SMU: Lecture 2

(Model-Free Policy Evaluation in RL + Intro to Model-Free Control)

Monday, February 27, 2023

(Heavily inspired by the Stanford RL Course of Prof. Emma Brunskill, but all potential errors are mine.)

Plan for The First Part

- Policy evaluation when we do not know the model (neither the state-transition probabilities, nor the reward functions).
- Two kinds of methods today (there are more out there):
    - Monte-Carlo Policy Evaluation
    - Temporal-Difference Learning

Part 0: Reminder from Last Lecture

Markov Reward Process

Markov reward process = Markov process + Reward

Formally, MRP is given by:

- A set of states S.
- A transition model $P[X_{t+1} = s' | X_t = s]$, which we also denote by $P(s' | s)$.
- A reward function $R(s) = \mathbb{E}[R_t | X_t = s]$, which is the expected reward the agent receives in state s, $(s \in S)$.
- A discount factor $\gamma \in [0; 1]$.

Return from an Episode

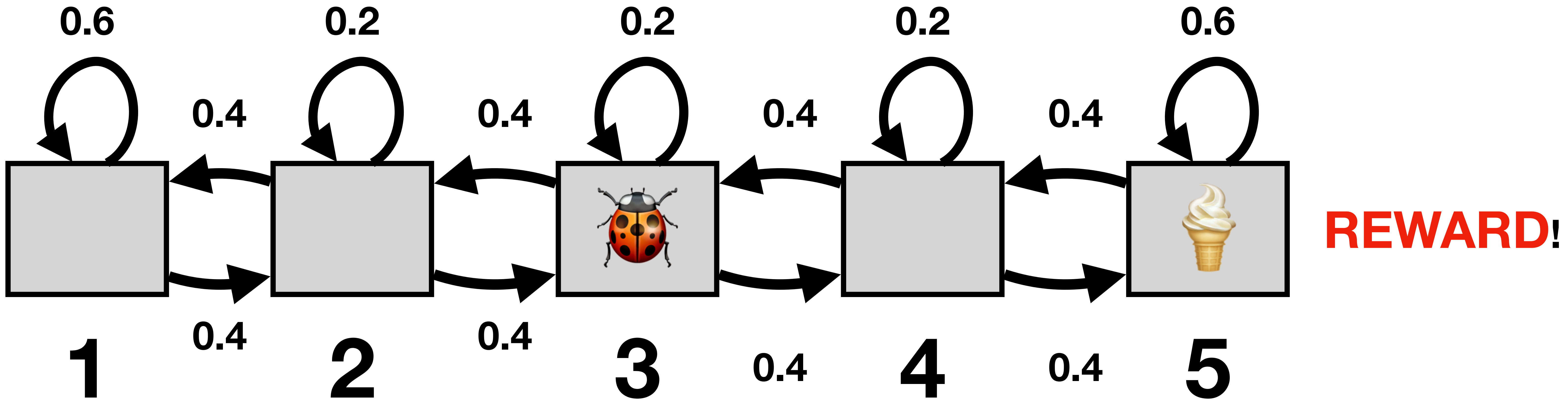
- **Horizon:**
  - Number of time steps in an episode (which can also be infinite). **We will first assume infinite horizons** (they are easier because they will lead to stationary, i.e. time-independent, policies!).

- **Return g_t:**
  - **Given:** An episode $s_1, s_2, s_3, s_4, \ldots, s_H$.
  - **Compute:** Return g_t = discounted sum of rewards from time t.
  - **As a formula:**

$$g_t = R(s_t) + R(s_{t+1}) \cdot \gamma + R(s_{t+2}) \cdot \gamma^2 + \ldots = R(s_t) + \sum_{i=1} R(s_{t+i}) \cdot \gamma^i$$

Markov Reward Process

Markov reward process = Markov process + Reward

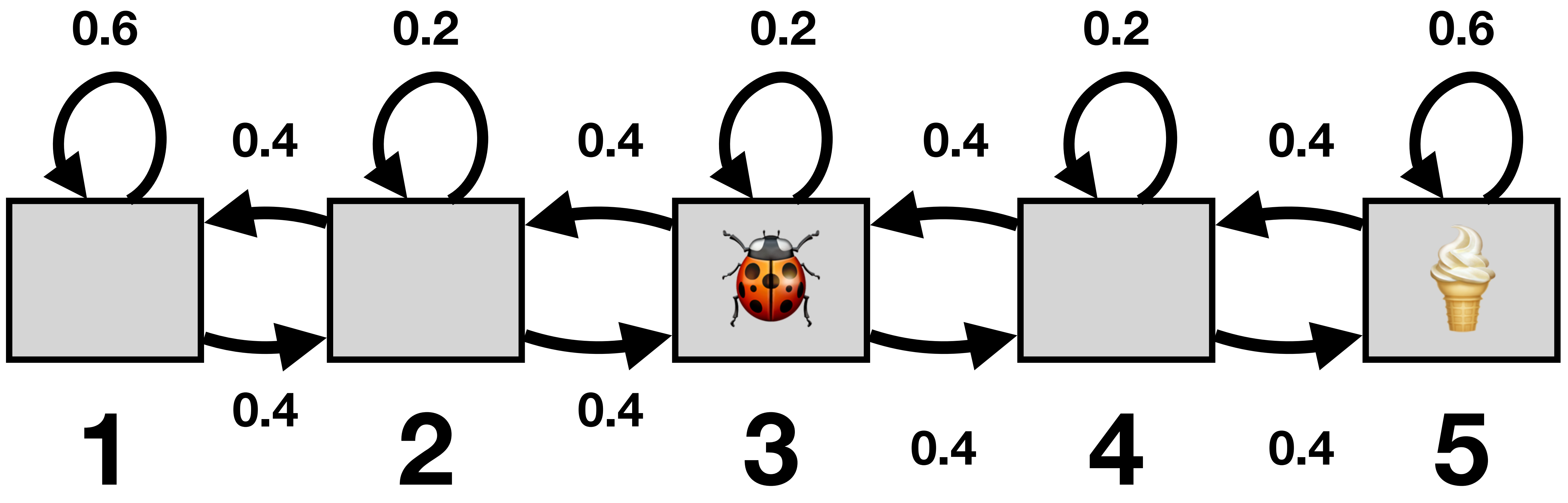


For example:

$$R(s) = \begin{cases} 0, & s = 1 \\ 0, & s = 2 \\ 0, & s = 3 \\ 0, & s = 4 \\ 10, & s = 5 \end{cases}$$

We expect that each time we visit s_5, there will be ice cream (i.e. we are not running out of it).

Episode (An Example)

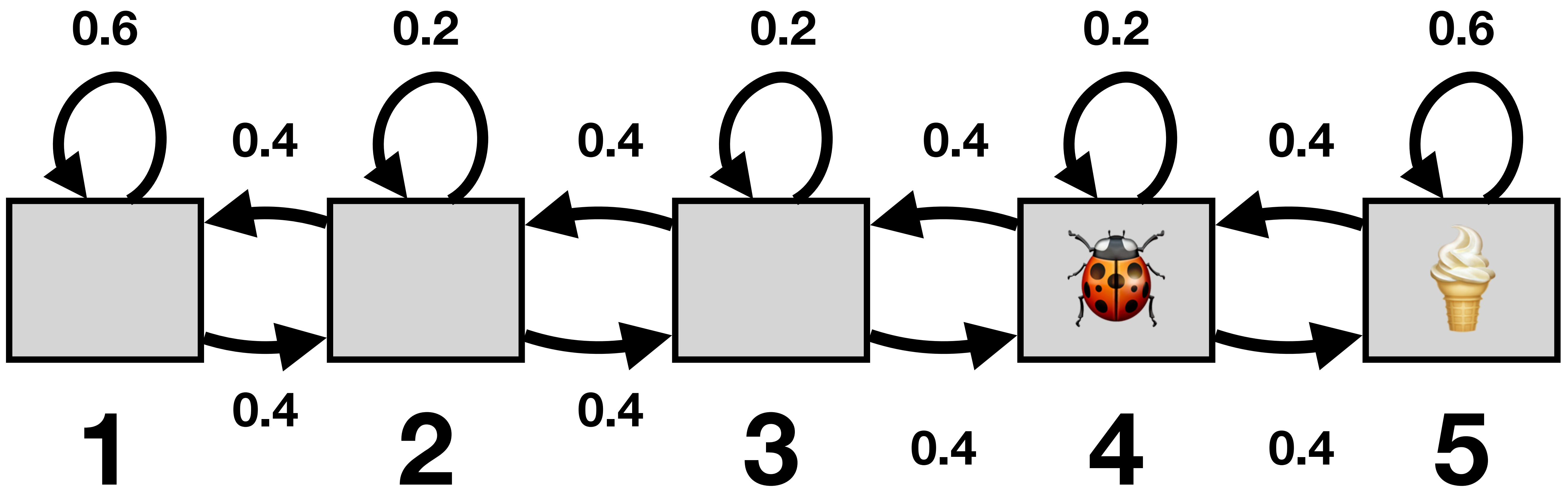


Time: $t = 1$

Current state: $s_1 = 3$, Current reward: $r_1 = 0$

Episode: 3

Episode (An Example)

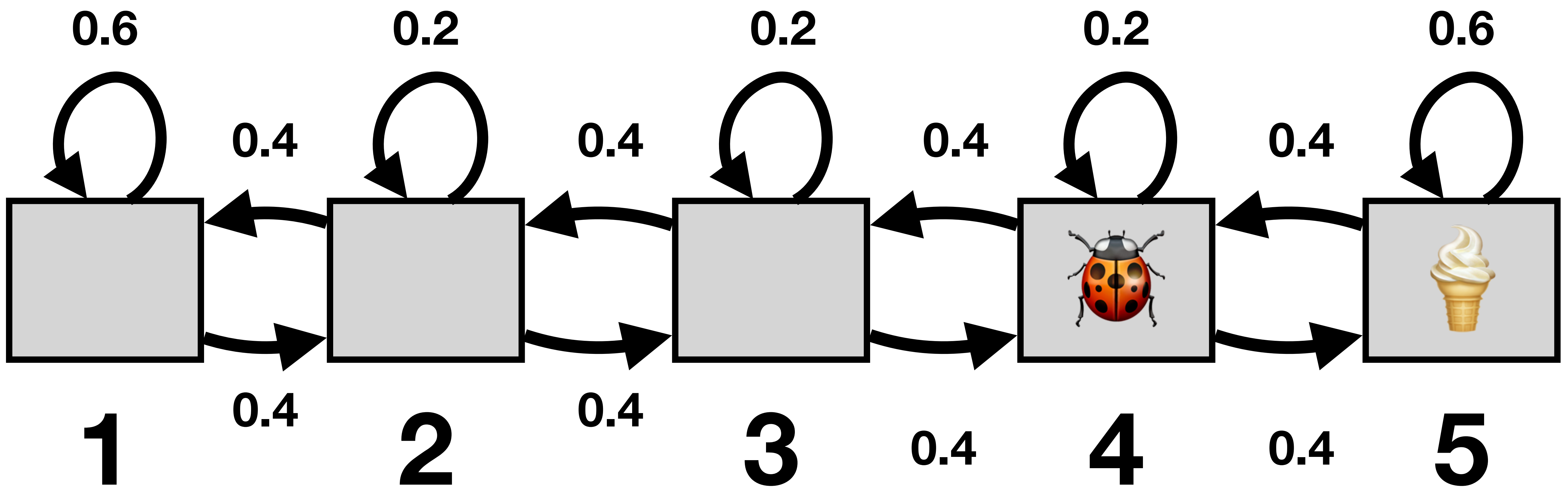


Time: $t = 2$

Current state: $s_2 = 4$, Current reward: $r_2 = 0$

Episode: $3, 4$

Episode (An Example)

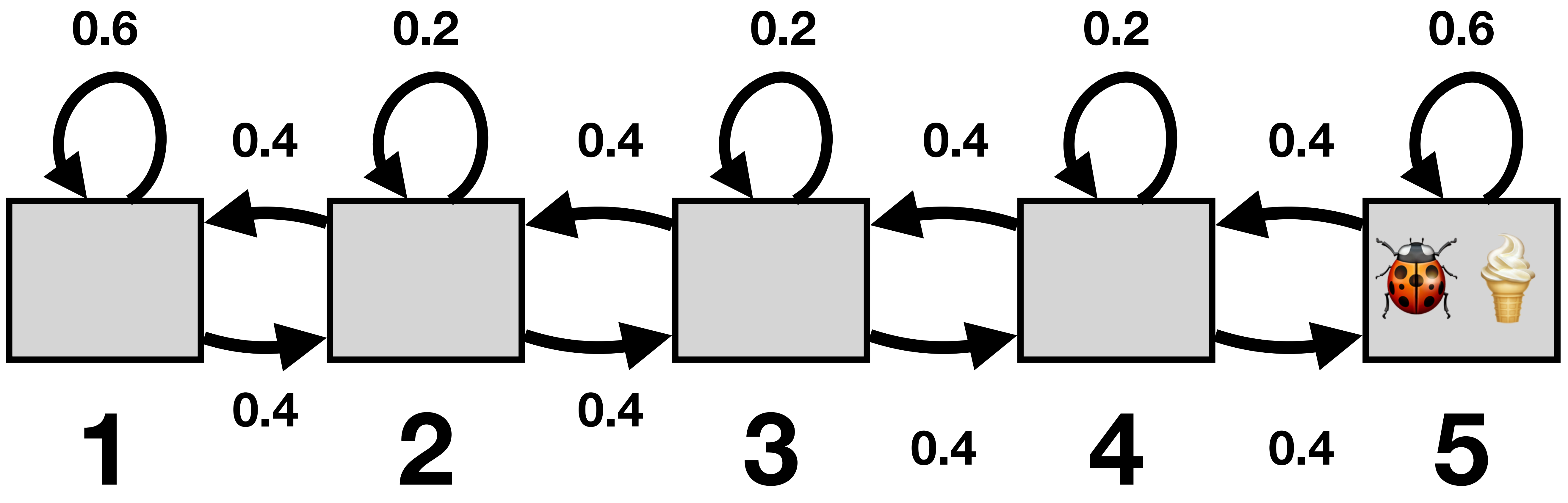


Time: $t = 3$

Current state: $s_3 = 4$, Current reward: $r_3 = 0$

Episode: $3, 4, 4$

Episode (An Example)

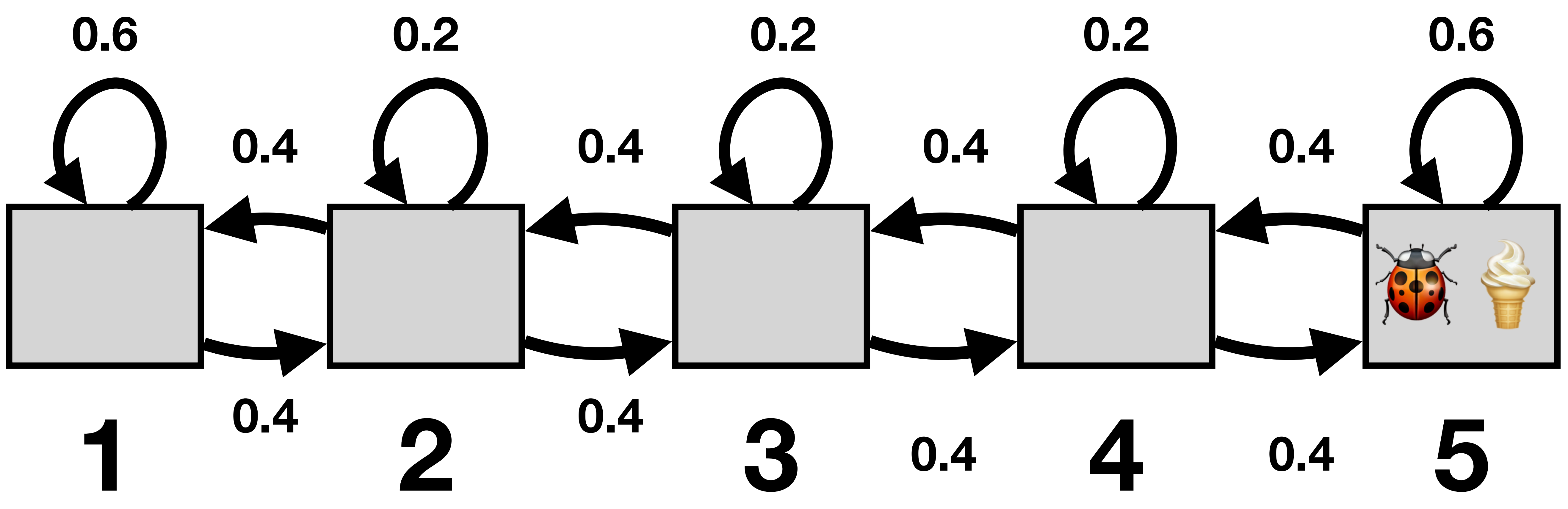


Time: $t = 4$

Current state: $s_4 = 5$, Current reward: $r_4 = 10$

Episode: $3, 4, 4, 5$

Episode (An Example)

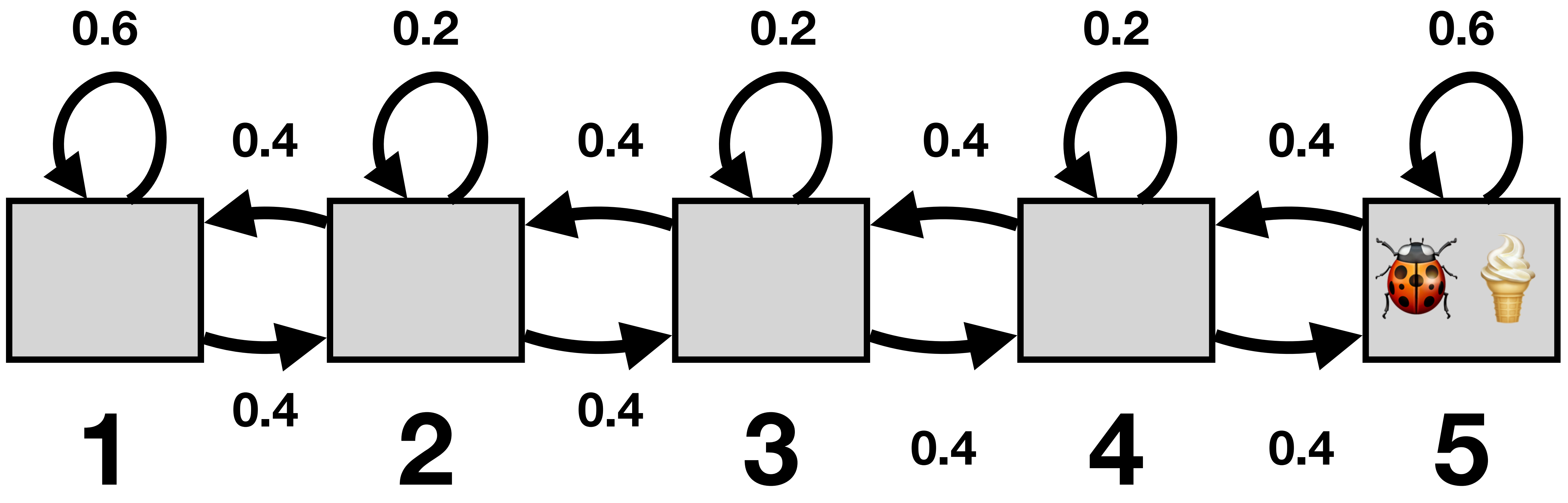


Time: $t = 5$

Current state: $s_4 = 5$, Current reward: $r_5 = 10$

Episode: $3, 4, 4, 5, 5$

Episode (An Example)



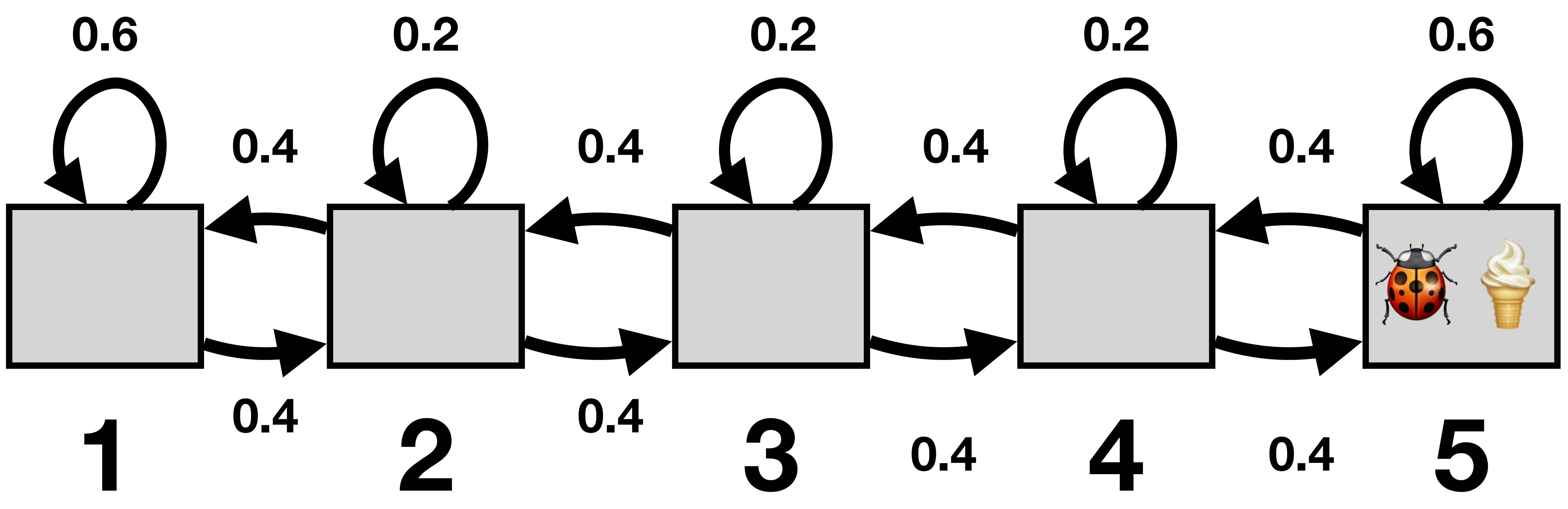
Time: $t = 5$

Current state: $s_4 = 5$

Episode: $3, 4, 4, 5, 5$

$$g_1 = 0 + 0 \cdot 0.5 + 0 \cdot 0.5^2 + 10 \cdot 0.5^3 + 10 \cdot 0.5^4 = 1.875$$

Episode (An Example)



Time: $t = 5$

Current state: $s_4 = 5$

Episode: $3, 4, 4, 5, 5$

$g_3 = 0 + 10 \cdot 0.5 + 10 \cdot 0.5^2 = 7.5$

Return (Random Variable)

- What we had on the previous slide was return from one specific sampled episode.
- Next we define **return** of a Markov reward process as a random variable (it is important to understand the distinction between the two):

$$G_t = R(X_t) + \gamma \cdot R(X_{t+1}) + \gamma^2 \cdot R(X_{t+2}) + \ldots = \sum_{i=0}^{\infty} R(X_{t+i}) \cdot \gamma^i$$

Markov Decision Process

- **Markov decision process = Markov reward process + Actions**
- **An MDP is given by:**
  - A set of states S.
  - A set of actions A.
  - A transition model $P[X_{t+1} = s' | X_t = s, A_t = a] = \underbrace{P(s' | s, a)}_{\text{notation}}$
  - A reward $R(s, a) = \mathbb{E}[R_t | X_t = s, A_t = a]$, i.e. the expected reward that the agent receives when performing action a in state s.
  - Discount factor γ.

Policy

- Policy determines which action to take in each state s.
- It can be either deterministic or random — that is also why policy will not simply be a function from states to actions.
- **We define policy:** $\pi(a \mid s) = P(A_t = a \mid X_t = s)$.
- **Example** (policy for our ladybug 🐞):
  - $A = \{\text{left}, \text{right}\}$
  - $\pi(\text{left} \mid 1) = 0$, $\pi(\text{right} \mid 1) = 1$, $\pi(\text{left} \mid 2) = 0.5$, $\pi(\text{right} \mid 1) = 0.5, \ldots$

MDP+Policy = MRP

- When we specify a policy for a given MDP, we are effectively turning the MDP into a corresponding MRP.

- **Formally:**

  - Given an MDP (A, S, P, R, γ), we turn it into an MRP $(S, P^\pi, R^\pi, \gamma)$ where

$$P^\pi(s' \mid s) = \sum_{a \in A} \pi(a \mid s) \cdot P(s' \mid s, a) \text{ *}$$

$$R^\pi(s) = \sum_{a \in A} \pi(a \mid s) \cdot R(s, a)$$

* In the more verbose notation: $P^\pi[X_{t+1} = s' \mid X_t = s] = \sum_{a \in A} \pi(a \mid s) \cdot P[X_{t+1} = s' \mid A_t = a, X_t = s]$.

MDP+Policy (An Example)

If we take the MDP with $S = \{1,2,3,4,5\}$, $A = \{\text{left, right, eat}\}$ and the state transition probabilties:

$$P(s' \mid s, \text{left}) = \begin{cases} 0.1 & s = s' \\ 0.9 & s - s' = 1, \\ 0 & \text{otherwise} \end{cases} \quad P(s' \mid s, \text{right}) = \begin{cases} 0.1 & s = s' \\ 0.9 & s' - s = 1, \\ 0 & \text{otherwise} \end{cases} \quad P(s' \mid s, \text{eat}) = \begin{cases} 1 & s = s' \\ 0 & \text{otherwise} \end{cases}$$

...and with the policy:

$$\pi(\text{left} \mid s) = \begin{cases} 0 & s = 1 \\ 0.5 & s \in \{2,3,4\}, \\ 0.5 & s = 5 \end{cases} \quad \pi(\text{right} \mid s) = \begin{cases} 1 & s = 1 \\ 0.5 & s \in \{2,3,4\}, \\ 0 & s = 5 \end{cases} \quad \pi(\text{eat} \mid s) = \begin{cases} 0 & s \in \{1,2,3,4\} \\ 0.5 & s = 5 \end{cases}$$

The states:

MDP+Policy (An Example)

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Now we will show the resulting Markov reward process:

MDP+Policy (An Example)

If we take the MDP with $S = \{1,2,3,4,5\}$, $A = \{\text{left}, \text{right}, \text{eat}\}$ and the state transition probabilties:

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...and with the policy:

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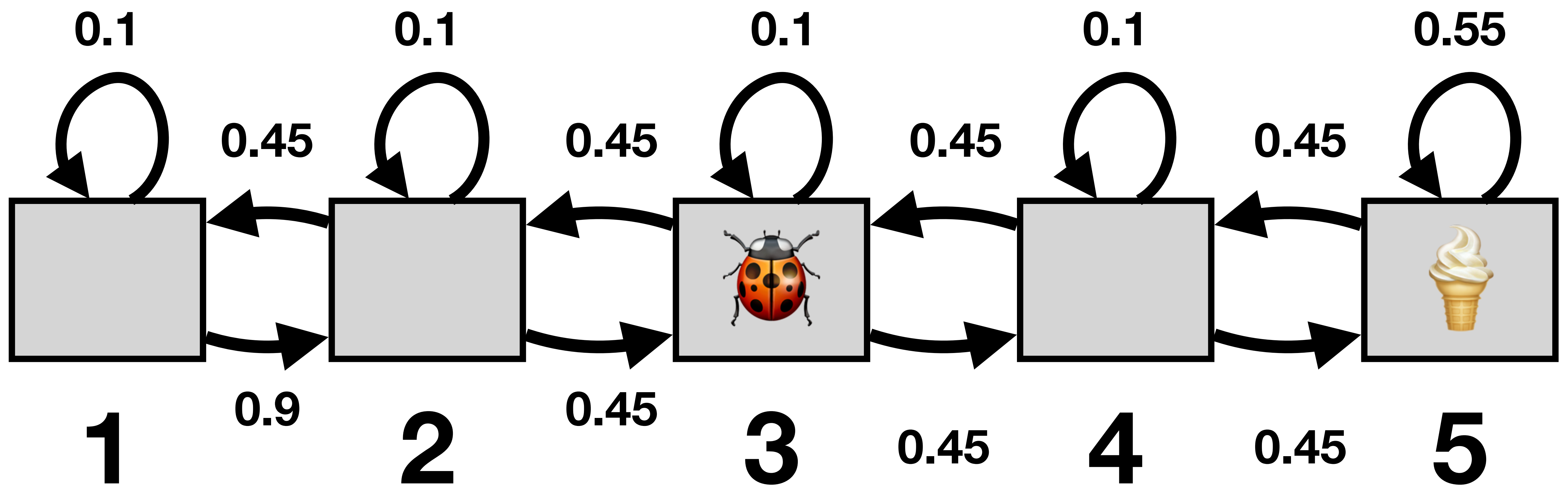
...then we get the following Markov reward process:



MDP+Policy (An Example)

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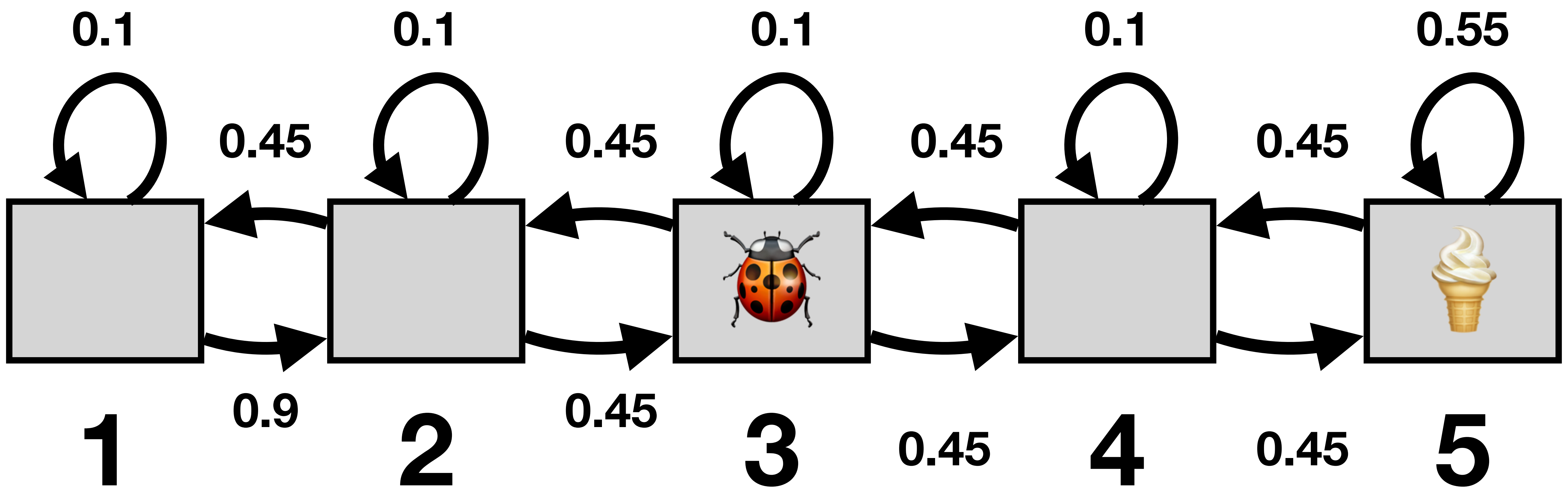

For example:

$P^{\pi}(2 \mid 3) = \pi(\textbf{left} \mid 3) \cdot P(2 \mid 3, \textbf{left}) +$

$+ \pi(\textbf{right} \mid 3) \cdot P(2 \mid 3, \textbf{right}) +$

$+ \pi(\textbf{eat} \mid 3) \cdot P(2 \mid 3, \textbf{eat}) =$

$= 0.5 \cdot 0.9 + 0.5 \cdot 0 + 0 \cdot 0 = 0.45$

MDP+Policy (An Example)

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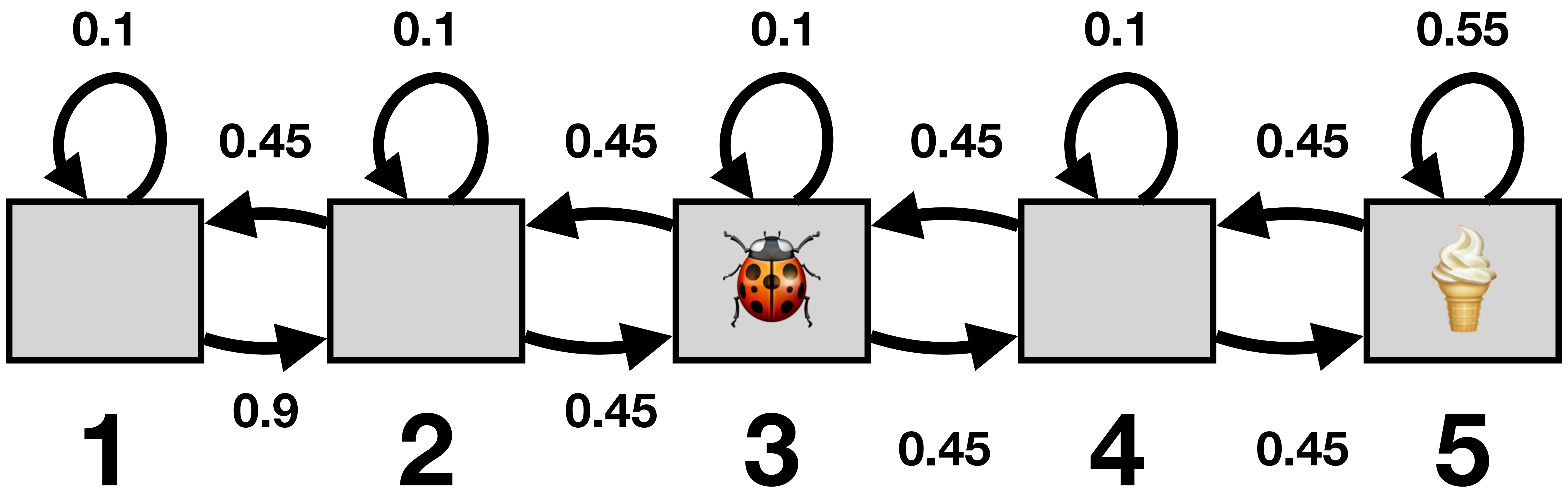


For example:

$P^\pi(2 \mid 2) = \pi(\textbf{left} \mid 2) \cdot P(2 \mid 2,\textbf{left}) +$

$+ \pi(\textbf{right} \mid 2) \cdot P(2 \mid 2,\textbf{right}) +$

$+ \pi(\textbf{eat} \mid 2) \cdot P(2 \mid 2,\textbf{eat}) =$

$= 0.5 \cdot 0.1 + 0.5 \cdot 0.1 + 0 \cdot 1 = 0.1$

MDP+Policy (An Example)

Now, for the rewards, suppose the reward function of the MDP is:

$$R(s,a) = \begin{cases} 10 & s = 5 \text{ and } a = \text{eat} \\ 0 & \text{otherwise} \end{cases}$$

and we still use the same policy:

$$\pi(\text{left} \mid s) = \begin{cases} 0 & s = 1 \\ 0.5 & s \in \{2,3,4\}, \\ 0.5 & s = 5 \end{cases} \quad \pi(\text{right} \mid s) = \begin{cases} 1 & s = 1 \\ 0.5 & s \in \{2,3,4\}, \\ 0 & s = 5 \end{cases} \quad \pi(\text{eat} \mid s) = \begin{cases} 0 & s \in \{1,2,3,4\} \\ 0.5 & s = 5 \end{cases}$$

then the reward function of the resulting Markov reward process is:

$$R^{\pi}(s) = \begin{cases} 5 & s = 5 \\ 0 & \text{otherwise} \end{cases},$$

MDP+Policy (An Example)

Now, for the rewards, suppose the reward function of the MDP is:

$$R(s,a) = \begin{cases} 10 & s = 5 \text{ and } a = \text{eat} \\ 0 & \text{otherwise} \end{cases}$$

and we still use the same policy:

$$\pi(\text{left}\,|\,s) = \begin{cases} 0 & s = 1 \\ 0.5 & s \in \{2,3,4\}, \\ 0.5 & s = 5 \end{cases} \qquad \pi(\text{right}\,|\,s) = \begin{cases} 1 & s = 1 \\ 0.5 & s \in \{2,3,4\}, \\ 0 & s = 5 \end{cases} \qquad \pi(\text{eat}\,|\,s) = \begin{cases} 0 & s \in \{1,2,3,4\} \\ 0.5 & s = 5 \end{cases}$$

then the reward function of the resulting Markov reward process is:

$$R^\pi(s) = \begin{cases} 5 & s = 5 \\ 0 & \text{otherwise} \end{cases},$$

here, for instance.

$$R^\pi(5) = \pi(\text{eat}\,|\,5) \cdot R(5,\text{eat}) + \pi(\text{left}\,|\,5) \cdot R(5,\text{left}) + \pi(\text{right}\,|\,5) \cdot R(5,\text{right}) = 0.5 \cdot 0 + 0.5 \cdot 10 + 0 \cdot 0 = 5$$

(State) Value Function

- **Definition:**

$$V(s) = \mathbb{E}[G_t \,|\, X_t = s] = \mathbb{E}[R(X_t) + \gamma \cdot R(X_{t+1}) + \gamma^2 \cdot R(X_{t+2}) + \ldots \,|\, X_t = s]$$

- **Intuition:** Value function $V(s)$ is the expected return when starting from state s.

(Bellman equation for MDP)

State Value Function of MDP

General case:

$$V^{\pi}(s) = \sum_{a \in A} \pi(a, s) \cdot \left[R(s, a) + \gamma \cdot \sum_{s' \in S} P(s' \,|\, s, a) \cdot V^{\pi}(s') \right]$$

Version for deterministic policy:

$$V^{\pi}(s) = R(s, \pi(s)) + \gamma \cdot \sum_{s' \in S} P(s' \,|\, s, \pi(s)) \cdot V^{\pi}(s')$$

Part 1: Problem Statement

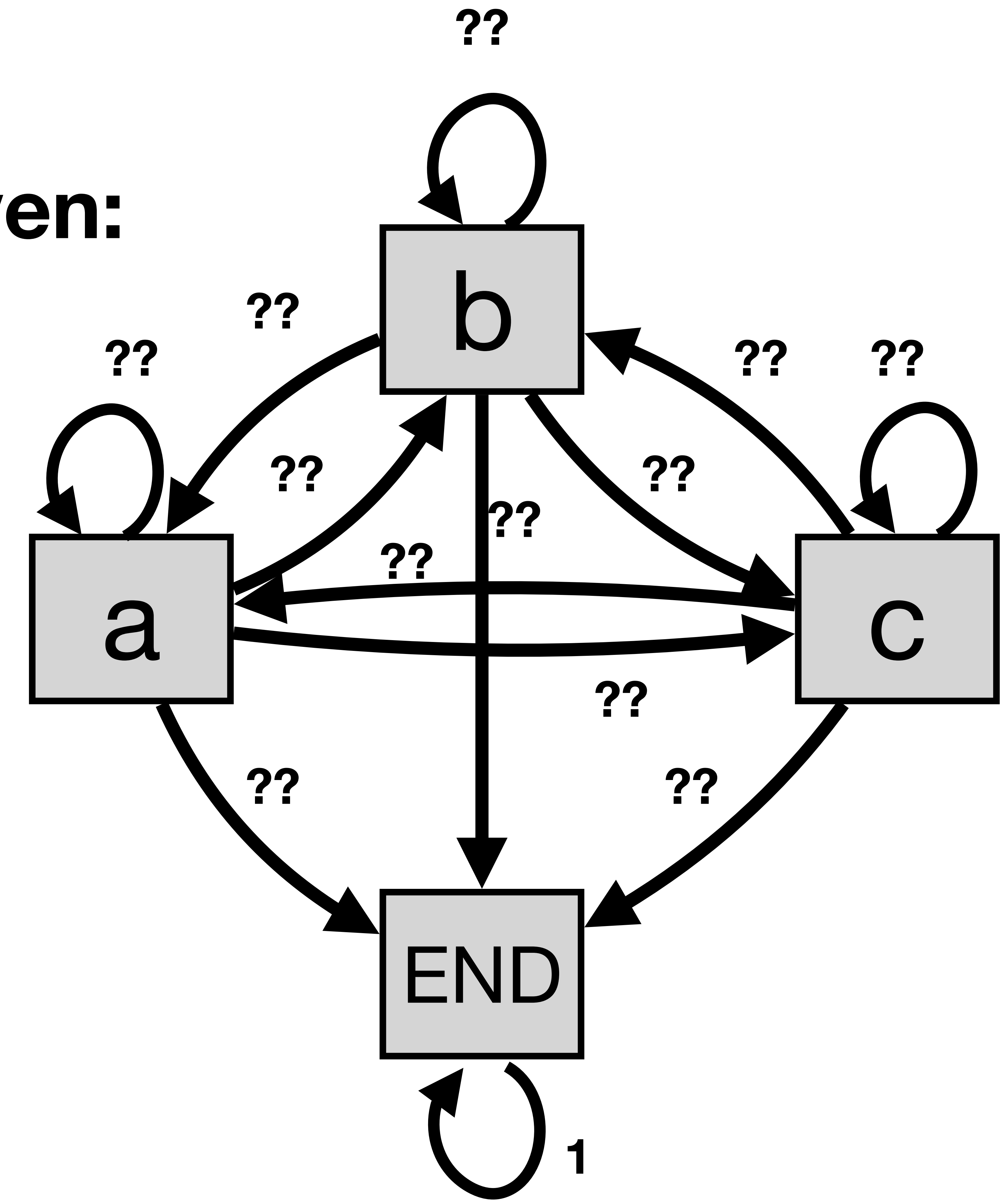
Problem: Model-Free Policy Evaluation

- Given a policy and an MDP with unknown parameters (or generally an environment with which we can interact), **estimate the value function.**

Example

Agent: 

States are given:



Rewards??

Actions are given:

$$A = \{l, r\}$$

Policy is given, e.g.:

$$\pi(l \mid a) = 0.2,\ \pi(r \mid a) = 0.8,$$

$$\pi(l \mid b) = 0.3,\ \pi(r \mid b) = 0.7,$$

...

Problem: Model-Free Policy Evaluation

- **Our task again:**
  - Given a policy and an MDP with unknown parameters (or generally an environment with which we can interact), **estimate the value function.**

An Assumption

- **Assumption:** In what follows we will assume that our MDP has terminal states and that the probability of infinitely long runs is zero.
- **Terminal states:** Once the system gets into a terminal state, it stays in it. The reward in the terminal state is always 0.
- **Why do we do this?** This assumption will allow us to use the formalism for infinite-horizon problems (which is mathematically simpler).

Part 2: Statistical Properties of Estimators

(An informal recap of what you already know from statistics)

Estimators (Statistics)

- **Typical setting:**
  - We are given a sample of random variables $X_1, X_2, \ldots, X_n$.
  - Suppose that we want to estimate some parameter θ, e.g., suppose all the X_i's are sampled independently from the same distribution and we want to estimate the mean of this distribution.
  - An **estimator of** θ is a function $\hat{\theta}$ that maps samples to estimates of the parameter θ.

Estimators as Random Variables

- **Example:** Let us have a normal distribution with mean μ and standard deviation σ. Denote by $\mathbf{X} = (X_1, X_2, \ldots, X_N)$ an independent sample from this distribution. Then the sample mean $\hat{\mu}(\mathbf{X}) = \frac{1}{N}\sum_{i=1}^{N} X_i$ is an estimator for the population mean μ.

- Note that, in this example, $\hat{\mu}(\mathbf{X})$ is a **random variable**.

Bias

Bias of an estimator $\hat{\theta}$ is defined as: $\text{BIAS}_\theta(\hat{\theta}) = \mathbb{E}_\theta[\hat{\theta}(\mathbf{X})] - \theta$.

If $\text{BIAS}_\theta(\hat{\theta}) = 0$ then we say that $\hat{\theta}$ is an unbiased estimator.

Example: $\dfrac{1}{N}\sum_{k=1}^{N} X_k$ is an unbiased estimator of population mean. Why?

Because we have $\mathbb{E}\left[\frac{1}{N}\sum_{k=1}^{N} X_k\right] = \frac{1}{N}\sum_{k=1}^{N}\mathbb{E}\left[X_k\right] = \frac{1}{N}\cdot N\cdot\mathbb{E}\left[X_k\right] = \mu$.

Mean Squared Error

Mean squared error of an estimator $\hat{\theta}$ is defined as: $\text{MSE}_\theta(\hat{\theta}) = \mathbb{E}_\theta[(\hat{\theta}(\mathbf{X}) - \theta)^2]$.

It holds $MSE_\theta(\hat{\theta}(\mathbf{X})) = \text{Var}_\theta(\hat{\theta}(\mathbf{X})) + \text{BIAS}(\hat{\theta}(\mathbf{X}))^2$.

Consistency

Let $\mathbf{X}_N = (X_1, \ldots, X_N)$ be an independent sample, used to estimate θ.

A sequence of estimators $\hat{\theta}_N(\mathbf{X}_N)$ is said to be consistent if for every $\varepsilon > 0$ it holds: $\lim_{N\to\infty} P[\,|\hat{\theta}_N(\mathbf{X}_N) - \theta| < \varepsilon] = 1.$

Why It Matters

- Estimators that we are going to study in this lecture can be analyzed in the same framework. After all, they are just statistical estimators.

Part 3: Monte-Carlo Policy Evaluation

Monte-Carlo Policy Evaluation (1/5)

Recall the definition of G_t, the return at time t *(we have not shown it explicitly for MDPs last time):*

$$G_t^\pi = R(X_t, A_t) + \gamma \cdot R(X_{t+1}, A_{t+1}) + \gamma^2 \cdot R(X_{t+2}, A_{t+2}) + \ldots = \sum_{i=0}^{\infty} R(X_{t+i}, A_{t+i}) \cdot \gamma^i$$

(for simplicity, we assume that the reward when R(a,s) is deterministic)

where X_i's and $A_i's$ are random variables — X_i is the state at time t and A_i is the action at time i. We suppose that these random variables are from an MDP with a policy π (which together define the distribution of these random variables).

Monte-Carlo Policy Evaluation (2/5)

The state value function $V^{\pi}(s)$ is:

$$V^{\pi}(s) = \mathbb{E}[G_t^{\pi} \,|\, X_t = s].$$

We were computing $V^{\pi}(s)$ by solving the Bellman equation (directly or iteratively):

$$V^{\pi}(s) = \sum_{a \in A} \pi(a, s) \cdot \left[R(s, a) + \gamma \cdot \sum_{s' \in S} P(s' \,|\, s, a) \cdot V^{\pi}(s') \right].$$

But there is also another way to approximate $V^{\pi}(s)$. *

*T*his method will not be very efficient for MDPs but bear with me… we are getting somewhere)*

Monte-Carlo Policy Evaluation (3/5)

An **episode** sampled from an MDP under a policy π is a sequence of states, actions and rewards which ends in a terminal state:

$$s_1, a_1, r_1, s_2, a_2, r_2, s_3, a_3, r_3, \ldots, s_T$$

where s_i is the state at time i, a_i is the action taken at time i and r_i is the corresponding reward obtained at time i.

The return at time t for a concrete episode $s_1, a_1, r_1, s_2, a_2, r_2 \ldots, s_T$

$$g_t = r_1 + \gamma \cdot r_2 + \gamma^2 \cdot r_3 + \ldots = \sum_{i=0}^{T-1} r_i \cdot \gamma^i$$

We can have bounds ∞, just remember that all rewards after T are 0.

Monte-Carlo Policy Evaluation (4/5)

We will now try to approximate $V^{\pi}(s)$ directly using $V^{\pi}(s) = \mathbb{E}[G_t^{\pi} \,|\, X_t = s]$ using sampled episodes. *After all, expectation can be approximated by an average of sampled values.*

We will sample finite episodes (after all we can't sample infinitely long episodes in practice). *This also means that MC policy estimation can only be used for episodic RL problems.*

Monte-Carlo Policy Evaluation (5/5)

Why the problem is not straightforward: *If we only wanted to estimate* $\mathbb{E}[G_t]$*, that would be easy, but we want to estimate* $\mathbb{E}[G_t \,|\, X_t = s]$ *that is we need to condition… but we cannot condition arbitrarily… we can only observe episodes sampled under the given policy…* ***so we will need to "wait" for*** s ***to occur.***

We will see two different MC algorithms to do that: First-Visit MC Estimation and Every-Visit MC Estimation.

First-Visit Monte-Carlo Evaluation

Initialize: $G(s) = 0$, $N(s) = 0$, $V^\pi(s) = undefined$ **for all** $s \in S$.

For $i = 1,\ldots,N$**:**

Sample episode $e_i := s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \ldots, s_{i,T_i}$.

For each time step $1 \leq t \leq T_i$:

If t is the first occurrence of state s in the episode e_i

$g_{i,t} := r_{i,t} + \gamma \cdot r_{i,t+1} + \gamma^2 \cdot r_{i,t+2} + \ldots + \gamma^{T_i - t} \cdot r_{i,T_i}$

$N(s) := N(s) + 1$ /* *Increment total visits counter* */

$G(s) := G(s) + g_{i,t}$ /* *Increment total return counter* */

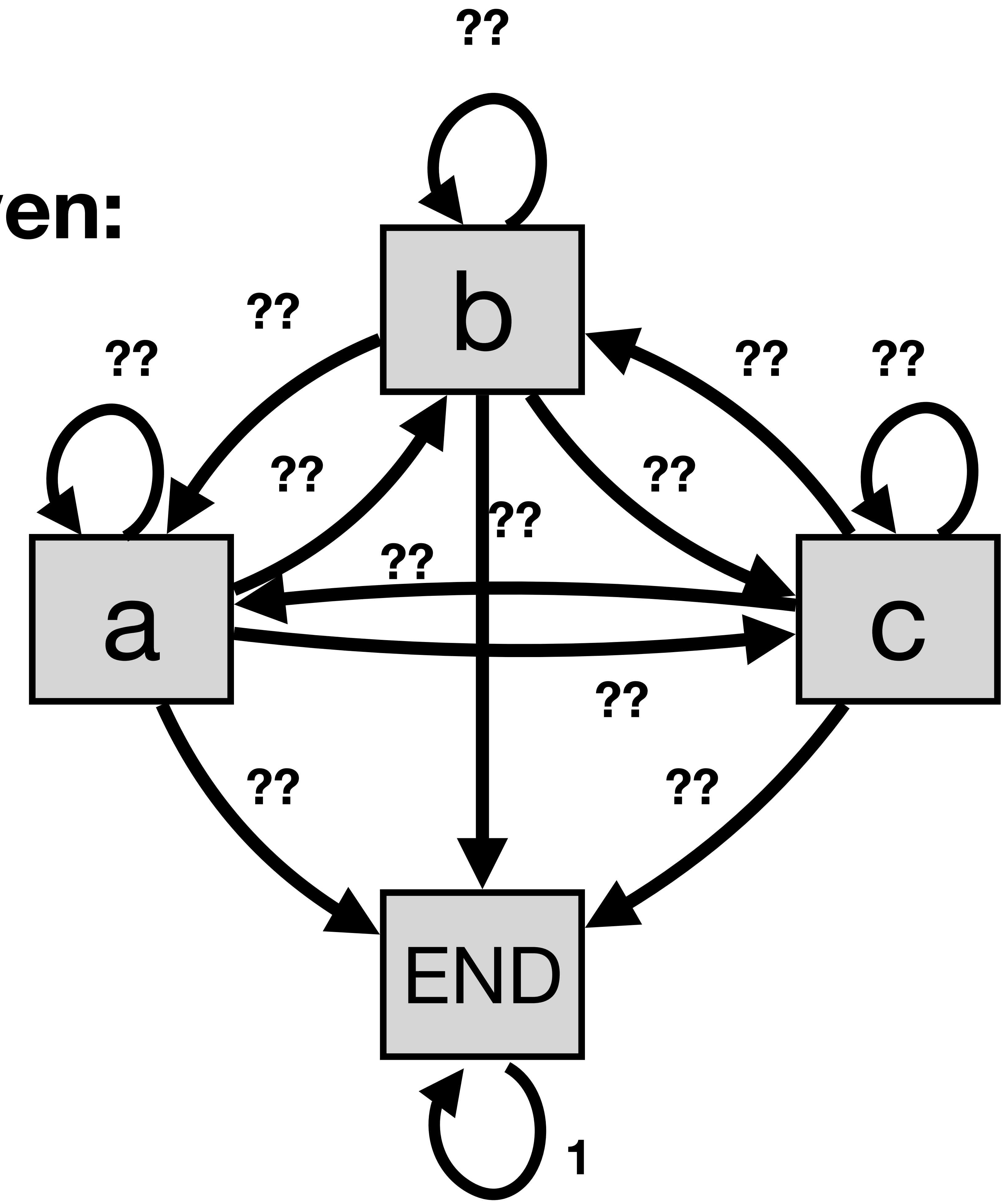
$V^\pi(s) := G(s)/N(s)$ /* *Update current estimate* */

Recall Our Example

Agent: 

Rewards??

States are given:



Actions are given:

$$A = \{L, R\}$$

Some policy π is given
(details not important now).

First-Visit MC Evaluation (Example)

Given: $S = \{a, b, c, \text{end}\}, A = \{L, R\}, \gamma = 1$

Sampled episodes (using given policy π):

$e_1 = a,\ R,\ 0,\ b,\ R,\ 10,\ c, L,\ 0,\ b,\ R,\ 0,\ c,\ R,\ 0,\ \text{end}$

$e_2 = a,\ R,\ 0,\ b,\ R,\ 10,\ c, L,\ 0,\ b,\ R,\ 10,\ a,\ L,\ 0,\ \text{end}$

After iteration 1:

$G(a) = 10,\ G(b) = 10,\ G(c) = 0,\ G(\textbf{end}) = 0$

$N(a) = 1,\ N(b) = 1,\ N(c) = 1,\ N(\textbf{end}) = 1$

$V^{\pi}(a) = 10,\ V^{\pi}(b) = 10,\ V^{\pi}(c) = 0,\ V^{\pi}(\textbf{end}) = 0$

After iteration 2:

Initialize: $G(s) = 0,\ N(s) = 0$ **for all** $s \in S$.

For $i = 1,\ldots,N$:

Sample episode

$e_i := s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \ldots, s_{i,T_i}$.

For each time step $1 \leq t \leq T_i$:

If t is the first occurrence of state s in the episode e_i

$g_{i,t} := r_{i,t} + \gamma \cdot r_{i,t+1} + \gamma^2 \cdot r_{i,t+2} + \ldots + \gamma^{T_i - t} \cdot r_{i,T_i}$

$N(s) := N(s) + 1$ */* Increment total visits counter */*

$G(s) := G(s) + g_{i,t}$ */* Increment total return counter */*

$V^{\pi}(s) := G(s)/N(s)$ */* Update current estimate */*

First-Visit MC Evaluation (Example)

Given: $S = \{a, b, c, \text{end}\}, A = \{L, R\}, \gamma = 1$

Sampled episodes (using given policy π):

$e_1 = a, R, 0, b, R, 10, c, L, 0, b, R, 0, c, R, 0, \text{end}$

$e_2 = a, R, 0, b, R, 10, c, L, 0, b, R, 10, a, L, 0, \text{end}$

After iteration 1:

$G(a) = 10, G(b) = 10, G(c) = 0, G(\textbf{end}) = 0$

$N(a) = 1, N(b) = 1, N(c) = 1, N(\textbf{end}) = 1$

$V^\pi(a) = 10, V^\pi(b) = 10, V^\pi(c) = 0, V^\pi(\textbf{end}) = 0$

After iteration 2:

$G(a) = 30, G(b) = 30, G(c) = 10, G(\textbf{end}) = 0$

$N(a) = 2, N(b) = 2, N(c) = 2, N(\textbf{end}) = 2$

$V^\pi(a) = 15, V^\pi(b) = 15, V^\pi(c) = 5, V^\pi(\textbf{end}) = 0$

Initialize: $G(s) = 0, N(s) = 0$ **for all** $s \in S$.

For $i = 1, \ldots, N$:

Sample episode

$e_i := s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \ldots, s_{i,T_i}$.

For each time step $1 \le t \le T_i$:

If t is the first occurrence of state s in the episode e_i

$g_{i,t} := r_{i,t} + \gamma \cdot r_{i,t+1} + \gamma^2 \cdot r_{i,t+2} + \ldots + \gamma^{T_i - t} \cdot r_{i,T_i}$

$N(s) := N(s) + 1$ /* *Increment total visits counter* */

$G(s) := G(s) + g_{i,t}$ /* *Increment total return counter* */

$V^\pi(s) := G(s)/N(s)$ /* *Update current estimate* */

Every-Visit Monte-Carlo Evaluation

Initialize: $G(s) = 0$, $N(s) = 0$ **for all** $s \in S$.

For $i = 1, \ldots, N$:

Sample episode $e_i := s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \ldots, s_{i,T_i}$.

For each time step $1 \leq t \leq T_i$:

~~**If** t is the first occurrence of state s in the episode e_i~~ /* This was for first-visit MC */

s is the state visited at time t in the episode e_i

$g_{i,t} := r_{i,t} + \gamma \cdot r_{i,t+1} + \gamma^2 \cdot r_{i,t+2} + \ldots + \gamma^{T_i - t} \cdot r_{i,T_i}$

$N(s) := N(s) + 1$ /* *Increment total visits counter* */

$G(s) := G(s) + g_{i,t}$ /* *Increment total return counter* */

$V^\pi(s) := G(s)/N(s)$ /* *Update current estimate* */

Every-Visit MC Evaluation (Example)

Given: $S = \{a, b, c, \text{end}\}, A = \{L, R\}, \gamma = 1$

Sampled episodes (using given policy π):

$e_1 = a,\ R,\ 0,\ b,\ R,\ 10,\ c, L,\ 0,\ b,\ R,\ 0,\ c,\ R,\ 0,$ end

$e_2 = a,\ R,\ 0,\ b,\ R,\ 10,\ c, L,\ 0,\ b,\ R,\ 10,\ a,\ L,\ 0,$ end

After iteration 1:

After iteration 2:

Initialize: $G(s) = 0, N(s) = 0$ **for all** $s \in S$**.**

For $i = 1, \ldots, N$**:**

Sample episode

$e_i := s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \ldots, s_{i,T_i}.$

For each time step $1 \le t \le T_i$:

s is the state visited at time t in the episode e_i

$g_{i,t} := r_{i,t} + \gamma \cdot r_{i,t+1} + \gamma^2 \cdot r_{i,t+2} + \ldots + \gamma^{T_i - t} \cdot r_{i,T_i}$

$N(s) := N(s) + 1$ /* *Increment total visits counter* */

$G(s) := G(s) + g_{i,t}$ /* *Increment total return counter* */

$V^{\pi}(s) := G(s)/N(s)$ /* *Update current estimate* */

Every-Visit MC Evaluation (Example)

Given: $S = \{a, b, c, \text{end}\}, A = \{L, R\}, \gamma = 1$

Sampled episodes (using given policy π):

$e_1 = a,\ R,\ 0,\ b,\ R,\ 10,\ c, L,\ 0,\ b,\ R,\ 0,\ c,\ R,\ 0,$ end

$e_2 = a,\ R,\ 0,\ b,\ R,\ 10,\ c, L,\ 0,\ b,\ R,\ 10,\ a,\ L,\ 0,$ end

After iteration 1:

$G(a) = 10,\ G(b) = 10,\ G(c) = 0,\ G(\textbf{end}) = 0$

$N(a) = 1,\ N(b) = 2,\ N(c) = 2,\ N(\textbf{end}) = 1$

$V^\pi(a) = 10,\ V^\pi(b) = 5,\ V^\pi(c) = 0,\ V^\pi(\textbf{end}) = 0$

After iteration 2:

Initialize: $G(s) = 0, N(s) = 0$ **for all** $s \in S$.

For $i = 1,\ldots,N$:

Sample episode

$e_i := s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \ldots, s_{i,T_i}$.

For each time step $1 \leq t \leq T_i$:

s is the state visited at time t in the episode e_i

$g_{i,t} := r_{i,t} + \gamma \cdot r_{i,t+1} + \gamma^2 \cdot r_{i,t+2} + \ldots + \gamma^{T_i - t} \cdot r_{i,T_i}$

$N(s) := N(s) + 1$ /* *Increment total visits counter* */

$G(s) := G(s) + g_{i,t}$ /* *Increment total return counter* */

$V^\pi(s) := G(s)/N(s)$ /* *Update current estimate* */

Every-Visit MC Evaluation (Example)

Given: $S = \{a, b, c, \text{end}\}, A = \{L, R\}, \gamma = 1$

Sampled episodes (using given policy π):

$e_1 = a,\ R,\ 0,\ b,\ R,\ 10,\ c, L,\ 0,\ b,\ R,\ 0,\ c,\ R,\ 0,$ end

$e_2 = a,\ R,\ 0,\ b,\ R,\ 10,\ c, L,\ 0,\ b,\ R,\ 10,\ a,\ L,\ 0,$ end

After iteration 1:

$G(a) = 10,\ G(b) = 10,\ G(c) = 0,\ G(\textbf{end}) = 0$

$N(a) = 1,\ N(b) = 2,\ N(c) = 2,\ N(\textbf{end}) = 1$

$V^\pi(a) = 10,\ V^\pi(b) = 5,\ V^\pi(c) = 0,\ V^\pi(\textbf{end}) = 0$

After iteration 2:

$G(a) = 30,\ G(b) = 40,\ G(c) = 10,\ G(\textbf{end}) = 0$

$N(a) = 3,\ N(b) = 4,\ N(c) = 3,\ N(\textbf{end}) = 2$

$V^\pi(a) = 10,\ V^\pi(b) = 10,\ V^\pi(c) = \frac{10}{3},\ V^\pi(\textbf{end}) = 0$

Initialize: $G(s) = 0,\ N(s) = 0$ **for all** $s \in S$.

For $i = 1,\dots,N$:

Sample episode

$e_i := s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \dots, s_{i,T_i}$.

For each time step $1 \le t \le T_i$:

s is the state visited at time t in the episode e_i

$g_{i,t} := r_{i,t} + \gamma \cdot r_{i,t+1} + \gamma^2 \cdot r_{i,t+2} + \ldots + \gamma^{T_i - t} \cdot r_{i,T_i}$

$N(s) := N(s) + 1$ /* *Increment total visits counter* */

$G(s) := G(s) + g_{i,t}$ /* *Increment total return counter* */

$V^\pi(s) := G(s)/N(s)$ /* *Update current estimate* */

Statistical Properties (1/7)

- First-visit MC Policy Evaluation is **unbiased** (and hence also consistent) estimator.
- Every-visit MC Policy Evaluation is a **biased** but **consistent** estimator, which often has better MSE.

Statistical Properties (2/7)

First-visit MC Policy Evaluation is **unbiased** (and hence also consistent) estimator.

Proof Sketch:

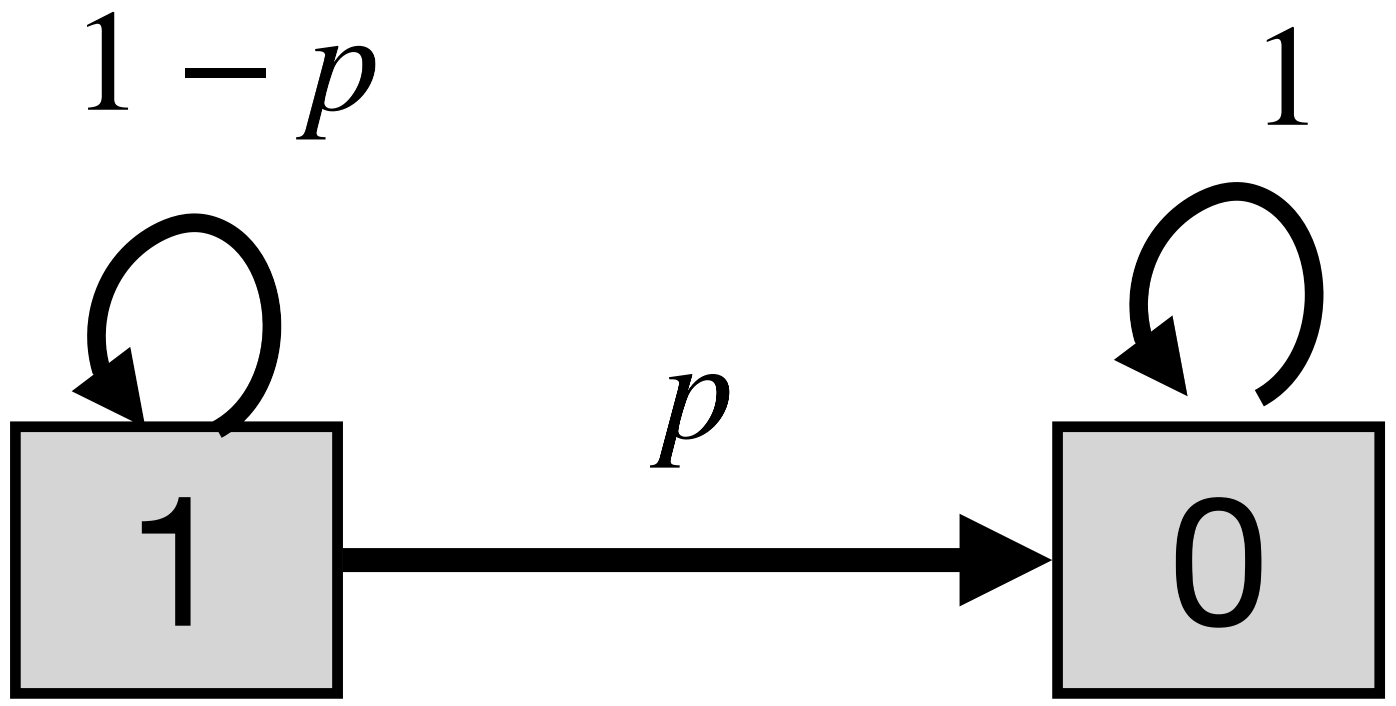
Assuming Markov property, the first occurrence* of the state s at time t together with the subsequence starting at t gives us an unbiased estimate of the return starting from s (this is practically from definition), i.e., $\mathbb{E}[G_t^\pi | X_t = s]$, which is by definition equal to $V^\pi(s)$. First-visit MC averages such independent samples from different episodes (different episodes => independence).

**Do you see why we cannot take, e.g., the last occurrence? Hint: Are subsequences starting with the last occurrence of s special in some way?*

Statistical Properties (3/7)

- Every-visit MC Policy Evaluation is a **biased** but **consistent** estimator, which often has **better MSE**.

- **Example (Showing that it is biased):**



$R(1) = 1$ $R(0) = 0$

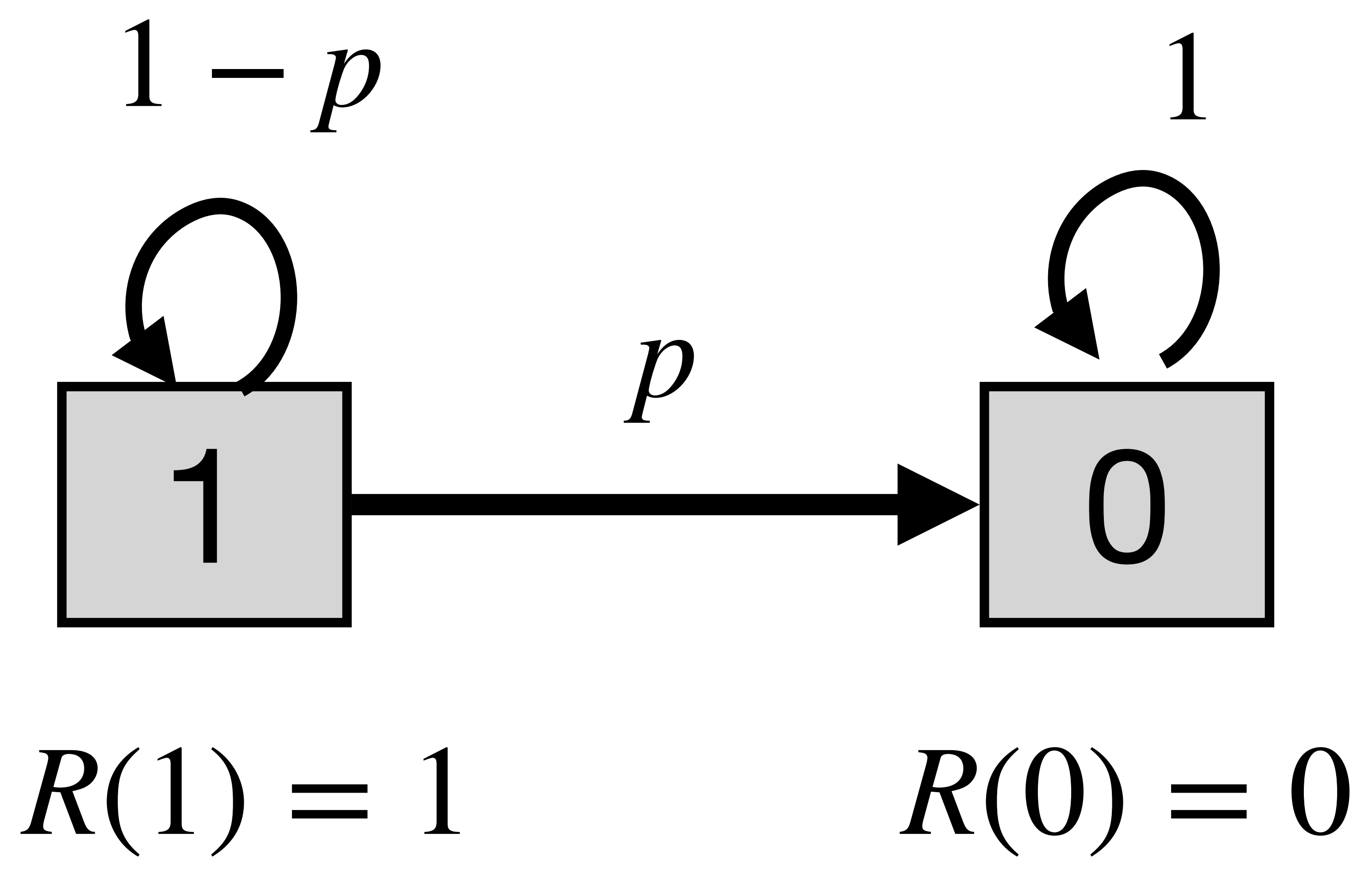
$\gamma = 1$

Statistical Properties (4/7)

- Every-visit MC Policy Evaluation is a **biased** but **consistent** estimator, which often has **better MSE**.

- **Example (Showing that it is biased):**

- Computing V explicitly using Bellman equation:
  $V(1) = 1 + (1 - p) \cdot V(1) + p \cdot 0$
  Hence, $V(1) = \dfrac{1}{p}$.



$\gamma = 1$

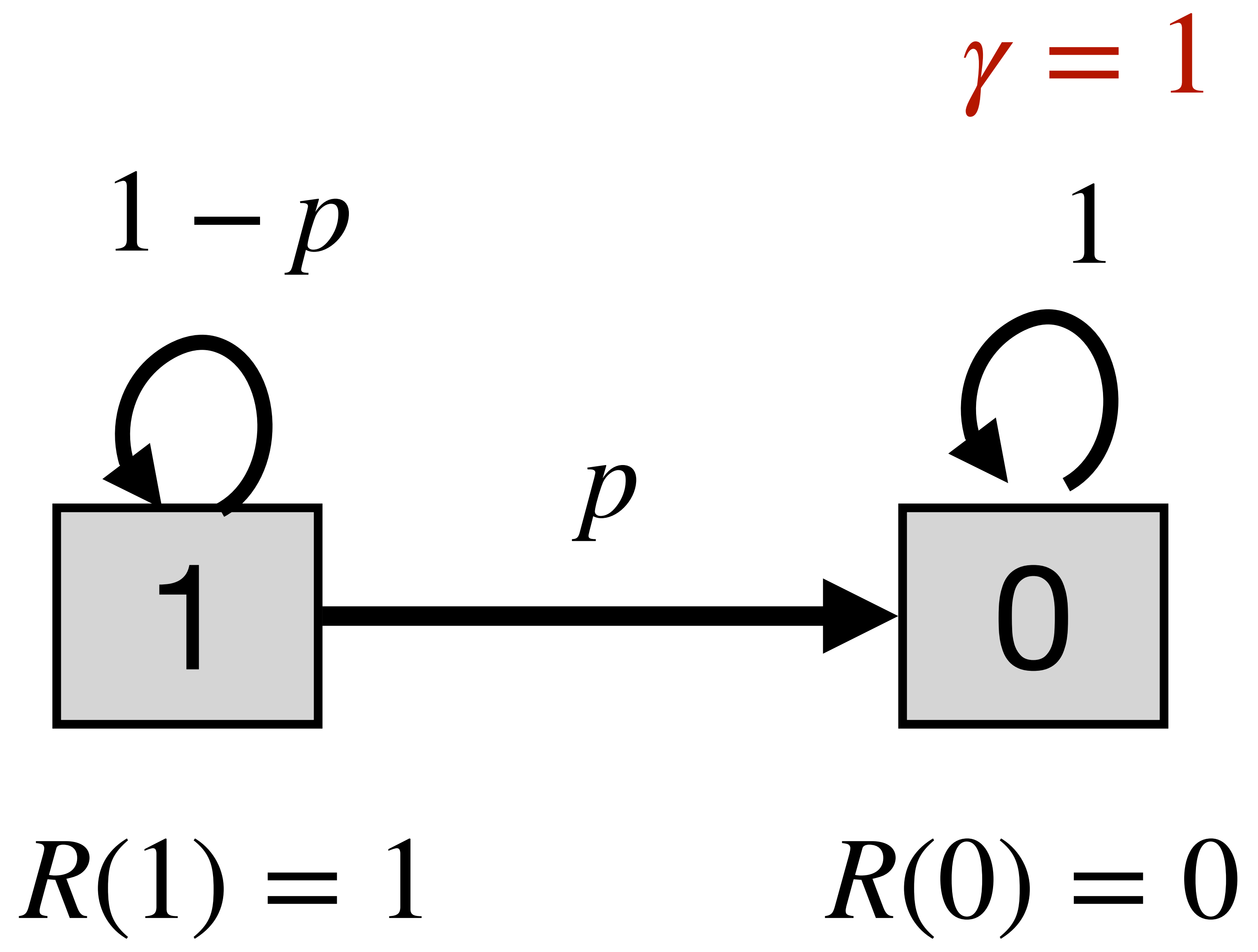
Statistical Properties (5/7)

- Every-visit MC Policy Evaluation is a **biased** but **consistent** estimator, which often has **better MSE**.

- **Example (Showing that it is biased):**

- Exact answer: $V(1) = \frac{1}{p}$.

- First-Visit MC:



$$\mathbb{E}[\hat{V}_{FV}(1)] = p + 2(p-1)p + 3(p-1)^2 p + \ldots = p \sum_{n=0}^{\infty} (n+1) \cdot (1-p)^n = p \cdot \frac{1}{p^2} = \frac{1}{p}$$

UNBIASED

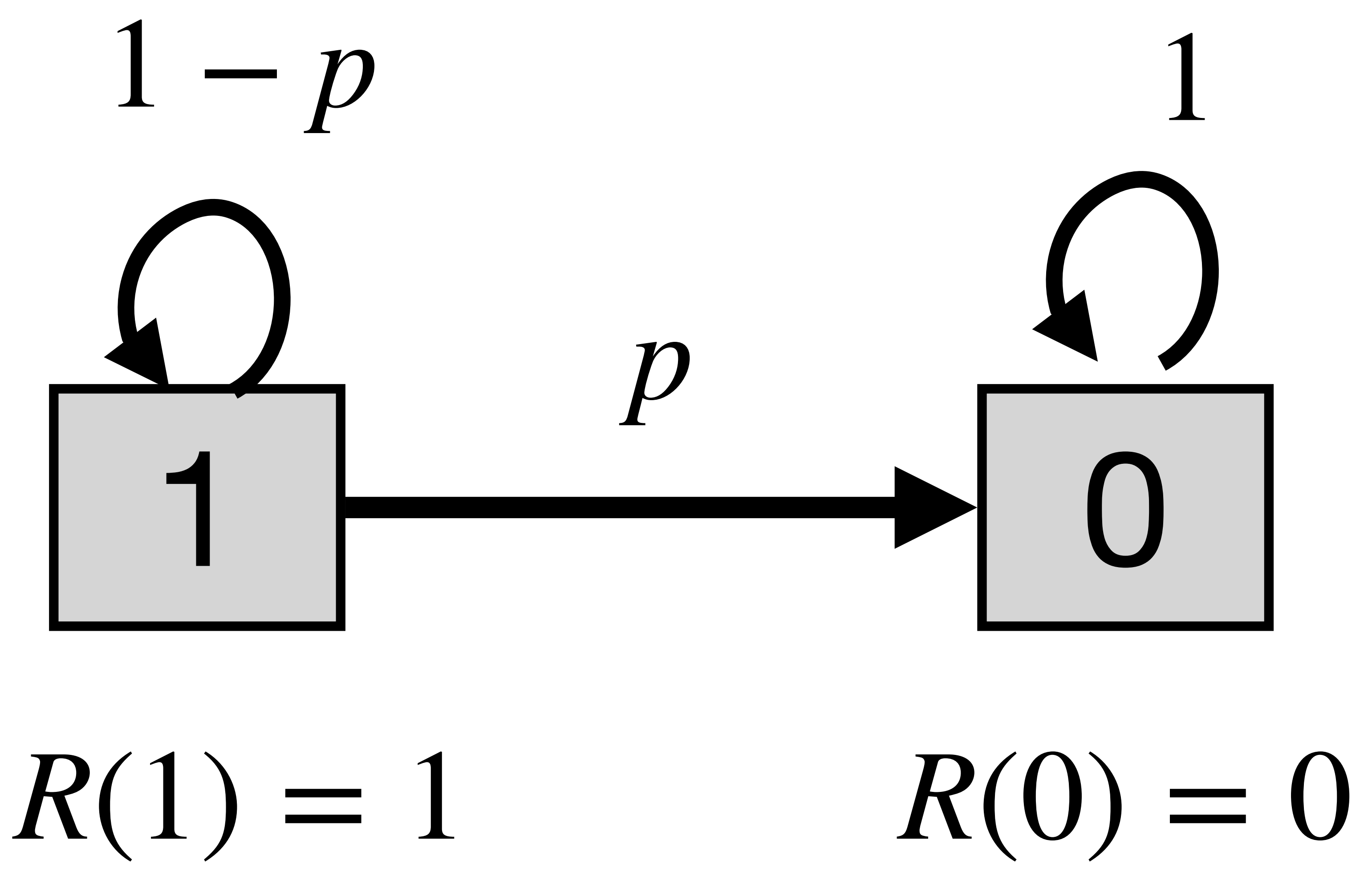
Statistical Properties (6/7)

- Every-visit MC Policy Evaluation is a **biased** but **consistent** estimator, which often has **better MSE**.

$\gamma = 1$

- **Example (Showing that it is biased):**

- Exact answer: $V(1) = \frac{1}{p}$.



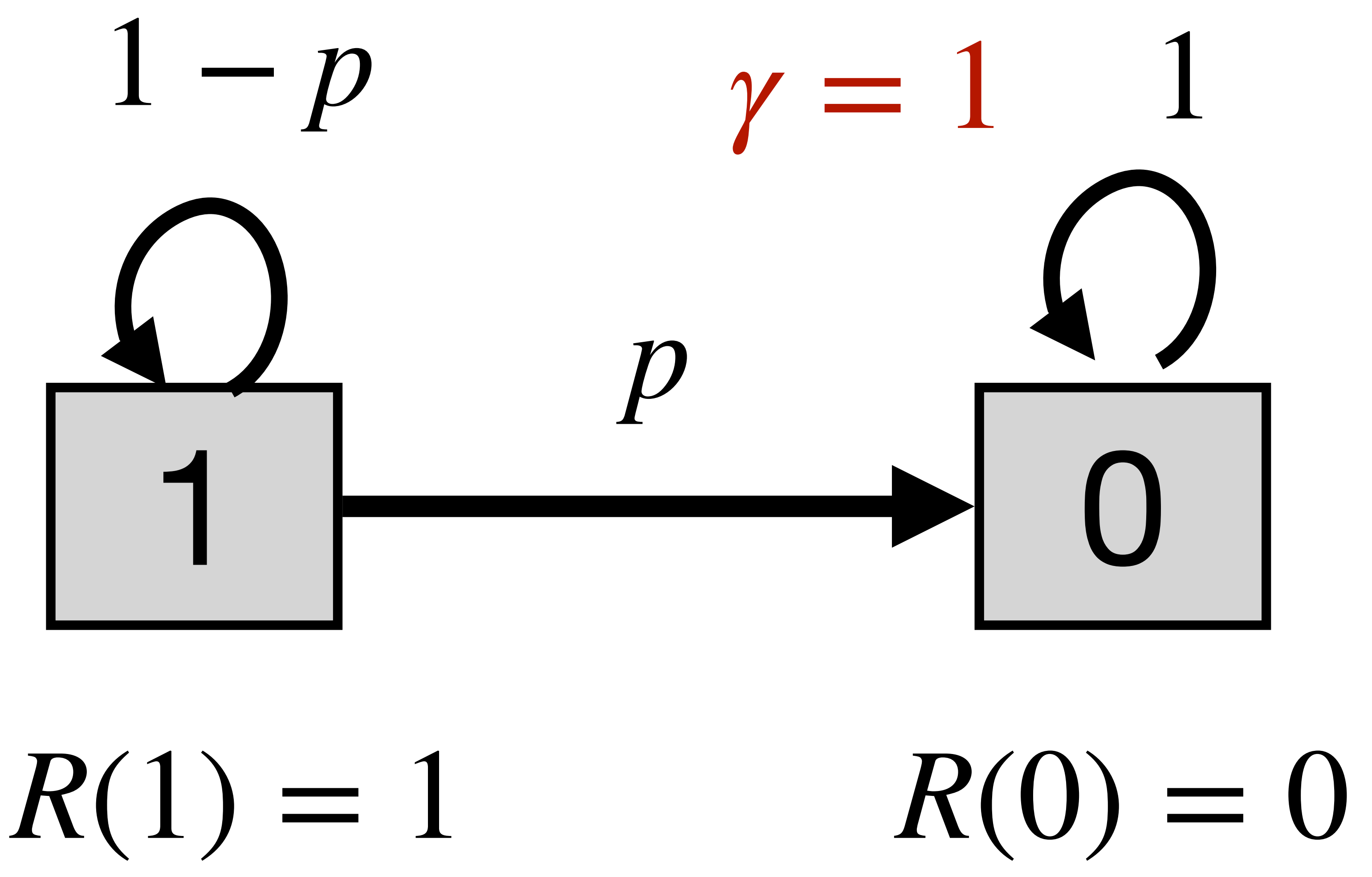
- Every-Visit MC (Bias):

$$\mathbb{E}[\hat{V}_{EV}(1)] = p + \frac{3}{2}(1-p)p + 2(1-p)^2 p + \ldots = p\sum_{n=0}^{\infty}\frac{n+2}{2}\cdot(1-p)^n = p\cdot\frac{p+1}{p^2} = \frac{p+1}{2p} \neq \frac{1}{p}$$

BIASED

Statistical Properties (7/7)

- Every-visit MC Policy Evaluation is a **biased** but **consistent** estimator, which often has **better MSE**.



- **Example (Showing that it is biased):**

- Exact answer: $V(1) = \frac{1}{p}$.

- Every-Visit MC (**Consistency**):

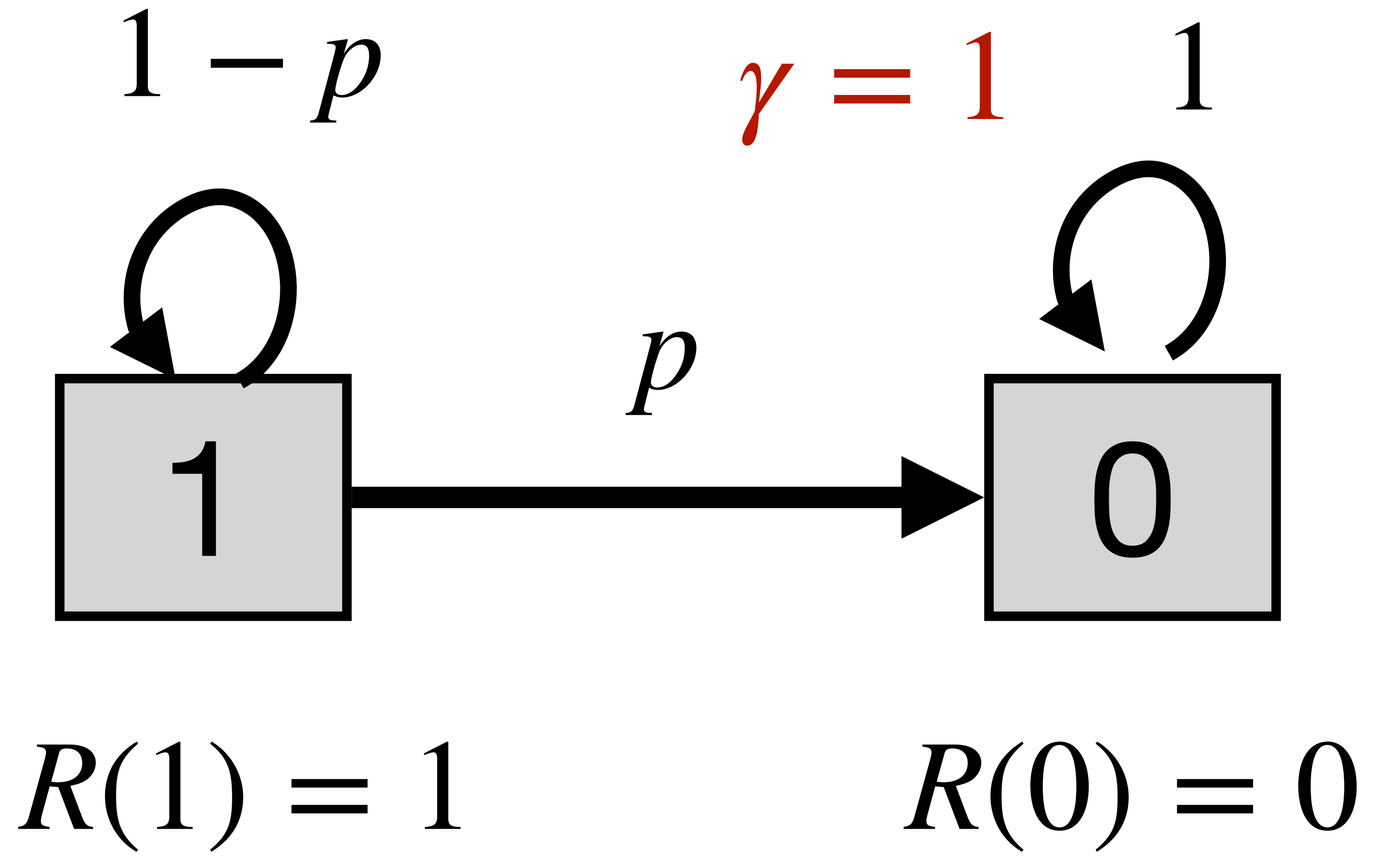
  $\hat{V}_{EV} = \frac{T+1}{2}$ where T is a geometrically distributed r.v. with expectation $\frac{1}{p}$.

  Averaging estimators over n independent episodes, one can show with a bit of algebraic manipulations that $P\left[\left|\hat{V}_n - \frac{1}{p}\right| < \varepsilon\right] = 1$ for all $0 < \varepsilon$.

Consistent

Statistical Properties (7/7)

- Every-visit MC Policy Evaluation is a **biased** but **consistent** estimator, which often has **better MSE**.



- **Example (Showing that it is biased):**

- Exact answer: $V(1) = \frac{1}{p}$.

- Every-Visit MC (**Consistency**):

  $\hat{V}_{EV} = \frac{T+1}{2}$ where T is a geometrically distributed r.v. with expectation $\frac{1}{p}$.

  Averaging estimators over n independent episodes, one can show with a bit of algebraic manipulations that $P\left[\left|\hat{V}_n - \frac{1}{p}\right| < \varepsilon\right] = 1$ for all $0 < \varepsilon$.

Consistent

Incremental Monte-Carlo Evaluation

Initialize: $N(s) = 0, V^{\pi}(s) = undefined$ **for all** $s \in S$.

For $i = 1, \ldots, N$:

Sample episode $e_i := s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \ldots, s_{i,T_i}$.

For each time step $1 \leq t \leq T_i$:

s is the state visited at time t in the episode e_i

$g_{i,t} := r_{i,t} + \gamma \cdot r_{i,t+1} + \gamma^2 \cdot r_{i,t+2} + \ldots + \gamma^{T_i - t} \cdot r_{i,T_i}$

$N(s) := N(s) + 1$ /* *Increment total visits counter* */

$V^{\pi}(s) := V^{\pi}(S) + \alpha \cdot (g_{i,t} - V^{\pi}(s))$ /* *Update value function* */

Special case: When we use $\alpha = \dfrac{1}{N(s)}$ then the resulting incremental MC becomes equivalent to every-visit MC.

Summary (So Far)

- **MC Methods:**
  - Try to estimate $V^\pi(s) = \mathbb{E}[G_t^\pi \,|\, X_t = s]$ directly as an average over sampled episodes (which is also why they need the episodic settings).
  - They do not use the Markov assumption!
  - Converge to the true values.
  - Can have high variance and some of them are also biased (first-visit MC is one which is not biased).

Part 4: Temporal Difference Learning

(We are still dealing with policy evaluation)

Temporal Difference Learning: A Teaser

- **TD learning** combines Monte-Carlo estimation and dynamic programming ideas.
- **TD learning** can be used both in episodic and infinite-horizon non-episodic settings,
- **TD learning** updates estimates of V^{π} continually, after every consecutive tuple *state-action-reward-state* (therefore we do not need to wait till the end of an episode).

….

TD-Learning: Basic Idea

Recall: $g_{i,t} := r_{i,t} + \gamma \cdot r_{i,t+1} + \gamma^2 \cdot r_{i,t+2} + \ldots + \gamma^{T_i - t} \cdot r_{i,T_i}$

Incremental MC:

$$V^\pi(s) := V^\pi(s) + \alpha \cdot (g_{i,t} - V^\pi(s)).$$

Temporal Difference Learning:

$$V^\pi(s_t) := V^\pi(s_t) + \alpha(r_{i,t} + \gamma \cdot V^\pi(s_{t+1}) - V^\pi(s_t))$$

$g_{i,t} \approx r_{i,t} + \gamma \cdot V^\pi(s_{t+1})$

TD-Learning: Relationship to Bellman Backup

Recall: $g_{i,t} := r_{i,t} + \gamma \cdot r_{i,t+1} + \gamma^2 \cdot r_{i,t+2} + \ldots + \gamma^{T_i - t} \cdot r_{i,T_i}$

Bellman equation update rule:

$$V^{\pi}_{k+1}(s) := \boxed{R(s, \pi(s)) + \gamma \cdot \sum_{s' \in S} P(s' \mid s, \pi(s)) \cdot V^{\pi}_k(s')}$$

Expectation

Temporal Difference Learning update rule:

$$\begin{aligned} V^{\pi}(s_t) &:= V^{\pi}(s_t) + \alpha(r_{i,t} + \gamma \cdot V^{\pi}(s_{t+1}) - V^{\pi}(s_t)) \\ &= (1 - \alpha) \cdot V^{\pi}(s_t) + \alpha \cdot \boxed{(r_{i,t} + \gamma \cdot V^{\pi}(s_{t+1}))} \end{aligned}$$

Sample

TD-Learning: Pseudocode

Initialize: $V^{\pi}(s) = 0$ for all $s \in S$

Loop:

Sample tuple (s_t, a_t, r_t, s_{t+1}).

Update $V^{\pi}(s_t) := V^{\pi}(s_t) + \alpha \cdot (\underbrace{r_{i,t} + \gamma \cdot V^{\pi}(s_{t+1})}_{\text{TD target}} - V^{\pi}(s_t))$

TD-Learning: Example 🐸

Initialize: $V^\pi(s) = 0$ for all $s \in S$

Loop:

Sample tuple (s_t, a_t, r_t, s_{t+1}).

Update $V^\pi(s_t) := V^\pi(s_t) + \alpha \cdot (\underbrace{r_{i,t} + \gamma \cdot V^\pi(s_{t+1})}_{\text{TD target}} - V^\pi(s_t))$

$$\alpha = 0.5, \gamma = 1$$

$e_1 = a, R, 0, b, R, 10, c, L, 0, b, R, 0, c, R, 0,$ end

Iteration 1: $V^\pi(a) := 0,$

Iteration 2: $V^\pi(b) := 5,$

Iteration 3: $V^\pi(c) := 0.5(0 + 5) = 2.5,$

Iteration 4: $V^\pi(b) := 5 + 0.5 \cdot (0 + 2.5 - 5) = 3.75,$

Iteration 5: $V^\pi(c) := 2.5 + 0.5 \cdot (0 + 0 - 2.5) = 1.25.$

TD-Learning: Example 🐸

Initialize: $V^\pi(s) = 0$ for all $s \in S$

Loop:

Sample tuple (s_t, a_t, r_t, s_{t+1}).

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$e_1 = a, R, 0, b, R, 10, c, L, 0, b, R, 0, c, R, 0,$ end

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TD-Learning: Example 🐸

Initialize: $V^\pi(s) = 0$ for all $s \in S$

Loop:

Sample tuple (s_t, a_t, r_t, s_{t+1}).

Update $V^\pi(s_t) := V^\pi(s_t) + \alpha \cdot (\underbrace{r_{i,t} + \gamma \cdot V^\pi(s_{t+1})}_{\text{TD target}} - V^\pi(s_t))$

$$\alpha = 0.5, \gamma = 1$$

$e_1 = a, R, 0, b, R, 10, c, L, 0, b, R, 0, c, R, 0,$ end

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TD-Learning: Example 🐸

Initialize: $V^\pi(s) = 0$ for all $s \in S$

Loop:

Sample tuple (s_t, a_t, r_t, s_{t+1}).

Update $V^\pi(s_t) := V^\pi(s_t) + \alpha \cdot (\underbrace{r_{i,t} + \gamma \cdot V^\pi(s_{t+1})}_{\text{TD target}} - V^\pi(s_t))$

$\alpha = 0.5, \gamma = 1$

$e_1 = a, R, 0, b, R, 10, c, L, 0, b, R, 0, c, R, 0,$ end

Iteration 1: $V^\pi(a) := 0,$

Iteration 2: $V^\pi(b) := 5,$

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TD-Learning: Example 🐸

Initialize: $V^\pi(s) = 0$ for all $s \in S$

Loop:

Sample tuple (s_t, a_t, r_t, s_{t+1}).

Update $V^\pi(s_t) := V^\pi(s_t) + \alpha \cdot (\underbrace{r_{i,t} + \gamma \cdot V^\pi(s_{t+1})}_{\text{TD target}} - V^\pi(s_t))$

$\alpha = 0.5, \gamma = 1$

$e_1 = a, R, 0, b, R, 10, c, L, 0, b, R, 0, c, R, 0,$ end

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$$\alpha = 0.5, \gamma = 1$$

$e_1 = a, R, 0, b, R, 10, c, L, 0, b, R, 0,$ $c, R, 0,$ end

Iteration 1: $V^\pi(a) := 0,$

Iteration 2: $V^\pi(b) := 5,$

Iteration 3: $V^\pi(c) := 0.5(0 + 5) = 2.5,$

Iteration 4: $V^\pi(b) := 5 + 0.5 \cdot (0 + 2.5 - 5) = 3.75,$

Iteration 5: $V^\pi(c) := 2.5 + 0.5 \cdot (0 + 0 - 2.5) = 1.25.$

TD-Learning: Example 🐸

Initialize: $V^\pi(s) = 0$ for all $s \in S$

Loop:

Sample tuple (s_t, a_t, r_t, s_{t+1}).

Update $V^\pi(s_t) := V^\pi(s_t) + \alpha \cdot (\underbrace{r_{i,t} + \gamma \cdot V^\pi(s_{t+1})}_{\text{TD target}} - V^\pi(s_t))$

$$\alpha = 0.5, \gamma = 1$$

$e_1 = a, R, 0, b, R, 10, c, L, 0, b, R, 0, c, R, 0,$ end

Iteration 1: $V^\pi(a) := 0,$

Iteration 2: $V^\pi(b) := 5,$

Iteration 3: $V^\pi(c) := 0.5(0 + 5) = 2.5,$

Iteration 4: $V^\pi(b) := 5 + 0.5 \cdot (0 + 2.5 - 5) = 3.75,$

Iteration 5: $V^\pi(c) := 2.5 + 0.5 \cdot (0 + 0 - 2.5) = 1.25.$

Every-Visit Monte-Carlo: $V^\pi(a) = 10, V^\pi(b) = 5, V^\pi(c) = 0, V^\pi($**end**$) = 0$

What About the α's?

- One thing we can do is to have α depend on the number of iterations so far, i.e., we can have α_k instead of just α.

- Convergence is guaranteed when $\alpha_k's$ satisfy the following conditions (follows from Robbins-Munro algorithm):

$$\sum_{k=1}^{\infty} \alpha_k = \infty, \quad \sum_{k=1}^{\infty} a_k^2 < \infty.$$

- A sequence which satisfies the above conditions is, e.g., $a_k = \frac{1}{k}$. However, in practice, similar sequences do not have to converge very fast…

- *Note: It was also proved by Sutton (1988) that, for tabular MDPs, there always exists some small enough learning rate α such that TD converges but this result is not very practical.*

Policy Evaluation: Summary

	DPCE	MC	TD
Can use w/out access to true MDP models	X	X	X
Usable in continuing (non-episodic) setting	X		X
Assumes Markov process	X		X
Converges to true value in limit[3]	X	X	X
Unbiased estimate of value		X	

- DPCE = Dynamic Programming w/certainty equivalence estimates, MC = Monte Carlo, TD = Temporal Difference

Table from slides by Prof. Emma Brunskill

Next Time: Model-Free Control

Model-Free Control

- Given an MDP with unknown parameters (or generally an environment with which we can interact), **find the optimal policy π.**

Important Concepts to Refresh…

- Besides the things we discussed today, in the next lecture, we will also again use the following concepts:
  - the state-action value function $Q^\pi(s, a)$,
  - policy iteration and policy improvement.