Fundamental Characteristics of Networks
Models of Random Graphs

Network Application Diagnostics B2M32DSA

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October 17, 2023

Outline

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The Network Perspective [Weh13]

Mainstream Social Science

- Society is a set of independent individuals.
- Individuals are the unit of analysis, treated as bundles of attributes.

Complex Network Analysis (CNA)

- **Relations** (dyads, triads) are the unit of analysis.
- Actions of **actors** are interdependent.
- **Static**: Structure is (first of all) thought to be a stable pattern.
- **Dynamic**: Choices/actions result in structures, but structures shapes decisions and actions, i.e. processes take place on networks.

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Networks Focused on Relations [Weh13]

RELATIONS MATTER!

Contrasted with both an *atomistic* perspective or a *whole-group* perspective

Social Network Analysis (SNA)

- Humanities and social science
- Activities and structures tied with people
  - Shopping basket analysis, targeted advertising
  - Enterprise processes analysis(people cooperation, good distribution)

Complex Network Analysis (CNA)

- Uses the same method as SNA
- Applied to all domains of human acting
- Biology, military, computer network, citations, telecommunication

Network Properties [Weh13]

- A graph $\mathcal{G}$ can be represented as sets or with matrices.
- Properties of vertices $\mathcal{P}$ and lines $\mathcal{W}$ can be measured in different scales:
  - *numerical* (mapped to real numbers),
  - *ordinal* (categorical value with an order), and
  - *nominal* (categorical value with no natural ordering).
- The size of a network/graph is expressed by two numbers:
  - number of vertices $N = |\mathcal{V}|$
  - number of lines $M = |\mathcal{L}|$.

How to Analyze Complex Networks [Erc15]

- Determination of what **properties** to search for.
- Which nodes of the complex networks are more **important** than others.
- Which **groups** of nodes are more closely related to each other.
- To see if some subgraph **pattern** is repeating itself significantly
  - an indication of a fundamental network **functionality**

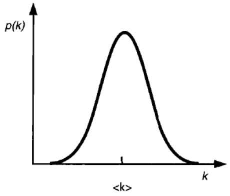
Typical Characteristics of Complex Networks [Erc15, Weh13]

- *Local (node) view*
  - **Degree Heterogeneity**
    - Actors differ in the number of ties they maintain.
    - Centrality measures help to identify prominent actors.
    - Presumption is that nodes or edges that are (in some sense) in the middle of a network are important for the network's function.
  - **Bridges and Small Worlds**
    - New information arrives over weak ties (Granovetter) or bridges (Burt).
    - Bridges tend to be short cuts in the networks,
    - ...are responsible for short average path lengths.
- *Global (community, structure, network) view*
  - Networks often have dense subgraphs.
  - Community detection helps to find them.
  - **Clusters**
  - **Modularity**
    - Based on a different null models.

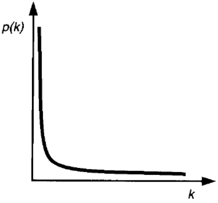
Outline

Degree Heterogeneity [Weh13]

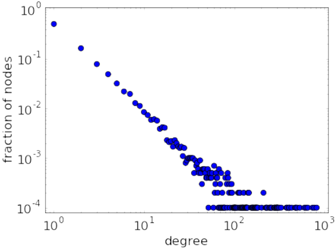
- Not all nodes show the same activity (degree) in networks.
- Some nodes show an astounding activity.
- Degree is most of all a question of tie formation cost.
    - Preferential attachment
    - Fitness model



Gaussian



Skewed Distributions



Vertex Degree Statistics [Erc15]

Theorem 1 (Theorem 4.1 [Erc15], p.64)

For any graph $G(V, E)$, the sum of the degrees of vertices is twice the number of its edges, stated formally as follows:

$$\sum_{v \in V} k(v) = 2M \tag{1}$$

where $k(v)$ is the degree of vertex x.

- The **average degree of a graph**

$$\bar{k} = \langle k \rangle = \frac{1}{N} \sum_{v \in V} k(v) = \frac{2M}{N} \tag{2}$$

Degree Variability [Erc15]

- The **degree variance** $\sigma(G)$ of a graph $G(E, V)$

$$\sigma(G) = \frac{1}{N-1} \sum_{v \in V} (k(v) - \bar{k})^2 \tag{3}$$

- The **mean** of absolute distance between node degrees and the average degress of a graph G

$$\tau(G) = \frac{1}{N} \sum_{v \in V} |k(v) - \bar{k}| \tag{4}$$

Graph Density [Die05, Weh13, Erc15]

- The **density** ρ of a graph is the proportion of present lines to the maximum possible number of lines.
- A **complete graph** is a graph with maximum density.
- There are $\binom{N}{2} = N(N-1)/2$ possible lines (unordered pairs).
- The **graph (edge) density** for *undirected simple* graphs

$$\rho_G = \frac{2|E|}{|V||V|-1)} = \frac{2M}{N(N-1)} = \frac{\bar{k}}{(N-1)} \tag{5}$$

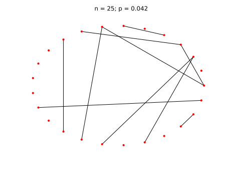
  - for large networks where $N >> 1$, $\rho = \bar{k}/N$
- The **graph (edge) density** for *directed simple* graphs

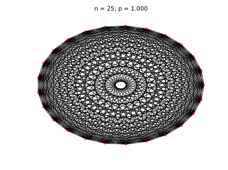
$$\rho_{\vec{G}} = \frac{|E|}{|V||V|-1)} = \frac{M}{N(N-1)} \tag{6}$$

Graph Sparsity [Die05, Erc15]

- The network is called **dense**
  - if ρ does not change significantly as $N \to \infty$ [Erc15], p. 65
  - the number of edges is about quadratic in their number of vertices, i.e. $|E| \approx |V|^2$ [Die05], p. 163
- The network is called **sparse**
  - if $\rho \to 0$ as $N \to \infty$ [Erc15], p. 65
  - the number of edges is about linear in their number of vertices, i.e. $|E| \approx \alpha|V|$ [Die05], p. 164 or $|E| \to$ const. as $N \to \infty$ [New10]
- A dramatic impact on processing of graphs.

A sparse graph and a dense graph with $N = 25$

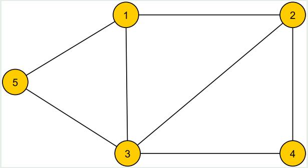


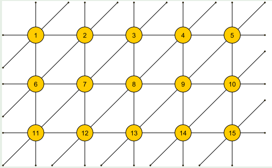


Degree Sequence [Erc15]

- The **degree sequence** of a graph G is the listing of the degrees of its vertices, usually in descending order.
- In **regular graphs** each vertex has the same degree.

Degree Sequence $[4, 3, 3, 2, 2]$





Degree Distribution [Erc15]

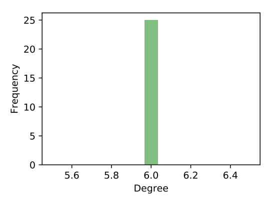
Definition 1 (Definition 3 [Erc15], p.65)

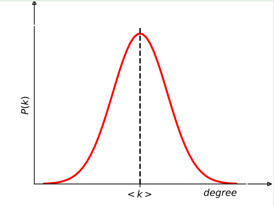
The degree distribution $P(k)$ of degree k in a graph G is given as the fraction of vertices with the same degree to the total number of vertices as below.

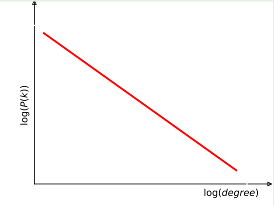
$$P(k) = \frac{n_k}{N} \tag{7}$$

where n_k is the number of vertices with degree k.

Degree distributions of regular, random, small-world graphs







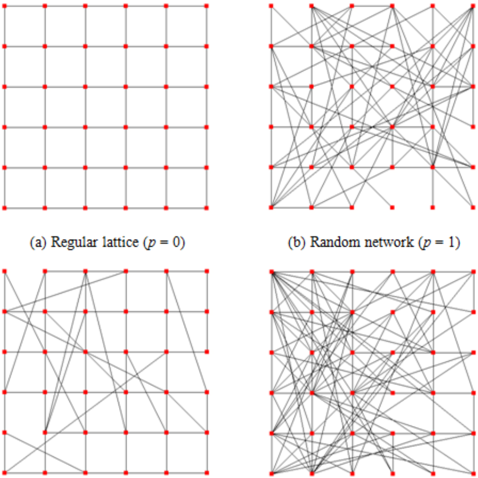
Outline

Random Graphs

- Basic idea
  - Edges are added at random between a fixed number N of vertices
  - Each instance is a snapshot at a particular time of a stochastic process, starting with unconnected vertices and for every time unit adding a new edge
- Four basic models of complex networks
  - **Regular lattices** (meshes) and trees
  - **Erdös-Renyi Random Graphs** (ER)
    - A disconnected set of nodes that are paired with a uniform probability.
  - Watts-Strogatz Models [WS98] (WS, SW)
    - **Small-world networks**
    - Connections between the nodes in a regular graph were rewired with a certain probability
  - Barabási-Albert Model [BAJ99] (BA, SF)
    - **Scale-free networks** characterized by a highly heterogeneous degree distribution, which follows a "power-law"

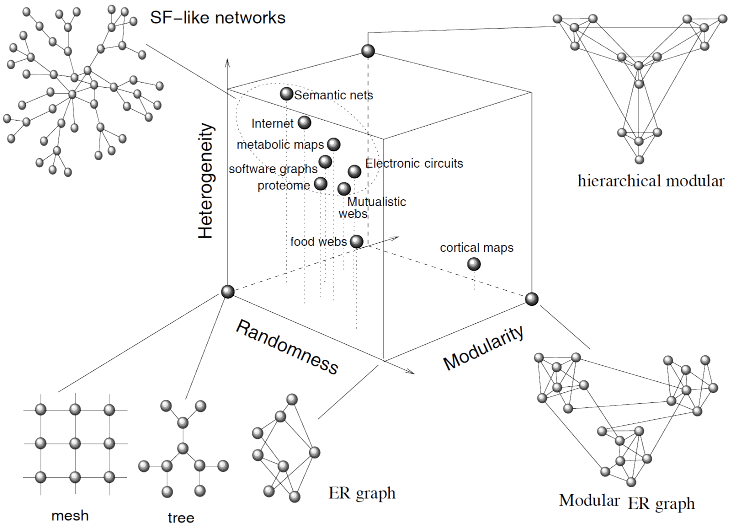
$$P(k) \sim k^{-\gamma}$$

Complex Network Models [GDZ+15]

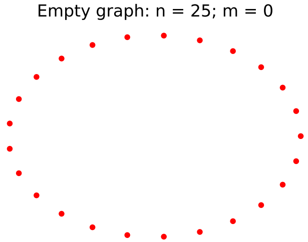


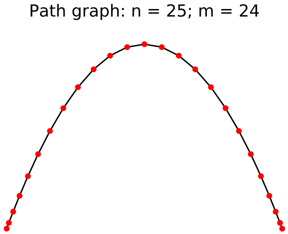
(a) Regular lattice ($p = 0$)
(b) Random network ($p = 1$)
(c) Small-world ($p = 0.01$)
(d) Scale-free ($n_0 = 3$, $m_0 = 3$)

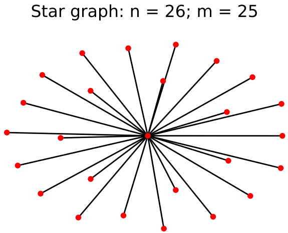
Zoo of Complex Networks [SV04]

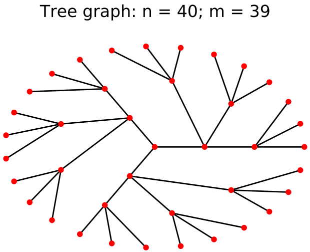


Basic Topologies of Graphs I



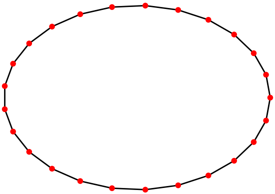




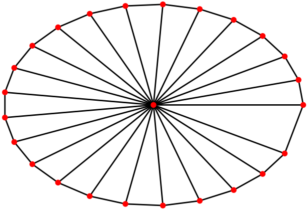


Basic Topologies of Graphs II

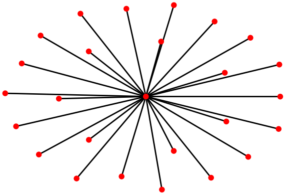
Cycle graph: n = 25; m = 25



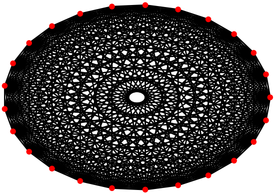
Wheel graph: n = 25; m = 48



Star graph: n = 26; m = 25

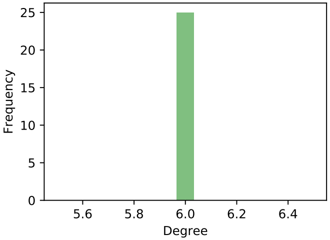


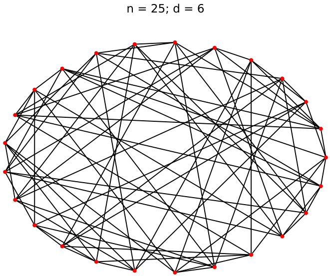
Complete graph: n = 25; m = 48

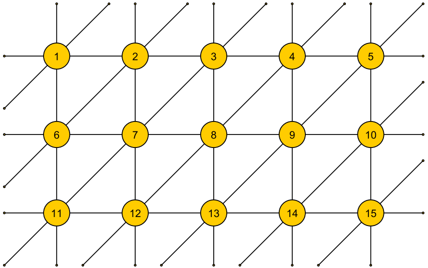


Regular Graph [Erc15]

- All vertices have the same degree.

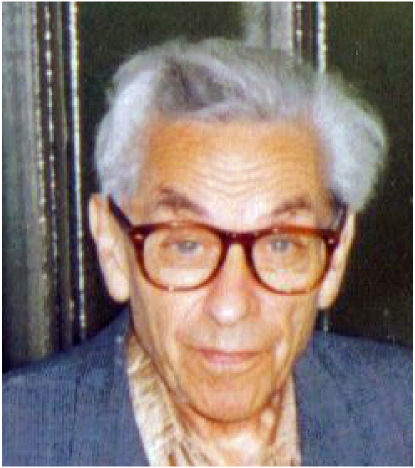






Outline

The Erdös and Renyi Model



Paul Erdös
(1913-1996)

Alfréd Rényi
(1921-1970)

Classical Random Graph (ER-model) [New10, Erc15]

- Proposed by Erdös and Renyi
- Let $G(V, E)$ be a simple graph with N vertices and M edges
- The propability to have an edge between any pair of nodes is distributed uniformly at random.

$$p = \frac{2M}{N(N-1)}$$

- The degree distribution of ER-model is binomial
  - A given vertex is connected with independent probability p to each of the $N-1$ other vertices.
  - The probability of being connected to a particular k other vertices and not to any of the others $p^k(1-p)^{N-1-k}$.
  - There are $\binom{N-1}{k}$ way to choose those k other vertices.
  - The total probability of being connected to exactly k others is

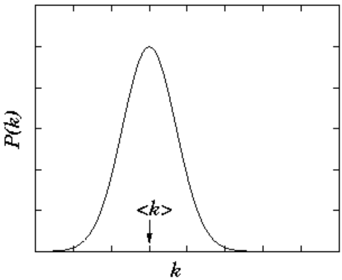
$$p_k = p(k) = \binom{N-1}{k} p^k (1-p)^{N-1-k}$$

ER-model Properties [New10, Erc15, EA15]

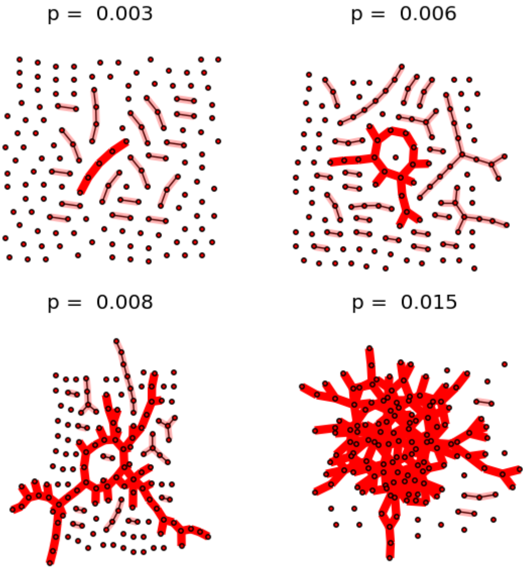
- It does not represent many real complex networks.
- It exhibits
  - homogeneous degree distribution.
  - a small diameter

- Approaching Poisson distribution as $N \to \infty$

$$P(k) \sim e^{-\langle k \rangle} \frac{\langle k \rangle^k}{k!}$$



ER-model. Giant Component [HSS08, New10]



Outline

Six Degree of Separation - Milgram Experiment 1967

- Random people from Nebraska were to send a letter (via intermediaries) to a stock broker in Boston.
- Could only send to someone with whom they were on a first-name basis.
- Among the letters that found the target, the average number of links was **six**.

  six degree of separation [Erc15]

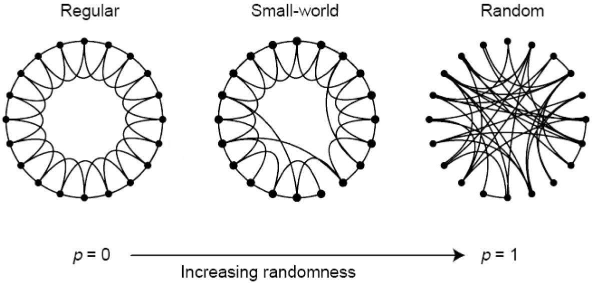
Stanley Milgram (1933 - 1984)

The Watts-Strogatz Model

Duncan J. Watts
(born 1971)

Steven Strogatz
(born 1959)

The Watts-Strogatz Small World Model



- A simple model for interpolating between regular and random networks
- Randomness controlled by a single tuning parameters

The Model

- Take a regular clustered network
- Rewire the endpoint of each link to a random node with probability p

Small World Model - Properties [Erc15, EA15]

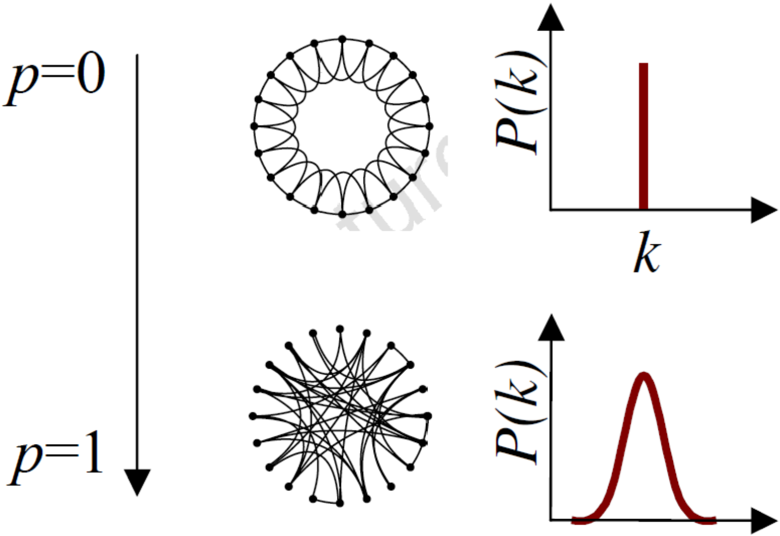
The Watts-Strogatz Model [WS98]

- Starting from the circulant network with n nodes connected to k neighbors.
- The diameter of the network increases with the logarithms of the network order:

$$d \approx \log N \text{ as } N \to \infty$$

- A high local clustering
  - The starting is a ring topology which each node is connected to its closest $k/2$ left neighbors and $k/2$ right neighbors

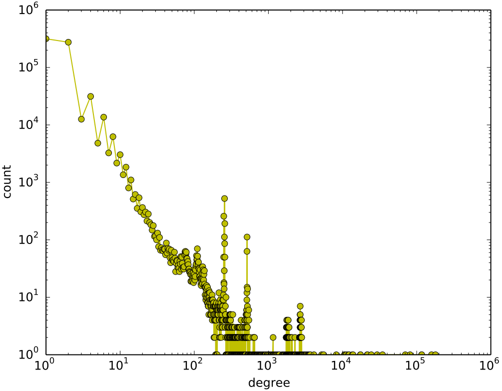
Small World Model - Degree Distributions [Erc15, EA15]



Outline

Real-world Networks with Fat-tail Distributions [Erc15, EA15]

- Many networks in the real-world have a fat-tailed degree distribution.
- Many real-life complex networks dynamically grow and change by adding and removing nodes and edges.
- Free-scale IP2IP network



The Barabási and Albert Model

Albert-László Barabási
(born 1967)

Réka Albert
(born 1972)

Scale-Free (BA) Network [BAJ99, Erc15, EA15]

Node Degree Distribution

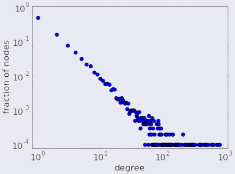
- a heavy-tailed distribution
- follows a power law (asymptotically)

$$P(k) \sim k^{-\gamma}$$

Assumptions:

- Preferential attachment
- Fitness model

Degree Distribution



Small network hub

Scale-Free (BA) Network [BAJ99, Erc15, EA15]

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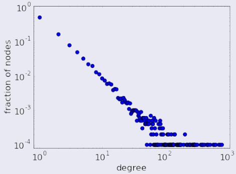
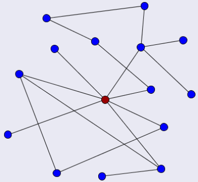
Degree Distribution



Small network hub



Barabási-Albert Model [BAJ99, Erc15, EA15]

The outline of the model:

- Begin with a small number, m_0, of nodes.
- At each step, add a new node v to the network, and connect it to $m \leq m_0$ of the existing nodes $u \in V$ with probability

$$p_{uv} = \frac{k_u}{\sum_{w \in V} k_w}$$

Algorithm BA_Generator

```
1: Input: G(V, E), V_new ...new vertices to joined to G
2: m_0 ← |E|
3: for all v ∈ V_new do
4:     V ← V ∪ {v}
5:     for m = 0; m ≤ m_0; m++ do
6:         attach v to u ∈ V with probability P_uv = k_u / Σ_{w∈V} k_w
7:     end for
8: end for
```

Scale-Free (BA) Network - Properties [BAJ99, Erc15, EA15]

- **Scale-free property**, c is a constant

$$p(k) = Ak^{-\gamma}$$
$$p(ck) = A(ck)^{-\gamma} = c^{-\gamma} p(k)$$

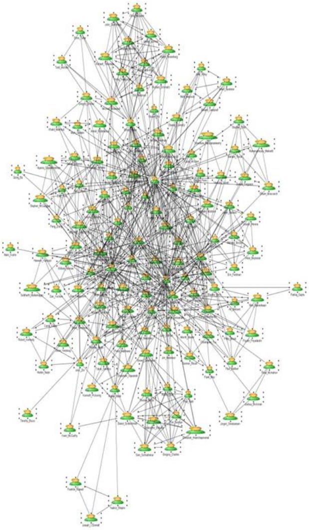
- The intercept and the slope is preserved on a logarithmic scale

$$\ln p(k) = -\gamma \ln k + \ln A$$
$$\ln p(ck) = -\gamma \ln(ck) + \ln A = -\gamma \ln(k) + \ln A - \gamma \ln(c)$$

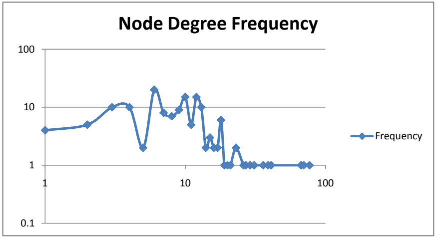
- Degree distribution follows power law, with the exhibition of very few high degree nodes and many low degree nodes. $P(k) \sim k^{-3}$
- The average clustering coefficient of these networks is low due to the large number of low-degree nodes. $C \sim N^{-0.75}$
- The average diameter is low due to the clustering of nodes around the high-degree nodes. $\ell \sim \frac{\ln N}{\ln \ln N}$

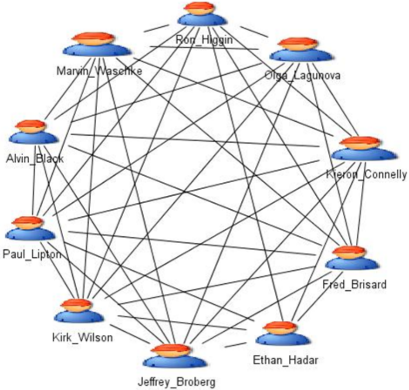
Outline

Example - Collaboration of People on Projects



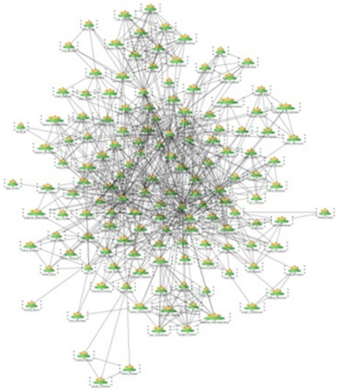
Collaboration SNA Analysis - Rich Club Problem



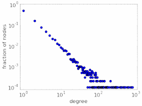


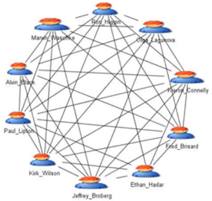
Facilitators

Collaboration Decomposition - Rich Clubs



Original network of people





Hubs (facilitators)

Teams

Outline

Assortativity [New02, New03b]

- the presence of non trivial correlations in network connectivity pattern.
- **Assortative mixing**, or **assortativity**, or **homophily** in SNA (CZ asortativní párování) (i.e., "love of the same") is the tendency of agents to associate and bond with similar others.
    - as in the proverb "birds of a feather flock together"
- **Disassortative mixing** is a bias in favor of connections between dissimilar nodes.
- **Degree correlations** . . . assortativity regarding to node degree.
- **Assortativity coefficient**: vertex is labeled with a scalar value or an enumerative/categorical value (e.g., shape, color) [New02, New03a].

Rich Club [ZM04, CFSV06]

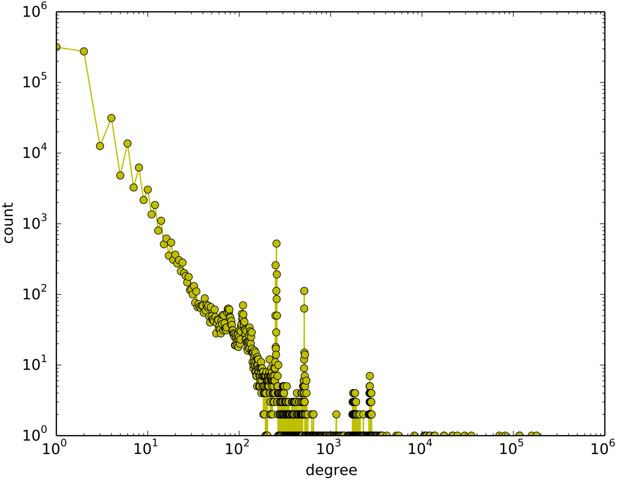
- **Rich-club phenomenon**: Hubs (nodes of high degree) tend to connect to other hubs (rich tends to connect to other rich)
- **Rich-club coefficient** . . . the fraction between the *actual* and the potential number of edges among $V_{>k}$.

$$\Phi(k) = \frac{2E_{>k}}{N_{>k}(N_{>k} - 1)}$$

where

- $V_{>k}$ is the set of vertices with degree larger than k,
- $N_{>k}$ is the number of such vertices, and
- $E_{>k}$ is the number of edges among such vertices.

Real-world Networks with Fat-tail Distributions



Summary

- Complex networks basic characteristics
- Topological forms
- Random Network Models
    - Classical Erdös-Renyi model
    - Small world model
    - Scale-free model
- Rich club detection

Competencies

- Describe the network perspective approach to problem solutions.
- What are the typical characteristics of complex networks?
- Describe the meaning of degree hetergeneity.
- Define graph density and sparsity.
- Define graph degree distribution and show some its typical examples.
- List the four basic models of complex networks and their characteristics.
- List basic graph topologies.
- Describe Erdös-Renyi graph model.
- Describe Watts-Strogatz graph model.
- Describe Barabási-Albert graph model and its scale-free property.
- What is the meaning of “the rich-club phenomenon”.

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