

Description Logics – Reasoning

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Outline

- 1 What can we conclude from description logics?
- 2 Inference problems
- 3 Inference Algorithms
 - Tableau Algorithm for \mathcal{ALC}



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What can we conclude from description logics?



Which clinical findings can occur on a head?

How: Get subclasses of $\textit{Finding} \sqcap \exists \textit{FindingSite} \cdot \textit{Head}$

e.g. *Heavyhead*, resulting from

$\textit{Headache} \equiv \textit{Pain} \sqcap \exists \textit{FindingSite} \cdot \textit{Head}$

$\textit{Pain} \sqsubseteq \textit{Finding}$

$\textit{HeavyHead} \sqsubseteq \textit{Headache}$



Which properties do I have to fill in when recording an allergic head?

How: For each property p check $AllergicHead \sqsubseteq \exists p \cdot T$

e.g. *FindingSite*, resulting from

$Pain \sqsubseteq \exists FindingSite \cdot T$

$ImmuneFunctionDisorder \sqsubseteq \exists PathologicalProcess \cdot T$

$AllergicHead \sqsubseteq Pain$

$AllergicHead \sqsubseteq ImmuneFunctionDisorder$



Is a Headache occurring in a Leg correct?

How: Check satisfiability of the concept $Headache \sqcap \exists FindingSite \cdot Leg$

No, because the concept is unsatisfiable, resulting from

$$Headache \sqsubseteq Pain \sqcap \exists FindingSite \cdot Head$$

$$Pain \sqsubseteq \leq 1 FindingSite \cdot T$$

$$Leg \sqsubseteq \neg Head$$


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- S is consistent, if S has at least one model



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Inference problems



Inference Problems in TBOX

We have introduced syntax and semantics of the language \mathcal{ALC} . Now, let's look on automated reasoning. Having a \mathcal{ALC} theory $\mathcal{K} = (\mathcal{T}, \mathcal{A})$. For TBOX \mathcal{T} and concepts $C_{(i)}$, we want to decide whether (unsatisfiability) concept C is *unsatisfiable*, i.e. $\mathcal{T} \models C \sqsubseteq \perp$?



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All these tasks can be reduced to unsatisfiability checking of a single concept ...



Reducing Subsumption to Unsatisfiability

Example

These reductions are straightforward – let's show, how to reduce subsumption checking to unsatisfiability checking. Reduction of other inference problems to unsatisfiability is analogous.

$$\begin{array}{ll}
 (\mathcal{T} \models C_1 \sqsubseteq C_2) & \text{iff} \\
 (\forall I)(I \models \mathcal{T} \implies I \models C_1 \sqsubseteq C_2) & \text{iff} \\
 (\forall I)(I \models \mathcal{T} \implies C_1^I \subseteq C_2^I) & \text{iff} \\
 (\forall I)(I \models \mathcal{T} \implies C_1^I \cap (\Delta^I \setminus C_2^I) \subseteq \emptyset) & \text{iff} \\
 (\forall I)(I \models \mathcal{T} \implies I \models C_1 \sqcap \neg C_2 \sqsubseteq \perp) & \text{iff} \\
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 \end{array}$$



Inference Problems for ABOX

... and for ABOX \mathcal{A} , axiom α , concept C , role R and individuals $a_{(i)}$ we want to decide whether



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All these tasks, as well as concept unsatisfiability checking, can be reduced to consistency checking. Under which condition and how ?



Reduction of concept unsatisfiability to theory consistency

Example

Consider an \mathcal{ALC} theory $\mathcal{K} = (\mathcal{T}, \mathcal{A})$, a concept C and a fresh individual a_f not occurring in \mathcal{K} :

$$\begin{aligned}
 & (\mathcal{T} \models C \sqsubseteq \perp) && \text{iff} \\
 & (\forall \mathcal{I})(\mathcal{I} \models \mathcal{T} \implies \mathcal{I} \models C \sqsubseteq \perp) && \text{iff} \\
 & (\forall \mathcal{I})(\mathcal{I} \models \mathcal{T} \implies C^{\mathcal{I}} \subseteq \emptyset) && \text{iff} \\
 & \neg [(\exists \mathcal{I})(\mathcal{I} \models \mathcal{T} \wedge C^{\mathcal{I}} \not\subseteq \emptyset)] && \text{iff} \\
 & \neg [(\exists \mathcal{I})(\mathcal{I} \models \mathcal{T} \wedge a_f^{\mathcal{I}} \in C^{\mathcal{I}})] && \text{iff} \\
 & (\mathcal{T}, \{C(a_f)\}) \text{ is inconsistent}
 \end{aligned}$$

Note that for more expressive description logics than \mathcal{ALC} , the ABOX has to be taken into account as well due to its interaction with TBOX.



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Inference Algorithms in Description Logics

Structural Comparison is polynomial, but complete just for some simple DLs *without full negation*, e.g. \mathcal{ALN} , see [dlh2003].



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other ... – e.g. resolution-based, transformation to finite automata, etc.



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We will introduce tableau algorithms.



Tableaux Algorithms

(TAs are not new in DL – they were known in predicate logics as well.)

Main idea

“ABOX \mathcal{A} is consistent w.r.t. TBOX \mathcal{T} if we find a model of $\mathcal{T} \cup \mathcal{A}$.” (similarly for theory \mathcal{K} as a whole)

Each TA can be seen as a *production system* :

state (\sim data base) containing a set of *completion graphs* (see next slides),



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strategy for picking the most suitable rule for application



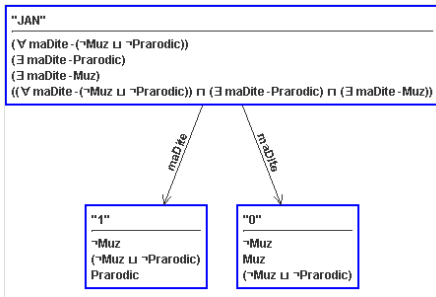
Completion Graphs

(Do not mix with *complete graphs* from the graph theory.)

Completion graph

is a labeled oriented graph $G = (V_G, E_G, L_G)$, where each

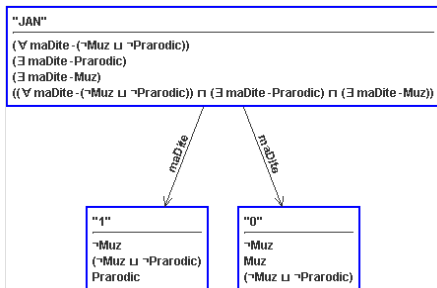
- node $x \in V_G$ is labeled with a set $L_G(x)$ of concepts and
- each edge $\langle x, y \rangle \in E_G$ is labeled with a set of edges $L_G(\langle x, y \rangle)$ (or shortly $L_G(x, y)$)



Completion Graphs

Direct Clash

occurs in a completion graph $G = (V_G, E_G, L_G)$, if $\{A, \neg A\} \subseteq L_G(x)$, or $\perp \in L_G(x)$ for some atomic concept A and a node $x \in V_G$



Completion Graphs

Complete Completion Graph

is a completion graph $G = (V_G, E_G, L_G)$, to which no inference rule can be applied (any more).

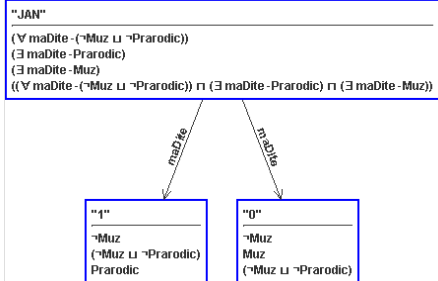


Tableau Algorithm for \mathcal{ALC}

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Tableau Algorithm for \mathcal{ALC} when $\mathcal{T} = \emptyset$

Let's have $\mathcal{K} = (\mathcal{T}, \mathcal{A})$, where $\mathcal{T} = \emptyset$ for now.

- 0 (Preprocessing) Transform all concepts appearing in \mathcal{K} to the “negational normal form” (NNF), “shifting” negation \neg to the atomic concepts (using equivalent operations known from propositional and predicate logics).



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 - Sets $V_{G_0}, E_{G_0}, L_{G_0}$ are smallest possible with these properties.



Tableau algorithm for \mathcal{ALC} without TBOX (2)

...

- 2 Current algorithm state is S . If each $G \in S$ contains a direct clash, terminate as “INCONSISTENT”.



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- 3 Let's choose one $G \in S$ that doesn't contain a direct clash. If G is *complete* w.r.t. rules shown next, terminate as “CONSISTENT”
- 4 Find a rule that is applicable to G and apply it. As a result, we obtain from the state S a new state S' . Jump to step 2.



TA for \mathcal{ALC} without TBOX – Inference Rules

\rightarrow_{\square} rule



TA for \mathcal{ALC} without TBOX – Inference Rules

\rightarrow_{\sqcap} rule

if $(C_1 \sqcap C_2) \in L_G(a)$ and $\{C_1, C_2\} \not\subseteq L_G(a)$ for some $a \in V_G$.



TA for \mathcal{ALC} without TBOX – Inference Rules \rightarrow_{\sqcap} rule

if $(C_1 \sqcap C_2) \in L_G(a)$ and $\{C_1, C_2\} \not\subseteq L_G(a)$ for some $a \in V_G$.

then $S' = S \cup \{G'\} \setminus \{G\}$, where $G' = (V_G, E_G, L_{G'})$, and $L_{G'}(a) = L_G(a) \cup \{C_1, C_2\}$
and otherwise is the same as L_G .



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\rightarrow_{\exists} rule



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TA Run Example

Example – Consistency Checking

$\mathcal{K}_2 = (\emptyset, \mathcal{A}_2)$, where $\mathcal{A}_2 = \{(\exists maDite \cdot Muz \sqcap \exists maDite \cdot Prarodic \sqcap \neg \exists maDite \cdot (Muz \sqcap Prarodic))(JAN)\}$.

- Let's transform the concept into NNF:

$$\exists maDite \cdot Muz \sqcap \exists maDite \cdot Prarodic \sqcap \forall maDite \cdot (\neg Muz \sqcup \neg Prarodic)$$



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 $\exists maDite \cdot Muz \sqcap \exists maDite \cdot Prarodic \sqcap \forall maDite \cdot (\neg Muz \sqcup \neg Prarodic)$
- Initial state G_0 of the TA is

"JAN"

$((\forall maDite \cdot (\neg Muz \sqcup \neg Prarodic)) \sqcap (\exists maDite \cdot Prarodic) \sqcap (\exists maDite \cdot Muz))$



TA Run Example (2)

Example

...

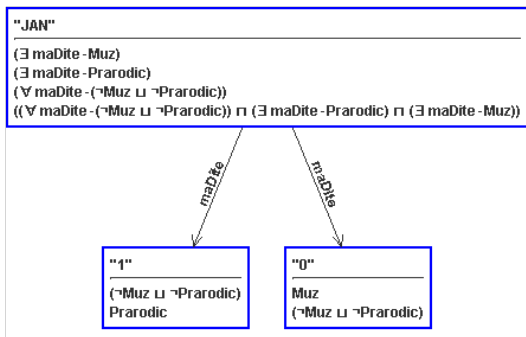
- Now, four sequences of steps 2,3,4 of the TA are performed. TA state in step 4, evolves as follows:

TA Run Example (2)

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- $\{G_0\} \xrightarrow{\neg\text{-rule}} \{G_1\} \xrightarrow{\exists\text{-rule}} \{G_2\} \xrightarrow{\exists\text{-rule}} \{G_3\} \xrightarrow{\forall\text{-rule}} \{G_4\}$, where G_4 is



TA Run Example (3)

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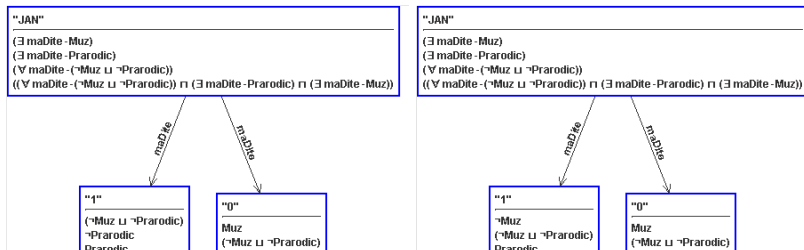
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- Now, we have to apply the \sqcup -rule to the concept $\neg Muz \sqcup \neg Rodic$ either in the label of node "0", or in the label of node "1". Its application e.g. to node "1" we obtain the state $\{G_5, G_6\}$ (G_5 left, G_6 right)

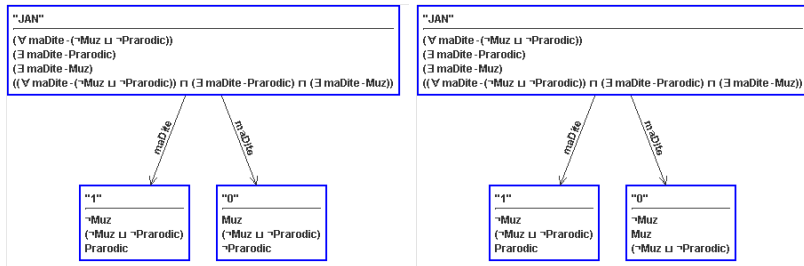


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- We see that G_5 contains a direct clash in node "1". The only other option is to go through the graph G_6 . By application of \sqcup -rule we obtain the state $\{G_5, G_7, G_8\}$, where G_7 (left), G_8 (right) are derived from G_6 :

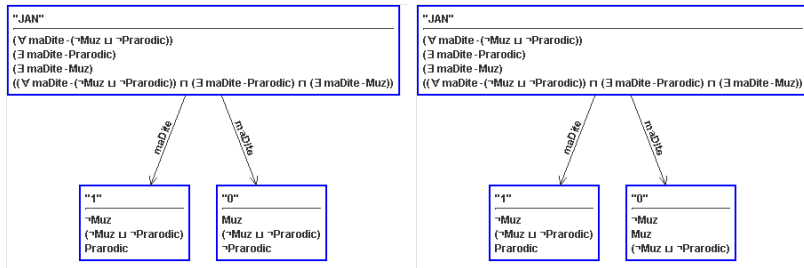


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- G_7 is complete and without direct clash.

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- $\Delta^{\mathcal{I}_2} = \{Jan, i_1, i_2\}$,



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- after application of any of the following rules $\rightarrow_{\sqcap}, \rightarrow_{\exists}, \rightarrow_{\forall}$ graph G is either enriched with a new node, new edge, or labeling of an existing node/edge is enriched. All these operations are finite.



Relation between ABOXes and Completion Graphs

We define also $\mathcal{I} \models G$ iff $\mathcal{I} \models \mathcal{A}_G$, where \mathcal{A}_G is an ABOX constructed from G , as follows

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- For other rules, the soundness is shown in a similar way.



Completeness

- To prove completeness of the TA, it is necessary to construct a model for each complete completion graph G that doesn't contain a direct clash. Canonical model \mathcal{I} can be constructed as follows:
 - the domain $\Delta^{\mathcal{I}}$ will consist of all nodes of G .
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 - P-SPACE (between NP and EXP-TIME).



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- consider \mathcal{T} containing axioms of the form $C_i \sqsubseteq D_i$ for $1 \leq i \leq n$.
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- for each model \mathcal{I} of the theory \mathcal{K} , each element of $\Delta^{\mathcal{I}}$ must belong to $\top_C^{\mathcal{I}}$. How to achieve this ?



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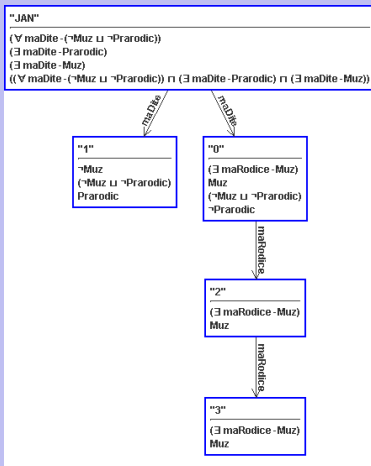
Example

Consider $\mathcal{K}_3 = (\{Muz \sqsubseteq \exists maRodice \cdot Muz\}, \mathcal{A}_2)$. Then \top_C is $\neg Muz \sqcup \exists maRodice \cdot Muz$. Let's use the introduced TA enriched by $\rightarrow_{\sqsubseteq}$ rule. Repeating several times the application of rules $\rightarrow_{\sqsubseteq}$, \rightarrow_{\sqcup} , \rightarrow_{\exists} to G_7 (that is not complete w.r.t. to $\rightarrow_{\sqsubseteq}$ rule) from the previous example we can get into an infinite loop



General Inclusions (3)

Example



... this algorithm doesn't necessarily terminate ☹.



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- \exists -rule is only applicable if the node a_1 in its definition is not blocked by another node.



Blocking in TA (2)

- In the previous example, the blocking ensures that node “2” is blocked by node “0” and no other expansion occurs. *Which model corresponds to such graph ?*



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- **Introduced TA with subset blocking is sound, complete and finite decision procedure for \mathcal{ALC} .**



Let's play ...

- <http://kbss.felk.cvut.cz/tools/dl>



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