Pseudorandom numbers

John von Neumann:

Any one who considers
arithmetical methods of producing random digits
is, of course, in a state of sin.
For, as has been pointed out several times,
there is no such thing as a random number
— there are only methods to produce random numbers, and
a strict arithmetic procedure of course is not such a method.

"Various Techniques Used in Connection with Random Digits,", in *Monte Carlo Method (*A. S. Householder, G. E. Forsythe, and H. H. Germond, eds.), National Bureau of Standards Applied Mathematics Series, 12, Washington, D.C.: U.S. Government Printing Office, 1951, pp. 36–38.

Pseudorandom number generator

Random vs. pseudorandom behaviour

Random behavior -- Typically, its outcome is unpredictable and the parameters of the generating process cannot be determined by any known method.

Examples:

Parity of number of passengers in a coach in rush hour.

Weight of a book on a shelf in grams modulo 10.

Direction of movement of a particular N_2 molecule in the air in a quiet room.

Pseudo-random -- Deterministic formula,

-- Local unpredictability, *"output looks like random*",

-- Statistical tests might reveal more or less "random behaviour"

Pseudorandom integer generator

A pseudo-random integer generator is an algorithm which produces a sequence

$$\{x_n\} = x_0, x_1, x_2, \ldots$$

of non-negative integers, which manifest pseudo-random behaviour.

Pseudorandom integer generator

Two important statistical properties:

- Uniformity
- Independence

Random number in a interval $[a, b]$ must be independently drawn from a uniform distribution with probability density function:

$$f(x) = \begin{cases} \frac{1}{b-a+1} & x \in [a, b] \\ 0 & \text{elsewhere} \end{cases}$$

Good generator

- Uniform distribution over large range of values:

  Interval $[a, b]$ is long, period = $b - a + 1,$ generates all integers in $[a, b]$.
- Speed

  Simple generation formula.

  Modulus (if possible) equal to a power of two – fast bit operations.

Pseudorandom number generator

Random floating point number generator

Task 1: Generate (pseudo) random integer values from an interval $[a, b]$.
Task 2: Generate (pseudo) random floating point values from interval $[0,1[$.

Use the solution of Task 1 to produce the solution of Task 2.
Let $\{x_n\}$ be the sequence of values generated in Task 1.
Consider a sequence $\{y_n\} = \{(x_n - a) / (b - a + 1)\}$.

Each value of $\{y_n\}$ belongs to $[0,1[$.
"Random" real numbers are thus approximated by "random" fractions.
Large length of $[a, b]$ guarantees sufficiently dense division of $[0,1[$.

Example 1

$[a, b] = [0, 1024]$.
$\{x_n\} = \{712, \quad 84, \quad 233, \quad 269, \quad 810, \quad 944, \dots\}$

$\{y_n\} = \{712/1023,\ 84/1023,\ 233/1023,\ 269/1023,\ 810/1023,\ 944/1023,\ \dots\}$
$= \{0.696,\ 0.082,\ 0.228,\ 0.263, 0.792,\ 0,923,\ \dots\}$

Linear Congruential Generator

Linear congruential generator

Linear congruential generator produces a sequence $\{x_n\}$ defined by relations

$$0 \le x_0 < M,$$

$$x_{n+1} = (Ax_n + C) \mod M, \quad n \ge 0.$$

Modulus M, seed x_0, multiplier and increment A, C.

Example 2

$$M = 18, \; A = 7, \; C = 5.$$

$$x_0 = 4,$$
$$x_{n+1} = (7x_n + 5) \mod 18, \quad n \ge 0.$$

$$\{x_n\} = \underbrace{4, 15, 2, 1, 12, 17, 16, 9, 14, 13, 6, 11, 10, 3, 8, 7, 0, 5}_{\text{sequence period, length = 18}}, 4, 15, 2, 1, 12, 17, 16, \ldots$$

Linear Congruential Generator

Example 3

$$M = 15, A = 11, \ C = 6.$$

$$x_0 = 8,$$
$$x_{n+1} = (11x_n + 6) \mod 15, \quad n \geq 0.$$

$$\{x_n\} = \underbrace{8, 14, 5, 11, 2,}_{\text{sequence period, length = 5}} 8, 14, 5, 11, 2, 8, 14, \ \ldots$$

sequence period, length = 5

Example 4

$$M = 13, A = 5, \ C = 11.$$

$$x_0 = 7,$$
$$x_{n+1} = (5x_n + 11) \mod 13, \quad n \geq 0.$$

$$\{x_n\} = \underbrace{7,}_{} 7, 7, 7, 7, \ \ldots$$

sequence period, length = 1

Linear Congruential Generator

Misconception

Prime numbers are "more random" than composite numbers, therefore using prime numbers in a generator improves randomness.
Counterexample: Example 4, all parameters are primes:

$$x_0 = 7, \quad x_{n+1} = (5x_n + 11) \mod 13.$$

Maximum period length

Hull-Dobell Theorem:
The lenght of period is maximum, i.e. equal to M, iff conditions 1. - 3. hold:

1. C and M are coprimes.
2. A–1 is divisible by each prime factor of M.
3. If 4 divides M then also 4 divides A–1.

Example 5

1. $M = 18,\ A = 7,\ C = 6.$ Condition 1. violated
2. $M = 20,\ A = 17,\ C = 7.$ Condition 2. violated
3. $M = 17,\ A = 7,\ C = 6.$ Condition 2. violated
4. $M = 20,\ A = 11,\ C = 7.$ Condition 3. violated
5. $M = 18,\ A = 7,\ C = 5.$ All three conditions hold

Linear Congruential Generator

Randomness issues

Example 6 $x_0 = 4,$
$x_{n+1} = (7x_n + 5) \bmod 18, \quad n \geq 0.$

$\{x_n\} =$ 4, 15, 2, 1, 12, 17, 16, 9, 14, 13, 6, 11, 10, 3, 8, 7, 0, 5, 4, 15, 2, 1, 12, 17, 16, ...

sequence period, length = 18

$\{x_n \bmod 2\} =$ 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, ...

$\{x_n \bmod 3\} =$ 1, 0, 2, 1, 0, 2, 1, 0, 2, 1, 0, 2, 1, 0, 2, 1, 0 ,2, 1, 0, 2, 1, 0, 2, 1, ...

$\{x_n \text{ div } 4\} =$ 0, 3, 0, 0, 3, 4, 4, 2, 3, 3, 1, 2, 2, 0, 2, 1, 0 ,1, 0, 3, 0, 0, 3, 4, 4, ...

Trouble

Low order bits of values generated by LCG exhibit significant lack of randomness.

Remedy

Disregard the lower bits in the output (not in the generation process!).
Output the sequence $\{y_n\} = \{x_n \text{ div } 2^H\}$, where $H \geq ¼ \log_2(M)$.

Linear Congruential Generator

Examples of LCGs in common use

Source	modulus m	multiplier a	increment c	output bits of seed in *rand()* or *Random(L)*
Numerical Recipes	2^{32}	1664525	1013904223	
Borland C/C++	2^{32}	22695477	1	bits 30..16 in *rand()*, 30..0 in *lrand()*
glibc (used by GCC)[15]	2^{31}	1103515245	12345	bits 30..0
ANSI C: Watcom, Digital Mars, CodeWarrior, IBM VisualAge C/C++ [16] C90, C99, C11: Suggestion in the ISO/IEC 9899,[17] C18	2^{31}	1103515245	12345	bits 30..16
Borland Delphi, Virtual Pascal	2^{32}	134775813	1	bits 63..32 of *(seed × L)*
Turbo Pascal	2^{32}	134775813 (8088405_{16})	1	
Microsoft Visual/Quick C/C++	2^{32}	214013 ($343FD_{16}$)	2531011 ($269EC3_{16}$)	bits 30..16
Microsoft Visual Basic (6 and earlier)[18]	2^{24}	1140671485 ($43FD43FD_{16}$)	12820163 ($C39EC3_{16}$)	
RtlUniform from Native API[19]	$2^{31} - 1$	2147483629 ($7FFFFFED_{16}$)	2147483587 ($7FFFFFC3_{16}$)	
Apple CarbonLib, C++11's `minstd_rand0` [20]	$2^{31} - 1$	16807	0	see MINSTD
C++11's `minstd_rand` [20]	$2^{31} - 1$	48271	0	see MINSTD
MMIX by Donald Knuth	2^{64}	6364136223846793005	1442695040888963407	
Newlib, Musl	2^{64}	6364136223846793005	1	bits 63..32
VMS's **MTH$RANDOM**,[21] old versions of glibc	2^{32}	69069 ($10DCD_{16}$)	1	
Java's java.util.Random, POSIX [ln]rand48, glibc [ln]rand48[_r]	2^{48}	25214903917 ($5DEECE66D_{16}$)	11	bits 47..16
`random0` [22][23][24][25][26]	$134456 = 2^3 7^5$	8121	28411	$\frac{X_n}{134456}$
POSIX[27] [jm]rand48, glibc [mj]rand48[_r]	2^{48}	25214903917 ($5DEECE66D_{16}$)	11	bits 47..15
POSIX [de]rand48, glibc [de]rand48[_r]	2^{48}	25214903917 ($5DEECE66D_{16}$)	11	bits 47..0
cc65[28]	2^{23}	65793 (10101_{16})	4282663 (415927_{16})	bits 22..8
cc65	2^{32}	16843009 (1010101_{16})	826366247 (31415927_{16})	bits 31..16
Formerly common: RANDU [9]	2^{31}	65539	0	

Sequence period

Many generators produce a sequence $\{x_n\}$ defined by the general recurrence rule

$$x_{n+1} = f(x_n) \qquad n \geq 0.$$

Therefore, if $x_n = x_{n+k}$ for some $k > 0$, then also

$$x_{n+1} = x_{n+k+1}, \; x_{n+2} = x_{n+k+2}, \; x_{n+3} = x_{n+k+3}, \ldots$$

Sequence period

Subsequence of minimum possible length $p > 0$, $\{x_n, x_{n+1}, \; x_{n+2}, \ldots \; x_{n+p-1}\}$ such that for any $n \geq 0$: $\quad x_n = x_{n+p}$.

Combined Linear Congruential Generator

Definition Let there be r linear congruential generators defined by relations

$$0 \le y_{k,0} < M_k$$
$$y_{k,n+1} = \left(A_k y_{k,n} + C_k\right) \bmod M_k, \quad n \ge 0,$$
$$1 \le k \le r.$$

The combined linear congruential generator is a sequence $\{x_n\}$ defined by

$$x_n = (y_{1,n} - y_{2,n} + y_{3,n} - y_{4,n} + \dots (-1)^{r-1} \cdot y_{r,n}) \bmod (M_1 - 1), \quad n \ge 0.$$

Fact Maximum possible period length (not always attained!) is $(M_1 - 1)(M_2 - 1) \dots (M_r - 1) \ / \ 2^{r-1}$.

Example 7 $\mathrm{r} = 2, \quad 1 \le y_{1,0} \le 2147483562, \quad 1 \le y_{2,0} \le 2147483398$

$$y_{1,n+1} = \left(40014 y_{1,n} + 0\right) \bmod 2147483563, \quad n \ge 0,$$
$$y_{2,n+1} = \left(40692 y_{2,n} + 0\right) \bmod 2147483399, \quad n \ge 0,$$
$$x_n = (y_{1,n} - y_{2,n}) \bmod 2147483562, \quad n \ge 0.$$

Period length is $\frac{(M_1-1)(M_2-1)}{2} = 2305842648436451838$.

Combined Linear Congruential Generator

Example 8 r = 3, $y_{1,0} = y_{2,0} = y_{3,0} = 1,$

$$y_{1,n+1} = (9y_{1,n} + 11) \bmod 16, \quad n \geq 0,$$
$$y_{2,n+1} = (7y_{2,n} + 5) \bmod 18, \quad n \geq 0,$$
$$y_{3,n+1} = (4y_{3,n} + 8) \bmod 27, \quad n \geq 0,$$
$$x_n = (y_{1,n} - y_{2,n} + y_{3,n}) \bmod 15, \quad n \geq 0.$$

$\{x_n\}$ = 1, 4, 0, 2, 7, 12, 2, 2, 6, 6, 7, 7, 5, 2, 0, 9, 1, 1, 9, 11, 7, 9, 2, 8, 9, 12, 1, 1, 14, 2, 12, 9, 7, 4, 9, 8, 1, 6, 14, 5, 9, 0, 1, 4, 8, 8, 6, 9, 4, 4, 3, 11, 4, 3, 11, 14, 9, 12, 1, 7, 11, 11, 0, 0, 1, 1, 0, 11, 10, 3, 11, 11, 3, 6, 1, 4, 11, 2, 3, 6, 10, 10, 9, 11, 7, 3, 2, 14, 3, 3, 10, 1, 8, 14, 3, 9, 10, 13, 3, 2, 1, 3, 14, 14, 12, 6, 13, 13, 5, 8, 3, 6, 10, 1, 6, 5, 10, 9, 11, 11, 9, 6, 4, 13, 5, 5, 12, 0, 10, 13, 6, 11, 13, 0, 5, 5, 3, 6, 1, 13, 11, 8, 12, 12, 4, 10, 3, 8, 13, 3, 5, 8, 12, 12, 10, 13, 8, 8, 6, 0, 7, 7, 0, 2, 13, 0, 5, 11, 0, 0, 4, 4, 5, 5, 3, 0, 13, 7, 0, 14, 7, 9, 5, 8, 0, 6, 7, 10, 14, 14, 12, 0, 10, 7, 6, 2, 7, 6, 14, 5, 12, 3, 7, 13, 14, 2, 6, 6, 4, 7, 3, 2, 1, 9, 2, 2, 9, 12, 7, 10, 14, 5, 9, 9, 13, 13, 0, 14, 13, 9, 8, 2, 9, 9, 1, 4, 14, 2, 9, 0, 1, 4, 9, 8, 7, 9, 5, 2, 0, 12, 1, 1, 8, 14, 6, 12, 1, 7, 9, 11, 1, 0, 14, 2, 12, 12, 10, 4, 11, 11, 3, 6, 1, 4, 9, 14, 4, 3, 8, 8, 9, 9, 7, 4, 2, 11, 3, 3, 10, 13, 9, 11, 4, 9, 11, 14, 3, 3, 1, 4, 14, 11, 9, 6, 10, 10, 3, 8, 1, 6, 11, 2, 3, 6, 10, 10, 8, 11, 6, 6, 4, 13, 6, 5, 13, 0, 11, 14, 3, 9, 13, 13, 2, 2, 3, 3, 1, 13, 12, 5, 13, 12, 5, 8, 3, 6, 13, 4, 5, 8, 12, 12, 10, 13, 9, 5, 4, 0, 5, 5, 12, 3, 10, 1, 5, 11, 12, 0, 4, 4, 3, 5, 1, 0, 14, 8, 0, 0, 7, 10, 5, 8, 12, 3, 7, 7, 12, 11, 13, 12, 11, 8, 6, 0, 7, 7, 14, 2, 12, 0, 7, 13, 0, 2, 7, 6, 5, 8, 3, 0, 13, 10, 14, 14, 6, 12, 4, 10, 0, 5, 7, 9, 14, 14, 12, 0, 10, 10, 8, 2, 9, 9, (sequence restarts:) 1, 4, 0, 2, 7, 12, 2, 2, 7, 7, 5, ...

Period length is $432 < 15 \cdot 17 \cdot 26 / 4$.

Lehmer generator produces sequence $\{x_n\}$ defined by relations

$$0 < x_0 < M, \quad x_0 \text{ coprime to } M.$$

$$x_{n+1} = Ax_n \bmod M, \quad n \geq 0.$$

Modulus M, seed x_0, multiplier A.

Example 9

$$x_0 = 1,$$

$$x_{n+1} = 6x_n \bmod 13.$$

$$\{x_n\} = \underbrace{1, 6, 10, 8, 9, 2, 12, 7, 3, 5, 4, 11,}_{\text{sequence period, length = 12}} 1, 6, 10, 8, 9, 2, 12, \ldots$$

Example 10

$$x_0 = 2,$$

$$x_{n+1} = 5x_n \bmod 13.$$

$$\{x_n\} = \underbrace{2, 10, 11, 3,}_{\text{sequence period, length = 4}} 2, 10, 11, 3, 2, 10, 11, 3, \ldots$$

$$0 < x_0 < M, \qquad x_0 \text{ coprime to } M.$$

$$x_{n+1} = Ax_n \bmod M, \qquad n \geq 0.$$

Fact

The sequence period length produced by a Lehmer generator is maximal and equal to $M-1$ if

M is prime and

A is a primitive root of $(\mathbb{Z}/M\mathbb{Z})^*$.

Notation Multiplicative group of integers modulo prime p: $(\mathbb{Z}/p\mathbb{Z})^*$

Primitive root G is a primitive root of $(\mathbb{Z}/p\mathbb{Z})^*$ if
$\{G, G^2, G^3, ..., G^{p-1}\} = \{1, 2, 3, \ldots, p-1\}$ (powers are taken modulo p).

Example 11

$p = 13$, $G = 2$ is a primitive root of $(\mathbb{Z}/13\mathbb{Z})^*$.
$\{G, G^2, ..., G^{12}\} = \{2, 4, 8, 3, 6, 12, 11, 9, 5, 10, 7, 1\} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$.

$p = 13$, $G = 6$ is a primitive root of $(\mathbb{Z}/13\mathbb{Z})^*$.
$\{G, G^2, ..., G^{12}\} = \{6, 10, 8, 9, 2, 12, 7, 3, 5, 4, 11, 1\} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$.

$p = 13$, $G = 5$ is not a primitive root of $(\mathbb{Z}/13\mathbb{Z})^*$.
$\{G, G^2, ..., G^{12}\} = \{5, 12, 8, 1, 5, 12, 8, 1, 5, 12, 8, 1\} = \{1, 5, 8, 12\}$.

Finding group primitive roots

No elementary and effective method is known. Some cases has been studied in detail.

8th Mersenne prime $M_{31} = 2^{31}-1 = 2\ 147\ 483\ 647$

Fact G is a primitive root of $(\mathbb{Z}/M_{31}\mathbb{Z})^*$ iff
$G \equiv 7^b \pmod{M^{31}}$, where b is coprime to $M_{31}-1$.

$$M_{31}-1 = 2\ 147\ 483\ 646 = 2 \cdot 3^2 \cdot 7 \cdot 11 \cdot 31 \cdot 151 \cdot 331$$

Example 12

$\mathrm{G} = 7^5 = 16807$ is a primitive root of $(\mathbb{Z}/M_{31}\mathbb{Z})^*$ because 5 is coprime to $M_{31}-1$.

$\mathrm{G} = 7^{1116395447} \equiv 48271 \pmod{M31}$ is a primitive root of $(\mathbb{Z}/M_{31}\mathbb{Z})^*$
because 1116395447 is a prime and therefore coprime to $M_{31}-1$.

$G = 7^{1058580763} \equiv 69621 \pmod{M31}$ is a primitive root of $(\mathbb{Z}/M_{31}\mathbb{Z})^*$
because $1058580763 = 19 \cdot 41 \cdot 61 \cdot 22277$ and therefore coprime to $M_{31}-1$.

Blum Blum Shub Generator

Blum Blum Shub generator produces sequence $\{x_n\}$ defined by relations

$$2 \le x_0 < M, \quad x_0 \text{ coprime to } M.$$

$$x_{n+1} = x_n^2 \bmod M$$

Modulus M, seed x_0.

Seed x_0 coprime to M.
Modulus M is a product of two large distinct primes P and Q.
$P \bmod 4 = Q \bmod 4 = 3$,
$\gcd((P-3)/2, (Q-3)/2)$ is small.

Example 13 $x_0 = 4, \ M = 11 \cdot 47,$ $\gcd(4, 22) = 2,$

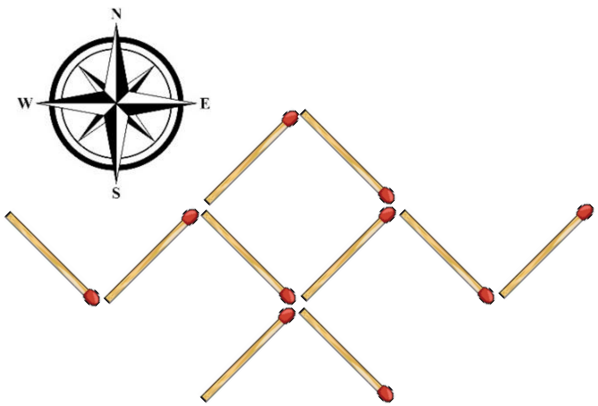
$$x_{n+1} = x_n^2 \bmod 517.$$

$\{x_n\} =$ **4,** 16, 256, 394, 136, 401, 14, 196, 158, 148, 190, 427, 345, 115, 300, 42, 213, 390, 102, 64, 477, 49, 333, 251, 444, 159, 465, 119, 202, 478, 487, 383, 378, 192, 157, 350, 488, 324, 25, 108, 290, 346, 289, 284, **4,** 16, 256, 394, 136, ...

sequence period, length = 44

Kvízová pauza

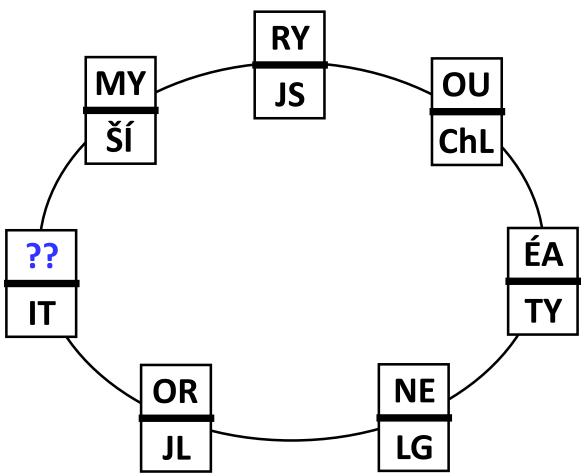
Přesuňte 3 sirky tak, aby vlaštovka letěla na jih.



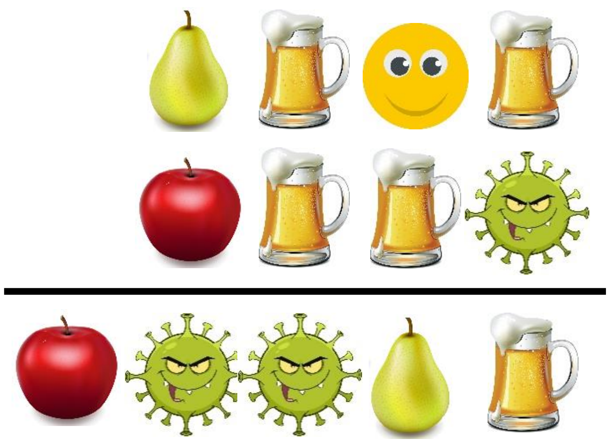
Přesuňte právě jednu z pěti modrých číslic, aby rovnost platila.

$$62 - 63 = 1$$

Jaká dvojice písmen logicky patří na místo otazníků?



Vyřešte algebrogram.



Primes related notions

Prime counting function $\pi(n)$

Counts the number of prime numbers less than or equal to n.

Example 14

$\pi(10)$ = 4. Primes less than or equal to 10: 2, 3, 5, 7.

$\pi(37)$ = 12. Primes less than or equal to 37: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37.

$\pi(100)$ = 25. Primes less than or equal to 100: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97.

Estimate

$$\frac{n}{\ln n} < \pi(n) < 1.25506\frac{n}{\ln n} \quad \text{for } n > 16.$$

Example 15

$$\frac{100}{\ln 100} < \pi(100) < 1.25506\frac{100}{\ln 100} \qquad \frac{10^6}{\ln 10^6} < \pi(10^6) < 1.25506\frac{10^6}{\ln 10^6}$$

$$21.715 < \pi(100) = 25 < 27.253 \qquad 72382.4 < \pi(10^6) = 78498 < 90844.3$$

Limit behaviour

Prime number theorem:

$$\lim_{n\to\infty}\frac{\pi(n)}{\frac{n}{\ln n}} = 1$$

Sieve of Eratosthenes

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Sieve of Eratosthenes

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Sieve of Eratosthenes

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Sieve of Eratosthenes

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Sieve of Eratosthenes

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Sieve of Eratosthenes

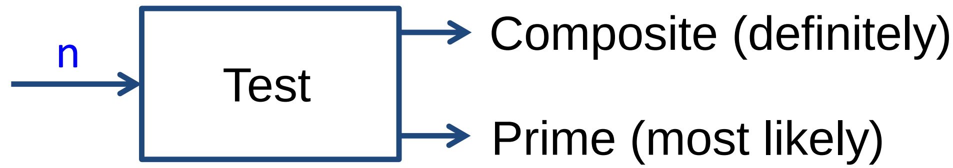
Algorithm

EratosthenesSieve (n)
    Let A be an array of Boolean values, indexed by integers 2 to n, initially all set to **true**
    for i = 2 **to** $\sqrt{n}$
        if $A[i]$ = **true then**
            for $j = i^2, i^2+i, i^2+2i, i^2+3i$, ..., not exceeding n
                $A[j]$:= **false**
    end
    output all i such that $A[i]$ is **true**
end

Time complexity: $\mathrm{O}(n \log \log n)$.

Randomized primality tests

General scheme



Fermat (little) theorem

If p is prime and $0 < a < p$, then $a^{p-1} \equiv 1 \pmod{p}$.

Fermat primality test

FermatTest (n, k)
    for i = 1 **to** k
        a = random integer in [2, n−2]
        if $a^{n-1} \not\equiv 1 \pmod{n}$ **then return** *Composite*
    end
    return *Prime*
end

Flaw There are infinitely many composite numbers for which the test always fails: Carmichael numbers: 561, 1105, 1729, 2465, 2821, 6601, 8911, 10585, ... (sequence A002997 in the OEIS)

Note OEIS = The On-Line Encyclopedia of Integer Sequences, (https://oeis.org)

Miller-Rabin primality test

Fermat: If p is prime and $0 < a < p$, then $a^{p-1} \equiv 1 \pmod{p}$.

Lemma: If p is prime and $x^2 \equiv 1 \pmod{p}$ then $x \equiv 1 \pmod{p}$ or $x \equiv -1 \pmod{p}$.

Example:
Is $n = 15$ prime?
Let $a = 4$.
Fermat test: $4^{15-1} \bmod 15 = 1$ … OK.
Apply the lemma to 4^{14} --> If 15 is prime, then $\sqrt{4^{14}} = 4^7 \bmod 15 \in \{1, -1\}$.
However, $4^7 \bmod 15 = 4$, hence 15 is a composite number.

Miller-Rabin primality test

Lemma: If p is prime and $x^2 \equiv 1 \pmod{p}$ then $x \equiv 1 \pmod{p}$ or $x \equiv -1 \pmod{p}$.

⇨ Let $n > 2$ be prime, $n-1 = 2^r \cdot d$ where d is odd, $1 < a < n-1$.

Then either $a^d \equiv 1 \pmod{n}$ or $a^{2^s \cdot d} \equiv -1 \pmod{n}$ for some $0 \le s \le r-1$.

```
MillerRabinTest (n, k)
    compute r, d such that d is odd and 2^r ·d = n−1
    for i = 1 to k   // WitnessLoop
        a = random integer in [2, n−2]
        x = a^d mod n
        if x = 1 or x = n−1 then goto EndOfLoop
        for j = 1 to r−1
            x = x^2 mod n
            if x = 1 then return Composite
            if x = n−1 then goto EndOfLoop
        end
        return Composite
        EndOfLoop:
    end
    return Prime
end
```

Examples:

$n = 1105 = 2^4 \cdot 69+1$
$a = 389$
$x_0 = 1039$
$x_1 = 1041$
$x_2 = 781$
$x_3 = 1$ -> Composite

$n = 1105 = 2^4 \cdot 69+1$
$a = 390$
$x_0 = 539$
$x_1 = 1011$
$x_2 = 1101$
$x_3 = 16$
-> Composite

$n = 13 = 2^2 \cdot 3+1$
$a = 7$
$x_0 = 5$
$x_1 = 12 \equiv -1 \pmod{13}$
WitnessLoop passes

Randomized primality tests

Miller-Rabin primality test

- Time complexity: $O(k \log^3 n)$.
- If n is composite then the test declares n prime with a probability at most 4^{-k}.
- A deterministic variant exists, however it relies on unproven generalized Riemann hypothesis.

AKS primality test

- First known deterministic polynomial-time primality test.
- **A**grawal, **K**ayal, **S**axena, 2002 - Gödel Prize in 2006.
- Time complexity: $O(\log^6 n)$.
- The algorithm is of immense theoretical importance, but not used in practice.

Integer factorization

Difficulty of the problem

- No efficient algorithm is known.
- The presumed difficulty is at the heart of widely used algorithms in cryptography (RSA).

Pollard's rho algorithm

- Effective for a composite number having a small prime factor.

```
PollardRho (n)
    x = y = 2; d = 1
    while d = 1
        x = g(x) mod n
        y = g(g(y)) mod n
        d = gcd (|x-y|, n)
    end
    if d = n return Failure
    else return d
end
```

g(x) .. a suitable polynomial function

For example, $g(x) = x^2 - 1$

gcd .. the greatest common divisor

Pollard's rho algorithm – analysis

- Assume $n = pq$.
- Values of x and y form two sequences $\{x_k\}$ and $\{y_k\}$, respectively, where $y_k = x_{2k}$ for each k. Both sequences enter a cycle. This implies there is t such that $y_t = x_t$.
- Sequences $\{x_k \bmod p\}$ and $\{y_k \bmod p\}$ typically enter a cycle of shorter length. If, for some $s < t$, $x_s \equiv y_s \pmod p$, then p divides $|x_s - y_s|$ and the algorithm halts.
- The expected number of iterations is $O(\sqrt{p}) = O(n^{1/4})$.

References

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