STRUCTURED MODEL LEARNING (SS2022)
4. SEMINAR

Assignment 1. Let $x = \{x_i \mid i \in V\}$ be a set of binary valued variables, i.e. $x_i = 0, 1$.

a) Prove by induction over the number of variables that every function $f(x)$ can be written as a polynomial

$$f(x) = \sum_{C \subset V} a_C \prod_{i \in C} x_i,$$

where the sum is over all subsets of V and a_C are some coefficients.

b) Conclude that the distribution for a binary valued Gibbs random field on a graph (V, E) can be written as

$$p(x) = \frac{1}{Z(u)} \exp\Big[\sum_{i \in V} u_i x_i + \sum_{ij \in E} u_{ij} x_i x_j\Big]$$

with some real numbers u_i, u_{ij}.

Assignment 2. Consider an Ising model on an undirected graph (V, E), i.e. a binary valued random field with joint distribution

$$p(x) = \frac{1}{Z} \exp\Big[-\alpha \sum_{\{i,j\} \in E} |x_i - x_j|\Big],$$

where $x_i = 0, 1$ and $\alpha > 0$.

a) Find the configurations $x \in \mathcal{B}^V$ with highest probability $p(x)$.

b) What are the marginal probabilities $p(x_i)$, $x_i = 0, 1$ of this model? *Hint:* use an symmetry argument.

c) Let us assume that the graph (V, E) is a two-dimensional rectangular grid and let us fix the states on its boundary to $x_i = 1$. How will this affect the marginal probabilities of the remaining nodes? Will this influence diminish if the size of the lattice increases?

Assignment 3. Let $X \in \mathbb{R}$ be a normally distributed random variable, i.e.

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{(x-\mu)^2}{2\sigma^2}}.$$

a) Prove the equality

$$\frac{\partial}{\partial \mu} \mathbb{E}_{\mathcal{N}(\mu,\sigma)} f(x) = \mathbb{E}_{\mathcal{N}(\mu,\sigma)} f'(x),$$

where $f'(x)$ denotes the derivative of f. Hint: use the substitution $\tilde{x} = (x - \mu)/\sigma$ in the integral for the expectation.

b) Prove the equality

$$\frac{\partial}{\partial \sigma} \mathbb{E}_{\mathcal{N}(\mu,\sigma)} f(x) = \mathbb{E}_{\mathcal{N}(\mu,\sigma)} f''(x).$$

Hint: use the same substitution as in a) and integration by parts.