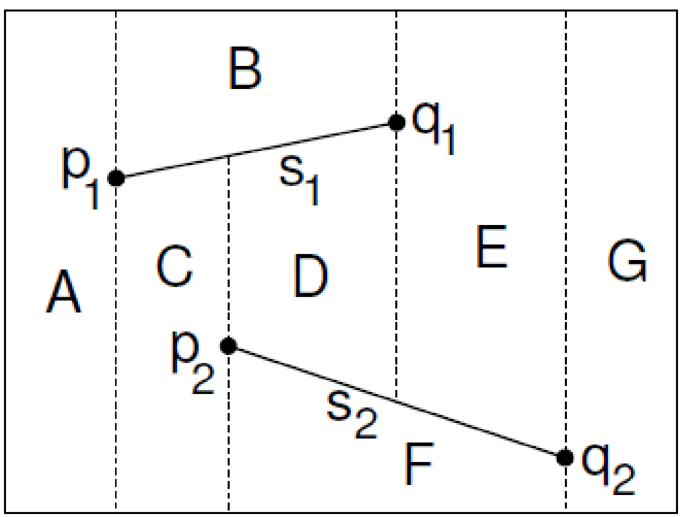
- While creating the trapezoidal map *T* construct the *Point location data structure D*
- Query this data structure

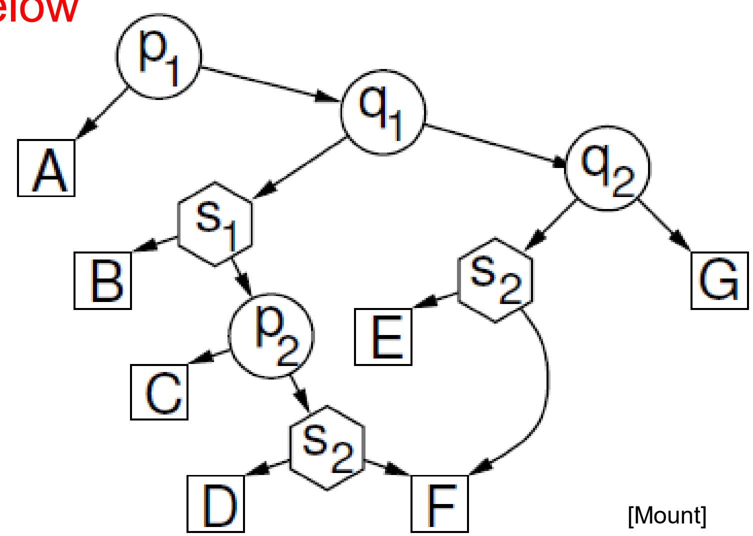
Point location data structure D

- Rooted directed acyclic graph (not a tree!!)
  - Leaves A – trapezoids, each appears exactly once
  - Internal nodes – 2 outgoing edges, guide the search

p_1 x-node – x-coord x_0 of segment start- or end-point
left child lies left of vertical line $x=x_0$
right child lies right of vertical line $x=x_0$
– used first to detect the vertical slab

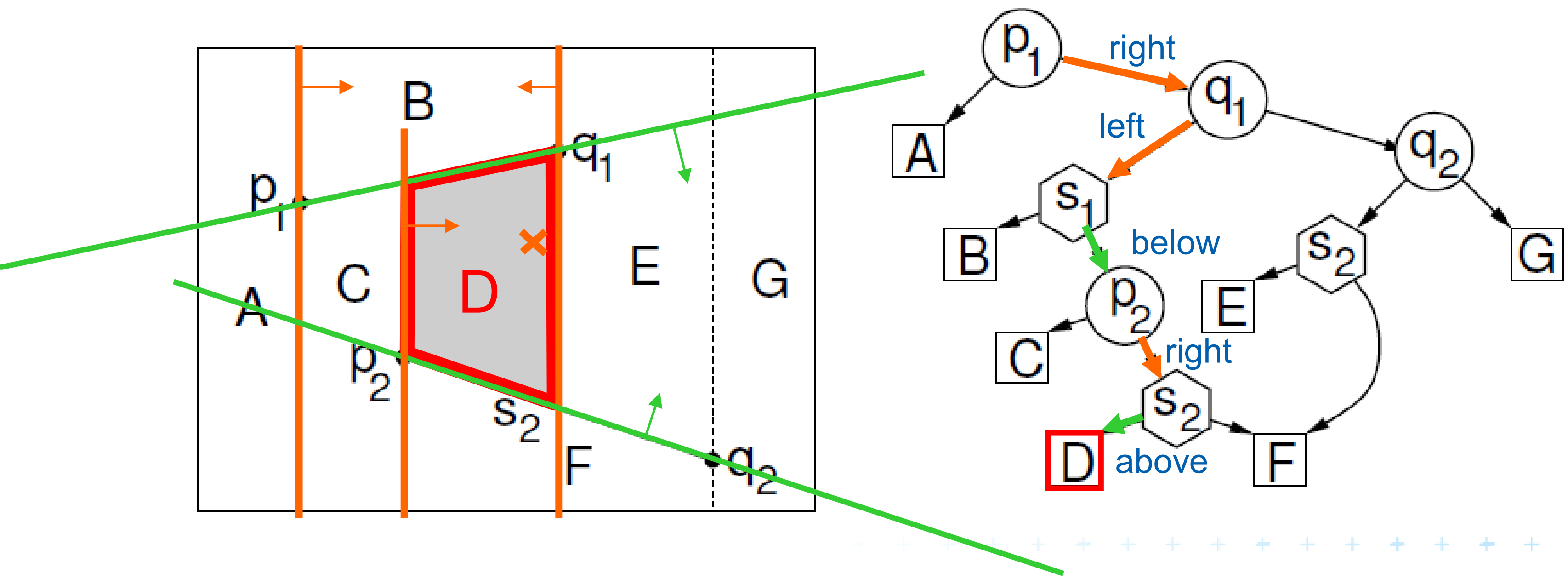
s_1 y-node – pointer to the line segment of the subdivision (not only its y!!!)
left – above, right – below





[Mount]

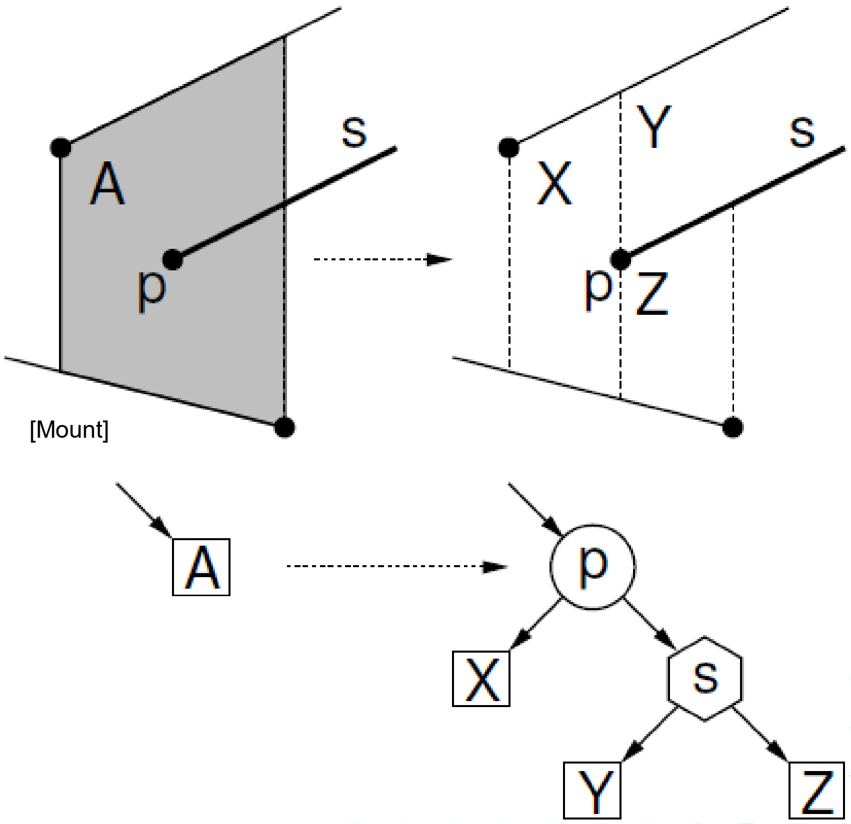
TM search example



[Mount]

Construction – addition of a segment

a) Single (left or right) endpoint - 3 new trapezoids

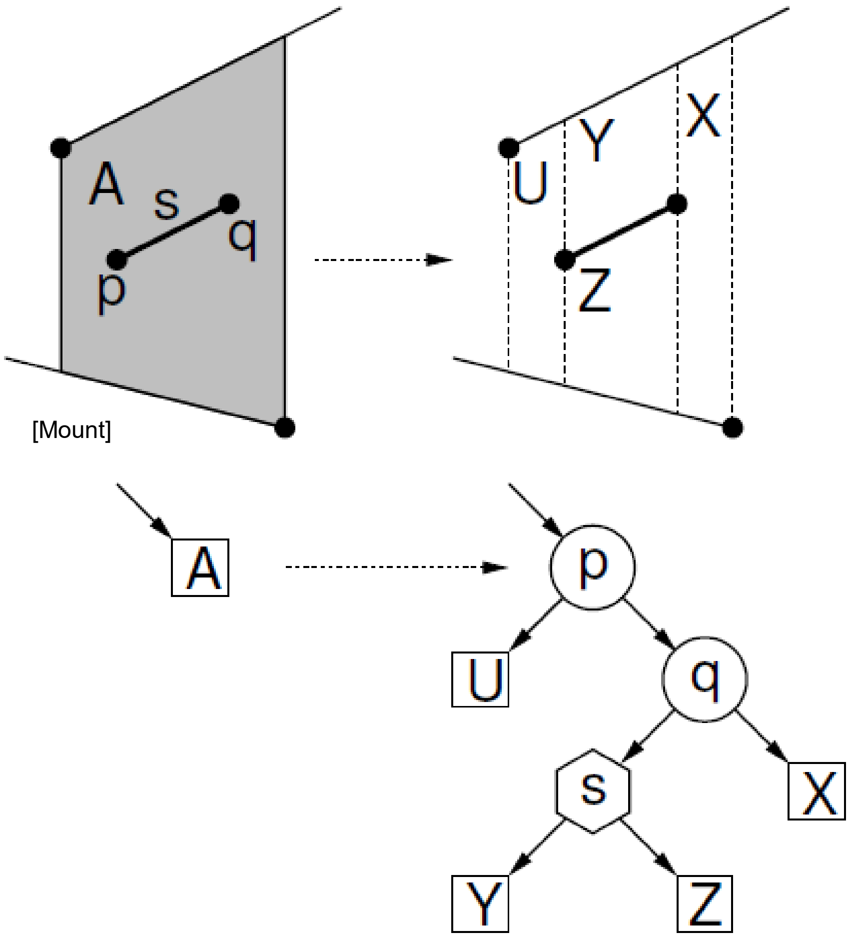


Trapezoid A replaced by

- * x-node for point *p*
- add left leaf for X Δ
- add right subtree
- * y-node for segment *s*
- add left leaf for Y Δ above
- add right leaf Z Δ below

Construction – addition of a segment

b) Two segment endpoints – 4 new trapezoids

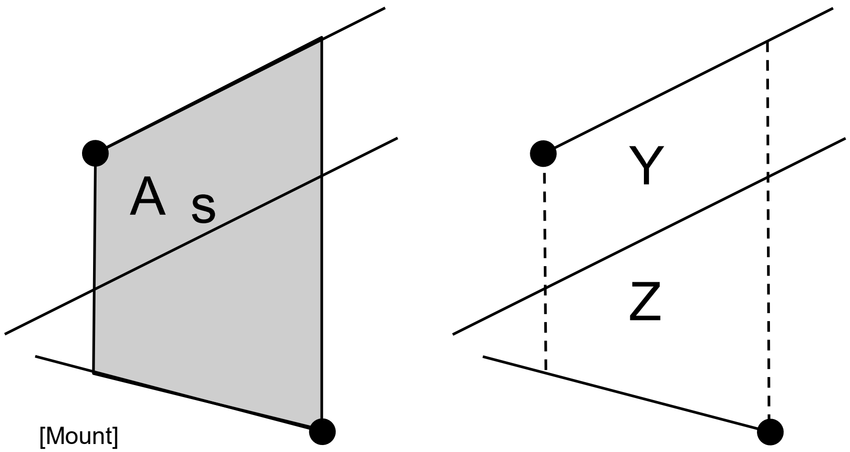


Trapezoid A replaced by

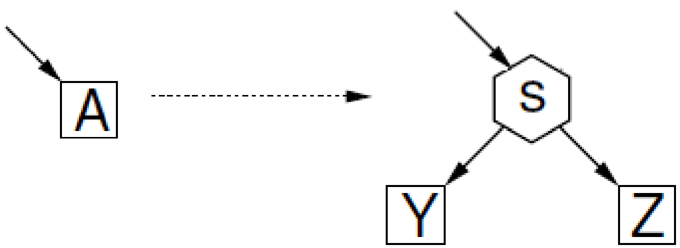
- * x-node for point *p*
- * x-node for point *q*
- * y-node for segment *s*
- add leaves for U, X, Y, Z

Construction – addition of a segment

c) No segment endpoint – create 2 trapezoids



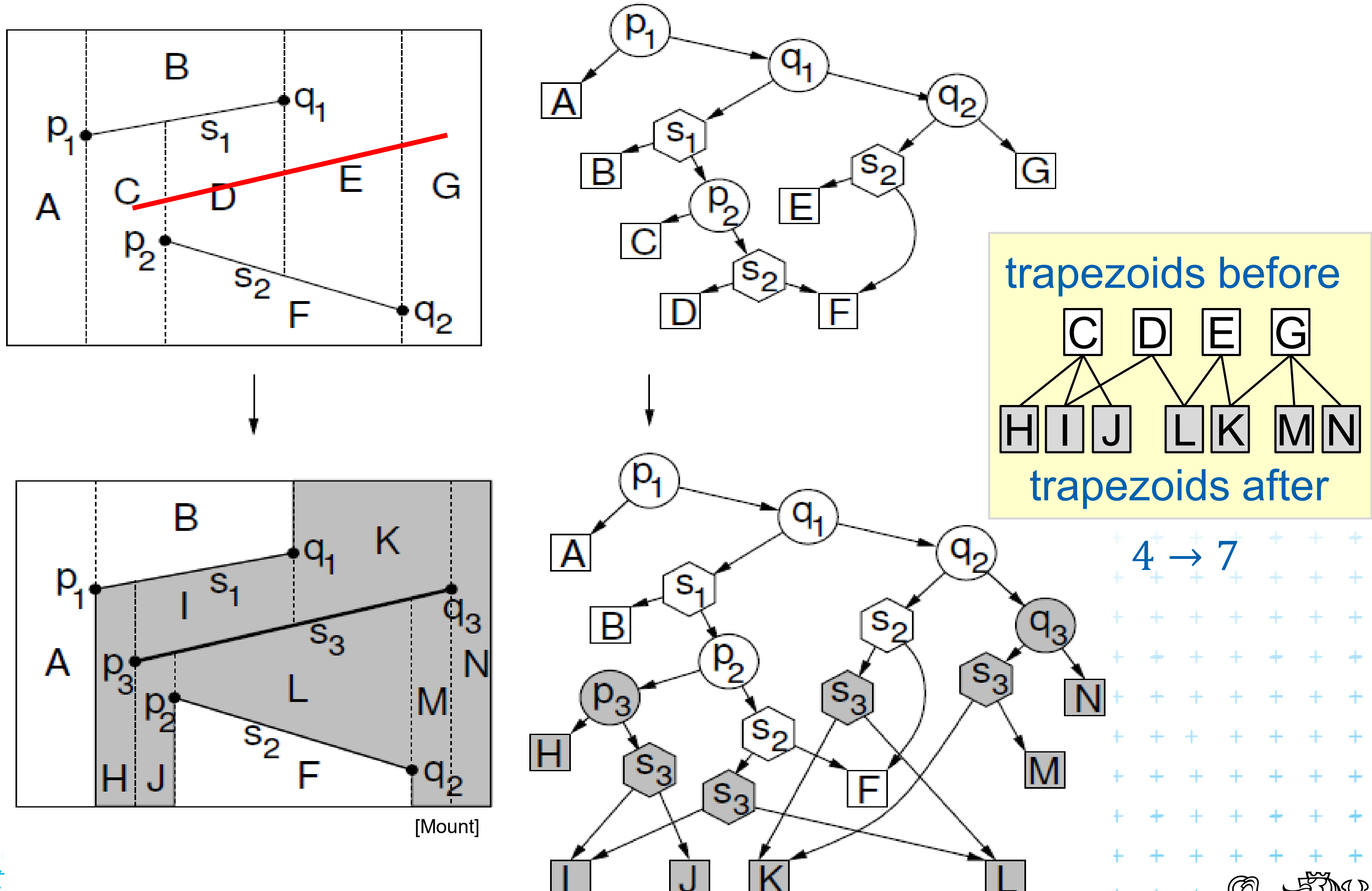
[Mount]



Trapezoid A replaced by

- * y-node for segment *s*
- add leaves for Y, Z

Segment insertion example



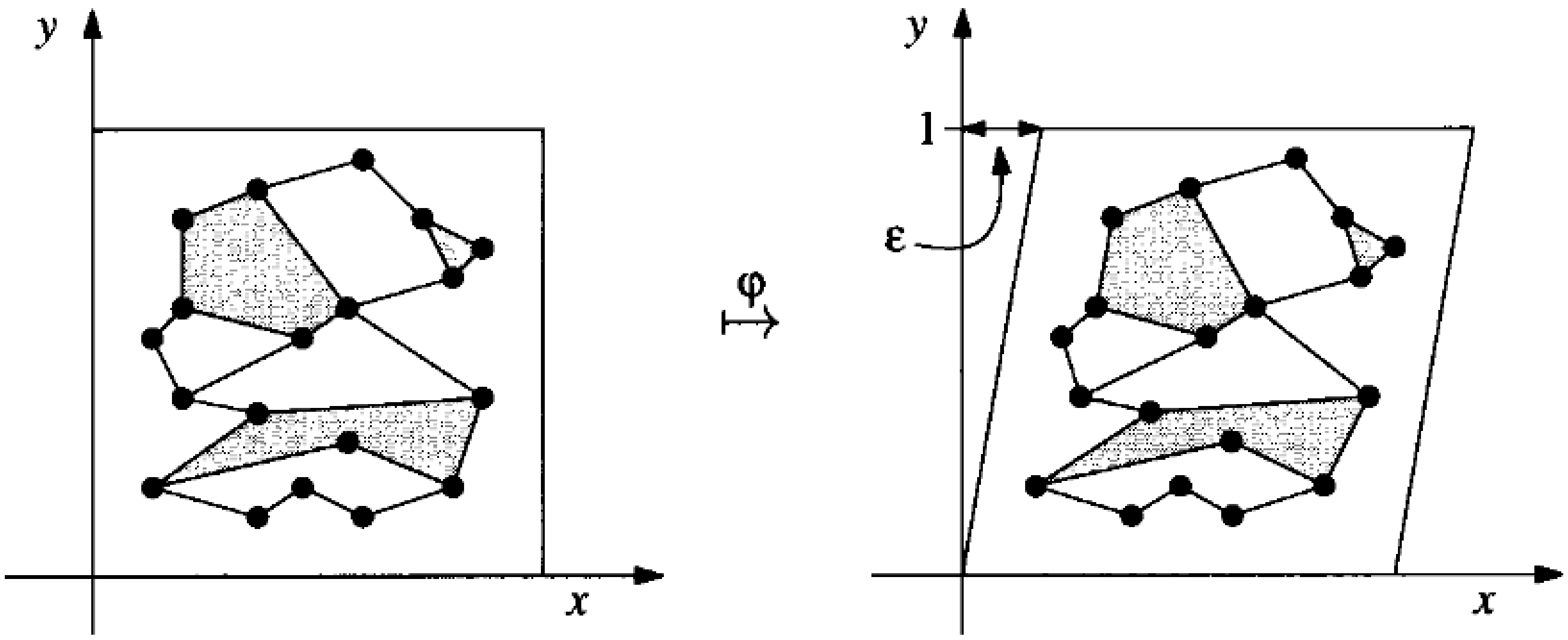
[Mount]

Analysis and proofs

- This holds:
  - Number of newly created Δ for inserted segment $O(1)$ (some added, some removed)
  - Search structure size is max $O(n^2)$, but $O(n)$ expected
  - Search point $O(\log n)$ in average
    => Expected construction $O(n(1 + \log n)) = O(n \log n)$
- For detailed analysis and proofs see
  - [Berg] or [Mount]

Handling of degenerate cases - principle

- No distinct endpoints lie on common vertical line
  - Rotate or shear the coordinates $x' = x + \varepsilon y$, $y' = y$



[Berg]

Handling of degenerate cases - realization

- Trick
  - store original (x, y), not the sheared x', y'
  - we need to perform just 2 operations:

1. For two points p, q determine if transformed point q is to the left, to the right or on vertical line through point p
   - If $x_p = x_q$ then compare y_p and y_q (on only for $y_p = y_q$)
   - => use the original coords (x, y) and **lexicographic order**
2. For segment given by two points decide if 3rd point q lies above, below, or on the segment $p_1 p_2$
   - Mapping preserves this relation
   - => use the original coords (x, y)

Point location summary

- Slab method [Dobkin and Lipton, 1976]
  - $O(n^2)$ memory $O(\log n)$ time
- Monotone chain tree in planar subdivision [Lee and Preparata,77]
  - $O(n^2)$ memory $O(\log^2 n)$ time
- Layered directed acyclic graph (Layered DAG) in planar subdivision [Chazelle , Guibas, 1986] [Edelsbrunner, Guibas, and Stolfi, 1986]
  - $O(n)$ memory $O(\log n)$ time => optimal algorithm of planar subdivision search (optimal but complex alg. => see elsewhere)
- Trapezoidal map
  - $O(n)$ expected memory $O(\log n)$ expected time
  - $O(n \log n)$ expected preprocessing (simple alg.)

References

- [Berg] Mark de Berg, Otfried Cheong, Marc van Kreveld, Mark Overmars: Computational Geometry: *Algorithms and Applications*, Springer-Verlag, 3rd rev. ed. 2008. 386 pages, 370 fig. ISBN: 978-3-540-77973-5 http://www.cs.uu.nl/geobook/
- [Mount] Mount, D.: *Computational Geometry Lecture Notes for Fall 2016*, University of Maryland, Lectures 9, 10 http://www.cs.umd.edu/class/fall2016/cmsc754/Lects/cmsc754-fall16-lects.pdf