Digital Image

(B4M33DZO, Summer 2024)

Lecture 2:

Fourier Transform

https://cw.fel.cvut.cz/wiki/courses/b4m33dzo/start

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Fourier Transform

time$\{t\}$ $\Longleftrightarrow$ **frequency**$\{u\}$

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forward: $$F(u) = \int_{-\infty}^{\infty} f(t) \cdot e^{-2\pi i u t} \mathrm{d}t$$

inverse: $$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(u) \cdot e^{+2\pi i u t} \mathrm{d}u$$

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basis functions: $$e^{i\alpha} = \cos(\alpha) + i \sin(\alpha)$$

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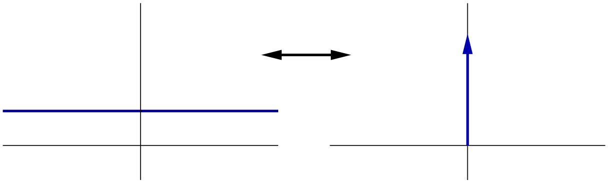
amplitude: $$|F(u)| = \sqrt{\mathbf{Re}(F(u))^2 + \mathbf{Im}(F(u))^2}$$

phase: $$\Phi(F(u)) = \tan^{-1}\left(\frac{\mathbf{Im}(F(u))}{\mathbf{Re}(F(u))}\right)$$

Fourier Transform

$$1 \iff \delta(u)$$



Fourier Transform

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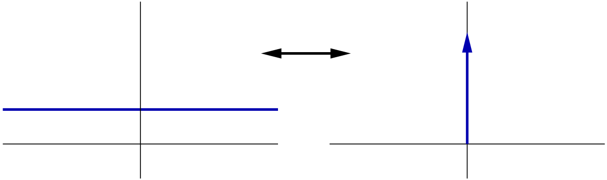
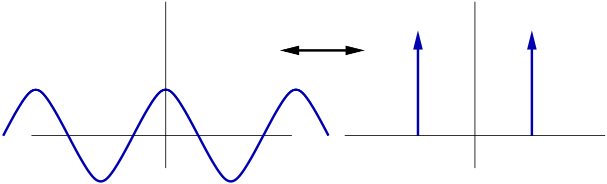
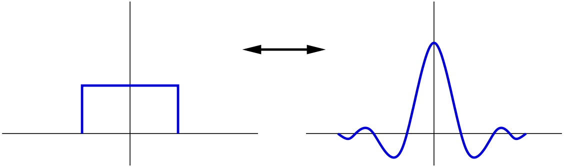


$$\cos(2\pi kx) \iff \frac{1}{2}\left(\delta(u+k)+\delta(u-k)\right)$$



Fourier Transform

$$\mathbf{1}(x+k) - \mathbf{1}(x-k) \quad \Longleftrightarrow \quad \frac{1}{\pi u}\sin(2\pi k u)$$



Fourier Transform

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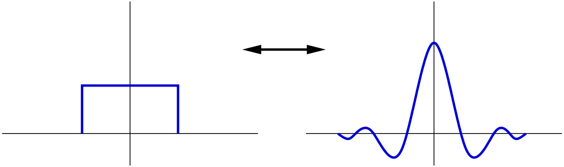
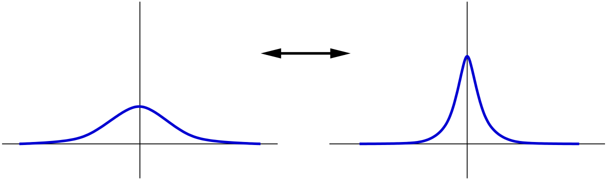


$$\exp\left(-kx^2\right) \quad \Longleftrightarrow \quad \sqrt{\frac{\pi}{k}}\exp\left(-\frac{\pi^2}{k}u^2\right)$$



Linearity:

$$a \cdot f(x) + b \cdot f(x) \iff a \cdot F(u) + b \cdot F(u)$$

Fourier Transform

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Parseval's theorem:

$$\int_{-\infty}^{\infty} |f(x)|^2 \mathrm{d}x = \int_{-\infty}^{\infty} |F(u)|^2 \mathrm{d}u$$

Fourier Transform

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Convolution theorem:

$$\int_{-\infty}^{\infty} f(x) \cdot g(t - x) \mathrm{d}x \quad \Longleftrightarrow \quad F(u) \cdot G(u)$$

Fourier Transform

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Convolution theorem:

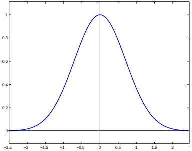
$$\int_{-\infty}^{\infty} f(x) \cdot g(t - x) \mathrm{d}x \iff F(u) \cdot G(u)$$

Shift theorem:

$$\int_{-\infty}^{\infty} f(x - a) \cdot e^{-2\pi j u x} \mathrm{d}x = F(u) \cdot e^{-2\pi j u a}$$

Fourier Transform

time

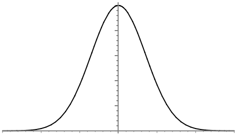


derivatives

0th:

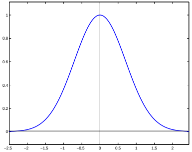
$$f(x) \Longleftrightarrow F(u)$$

frequency



Fourier Transform

time | **derivatives** | **frequency**



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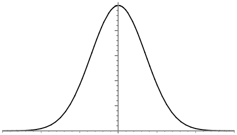
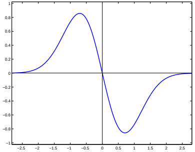
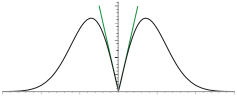
$$f(x) \Longleftrightarrow F(u)$$





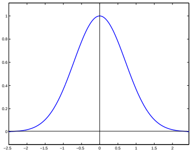
1st:

$$f'(x) \Longleftrightarrow u \cdot F(u)$$



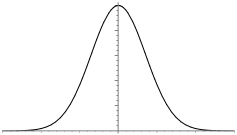
Fourier Transform

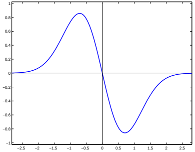
time | **derivatives** | **frequency**



0th:

$$f(x) \iff F(u)$$





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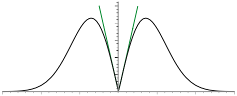
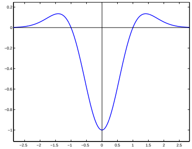
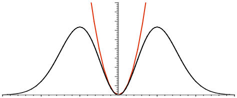




2nd:

$$f''(x) \iff u^2 \cdot F(u)$$



Fourier Transform (in 2D)

position$\{x, y\}$ $\Longleftrightarrow$ **frequency & orientation**$\{u, v\}$

Fourier Transform (in 2D)

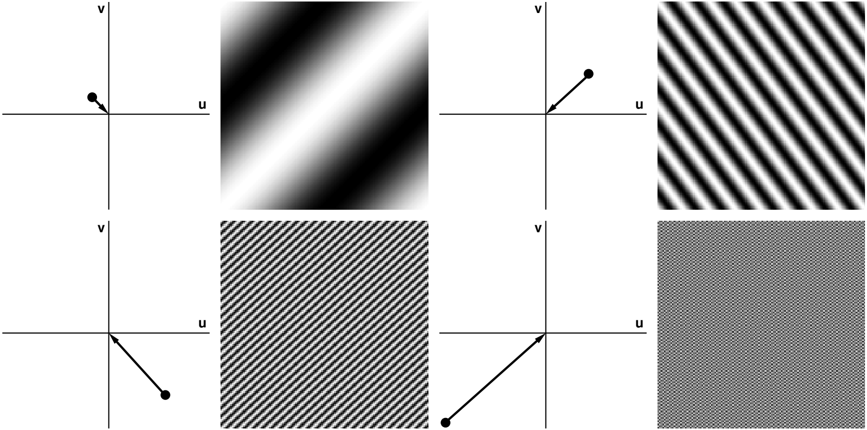
position $\{x, y\}$ $\Longleftrightarrow$ **frequency & orientation** $\{u, v\}$

forward: $$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) \cdot e^{-2\pi i(ux+vy)} \mathrm{d}x\mathrm{d}y$$

inverse: $$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v) \cdot e^{+2\pi i(ux+vy)} \mathrm{d}u\mathrm{d}v$$

Fourier Transform (in 2D)

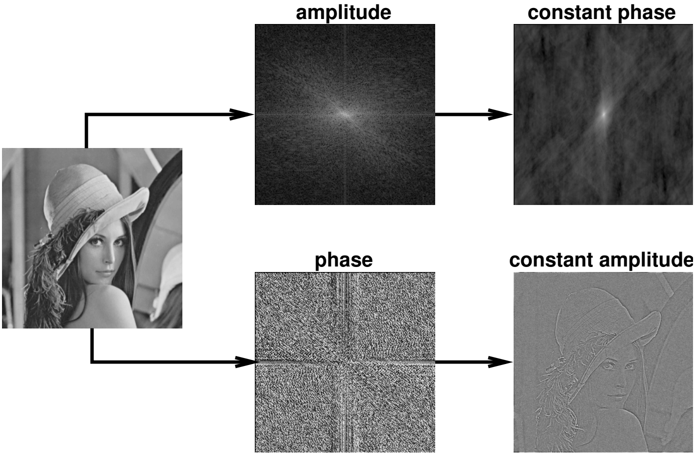
basis functions:



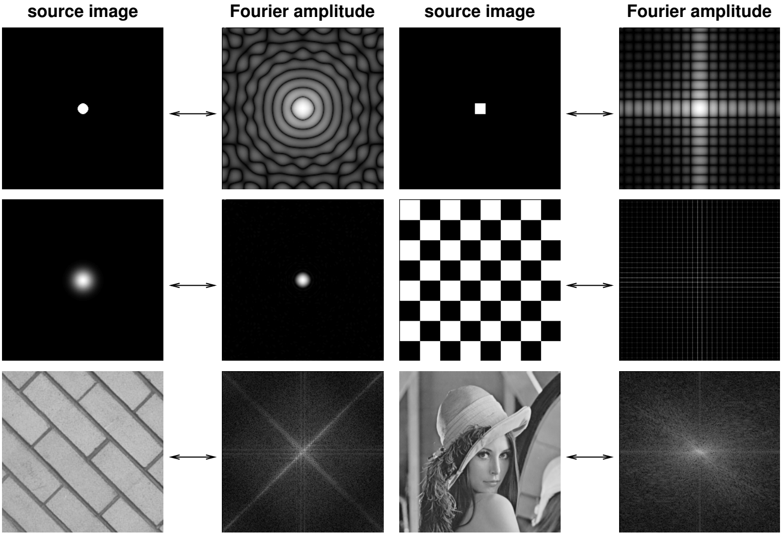
Fourier Transform (in 2D)

Edges with orientation $\arctan(v/u)$ **and frequency** $\sqrt{u^2 + v^2}$**:**

Amplitude $\Rightarrow$ **intensity** **Phase** $\Rightarrow$ **“location”**



Fourier Transform (in 2D)



Fourier Transform (in 2D)

Discrete Fourier Transform (DFT):

forward: $$\mathrm{F}[\mathrm{u},\mathrm{v}] = \frac{1}{\mathrm{N}} \sum_{\mathrm{x}=0}^{\mathrm{N}-1} \sum_{\mathrm{y}=0}^{\mathrm{N}-1} \mathrm{f}[\mathrm{x},\mathrm{y}] \cdot \mathrm{e}^{-2\pi \mathrm{i}(\mathrm{ux}+\mathrm{vy})/\mathrm{N}}$$

inverse: $$\mathrm{f}[\mathrm{x},\mathrm{y}] = \frac{1}{\mathrm{N}} \sum_{\mathrm{u}=0}^{\mathrm{N}-1} \sum_{\mathrm{v}=0}^{\mathrm{N}-1} \mathrm{F}[\mathrm{u},\mathrm{v}] \cdot \mathrm{e}^{+2\pi \mathrm{i}(\mathrm{ux}+\mathrm{vy})/\mathrm{N}}$$

Computation complexity: $\mathcal{O}(N^4)$. Can we do it faster?

Fast Fourier Transform (FFT):

separability: $$F[u,v] = \frac{1}{N}\sum_{x=0}^{N-1} e^{-2\pi i ux/N} \cdot \left(\sum_{y=0}^{N-1} f[x,y] \cdot e^{-2\pi i vy/N}\right)$$

recursion:
$$\begin{aligned} F[u] &= F_{even}[u] + F_{odd}[u] \cdot e^{-2\pi i u/N} \\ F[u+N/2] &= F_{even}[u] - F_{odd}[u] \cdot e^{-2\pi i u/N} \end{aligned}$$

Computation complexity: $\mathcal{O}(N^2 \log N)$

Sampling Theorem

Nyquist frequency: $f_{max} \leq \frac{1}{2\Delta}$

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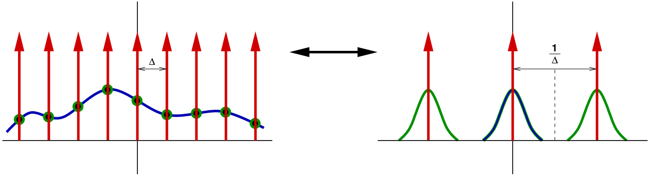
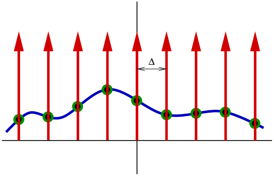
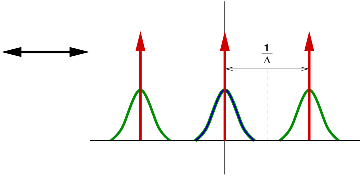


Sampling Theorem

Nyquist frequency: $f_{max} \leq \frac{1}{2\Delta}$

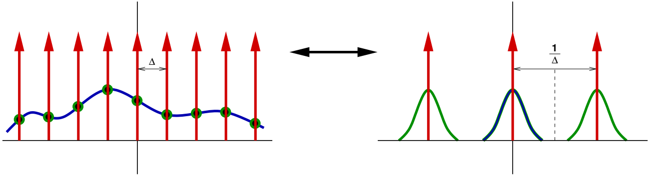




$$s(x) = \sum_k \delta\left(x - k\Delta\right) \iff S(u) = \sum_k \delta\left(u - \frac{k}{\Delta}\right)$$

Sampling Theorem

Nyquist frequency: $f_{max} \leq \frac{1}{2\Delta}$

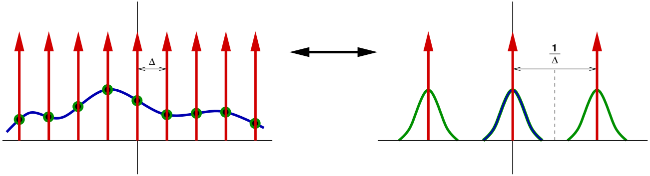


$$s(x) = \sum_k \delta(x - k\Delta) \iff S(u) = \sum_k \delta\left(u - \frac{k}{\Delta}\right)$$

$$d(x) = f(x) \cdot s(x) \iff D(u) = (F * S)(u)$$

Sampling Theorem

Nyquist frequency: $f_{max} \leq \frac{1}{2\Delta}$



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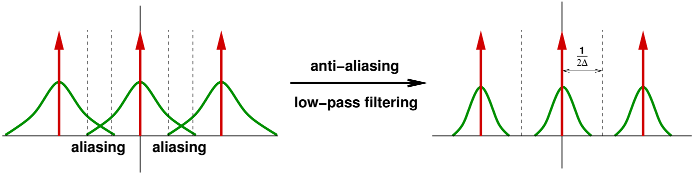
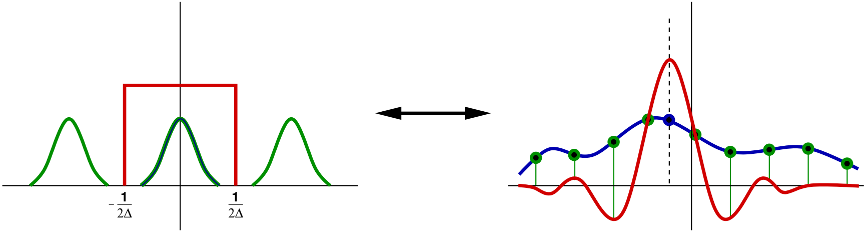


Aliasing Example

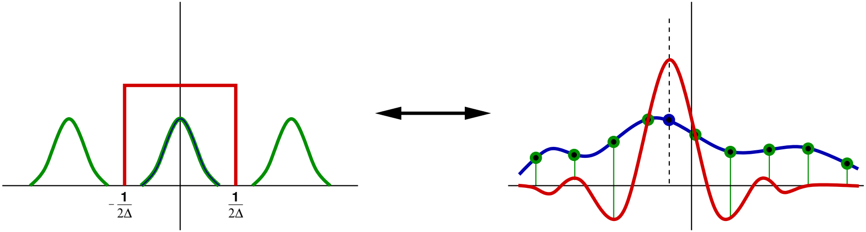
Sampling Theorem

Signal reconstruction:



Sampling Theorem

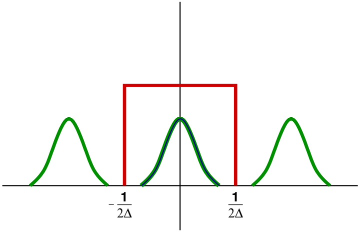
Signal reconstruction:

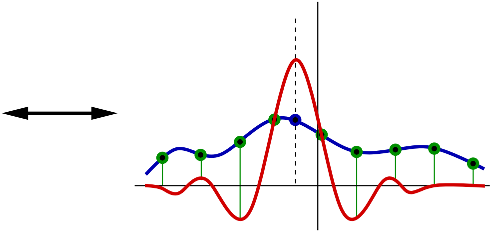


$$\textcolor{red}{B}(u) = \mathbf{1}\left(u + \frac{1}{2\Delta}\right) - \mathbf{1}\left(u - \frac{1}{2\Delta}\right) \iff \textcolor{red}{b}(x) = \frac{1}{\pi x}\sin\left(\frac{\pi x}{\Delta}\right)$$

Sampling Theorem

Signal reconstruction:





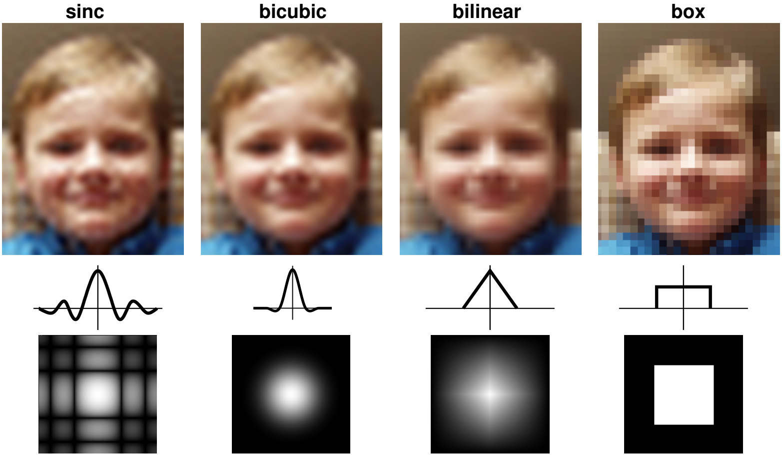
$$B(u) = \mathbf{1}\left(u + \frac{1}{2\Delta}\right) - \mathbf{1}\left(u - \frac{1}{2\Delta}\right) \iff b(x) = \frac{1}{\pi x}\sin\left(\frac{\pi x}{\Delta}\right)$$

$$F(u) = D(u) \cdot B(u) \iff f(x) = (d * b)(x)$$

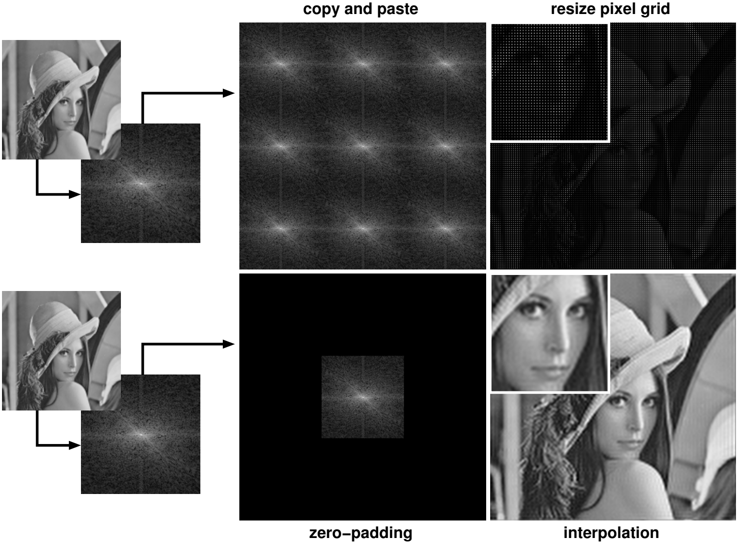
Image Resampling

Problem: convolution with sinc ⇒ time consuming, ringing artifacts.

Sinc approximations with narrow support (bicubic, bilinear, box):



Example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resize pixel grid
zero−padding
interpolation