Digital Image

(B4M33DZO, Summer 2024)

Lecture 3:

Convolution

https://cw.fel.cvut.cz/wiki/courses/b4m33dzo/start

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Sliding average of function f weighted by function g:

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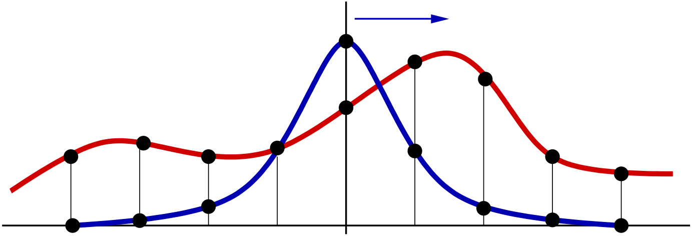

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Convolution theorem:

$$f * g = \mathcal{F}^{-1}\{\mathcal{F}\{f\} \cdot \mathcal{F}\{g\}\}$$

Convolution (in 2D)

Extension of 1D case (image F, convolution kernel G):

$$(F * G)(s,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x,y) \cdot G(s-x, t-y)\, \mathrm{d}x\mathrm{d}y$$

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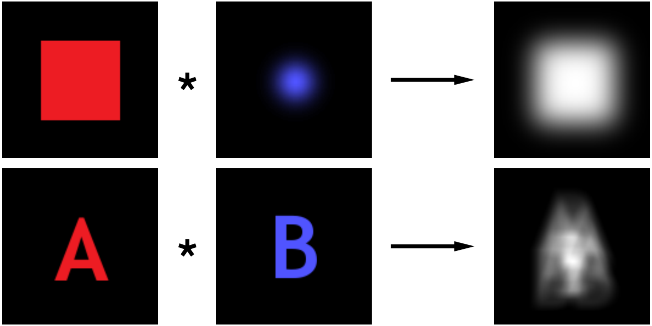

Convolution (in 2D)

Discrete case (image F, convolution kernel G):

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How to compute it faster?

method	accuracy	kernel	complexity
Fourier transform	exact	arbitrarily	$\mathcal{O}(\|\mathrm{F}\| \cdot \log \|\mathrm{F}\|)$
Separable kernel	exact	limited	$\mathcal{O}(\|\mathrm{F}\| \cdot \sqrt{\|\mathrm{G}\|})$
Integral image	high	limited	$\mathcal{O}(\|\mathrm{F}\|)$
Mip-mapping	low	arbitrarily	$\mathcal{O}(\|\mathrm{F}\|)$

Fourier Transform

Convolution theorem:

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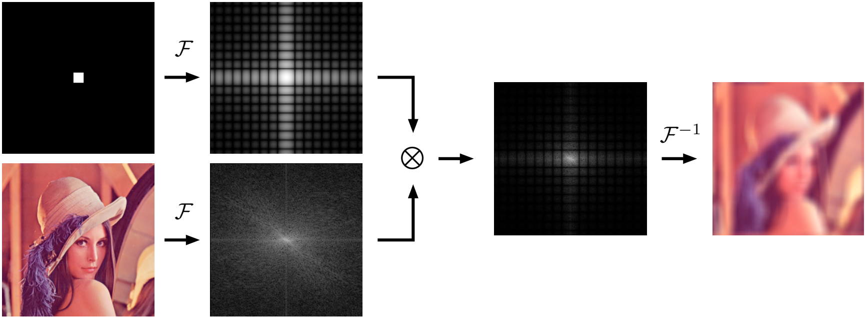
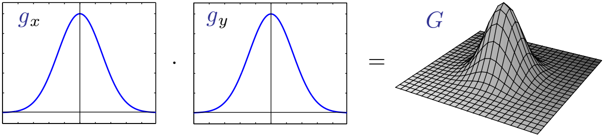


Separable Kernel

$$G(x, y) = g_x(x) \cdot g_y(y)$$



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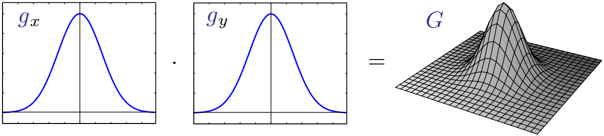

$$(F * G)(s,t) = \int_{-\infty}^{\infty} g_y(t-y) \cdot \left(\int_{-\infty}^{\infty} F(x,y) \cdot g_x(s-x)\,\mathrm{d}x \right) \mathrm{d}y$$

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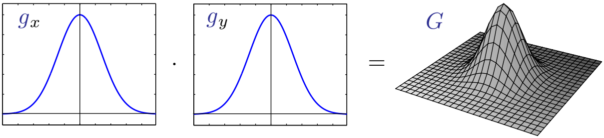

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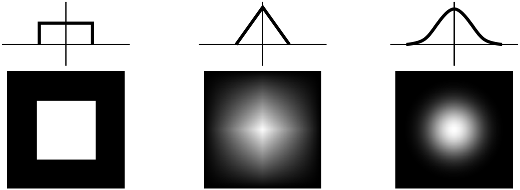
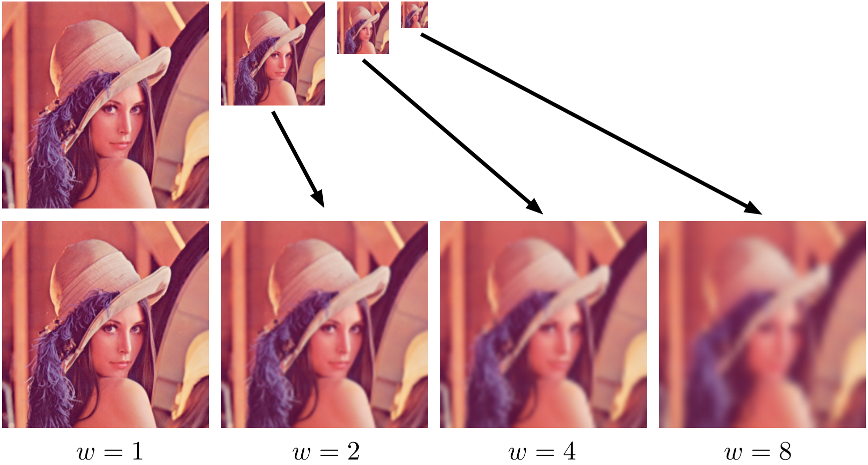


Mip-Mapping

Precalculate output for increasing kernel size:



Integral Image

$$f * g = \int f * g'$$

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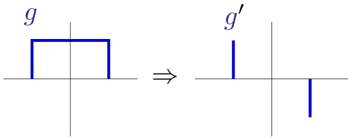

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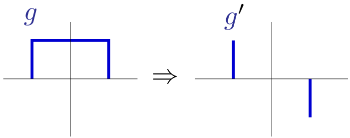

2	3	2	1
3	0	1	2
1	3	1	0
1	4	2	2

image I

Summed area table:

$$\mathrm{S}[x,y] = \sum_x \sum_y \mathrm{I}[x,y]$$

2	5	7	8
5	8	11	14
6	12	16	19
7	17	23	28

SAT S

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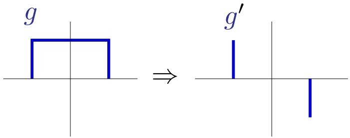

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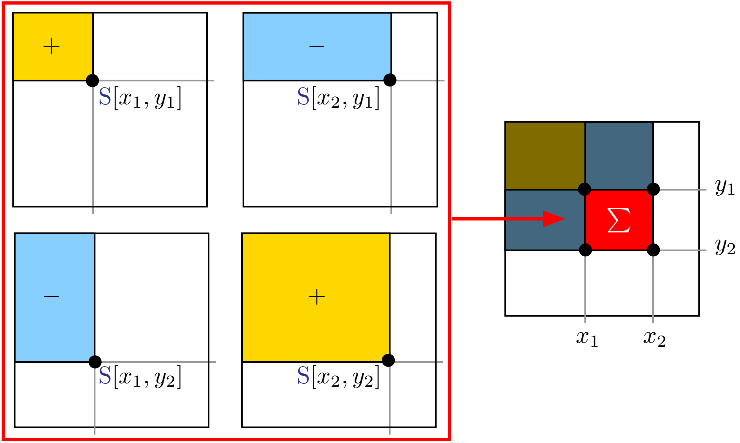
$$\mathrm{S}[x,y] = \mathrm{I}[x,y] + \mathrm{S}[x-1,y] + \mathrm{S}[x,y-1] - \mathrm{S}[x-1,y-1]$$

Integral Image

$$\sum = \mathrm{S}[x_2, y_2] - \mathrm{S}[x_1, y_2] - \mathrm{S}[x_2, y_1] + \mathrm{S}[x_1, y_1]$$



Integral Image

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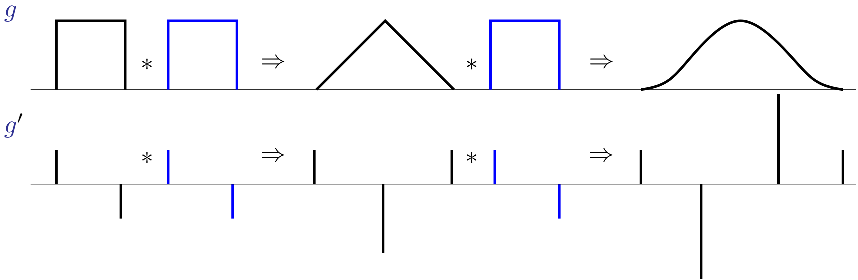

Integral Image

Examples of convolution kernels for integral image:

-1	1
1	-1

1	-2	1
-2	4	-2
1	-2	1

-1	3	-3	1
3	-9	9	-3
-3	9	-9	3
1	-3	3	-1