GVG'2022 Exercise-06 solution

1. We're observing a plane $(\vec{n}, d)$ by two cameras $(\mathtt{R}, \mathtt{C}, \mathtt{K})$, $(\mathtt{R}', \mathtt{C}', \mathtt{K}'))$. The world coordinate frame (O, δ) is selected in such way, that $\mathtt{R} = \mathtt{I}$ and $\mathtt{C}_\delta = [0,0,0]^\top$.

$$\begin{bmatrix}\mathtt{R} & -\mathtt{R}\vec{C}_\delta\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}, \quad \begin{bmatrix}\mathtt{R}' & -\mathtt{R}'\vec{C}'_\delta\end{bmatrix} = \begin{bmatrix}0 & 0 & 1 & 0\\ 1 & 0 & 0 & 2\\ 0 & 1 & 0 & 1\end{bmatrix}, \quad \mathtt{K} = \mathtt{K}' = \begin{bmatrix}1 & 0 & 3\\ 0 & 1 & 3\\ 0 & 0 & 1\end{bmatrix}$$

$$\vec{n}_\delta = [1,0,2]^\top, \quad d = 1, \quad \vec{x}_\alpha = [u,v]^\top = [1,2]^\top, \quad \vec{x}'_{\alpha'} = [u',v']^\top = ???$$

$$\frac{\zeta'}{\zeta}\vec{x}'_{\gamma'} = \left(\mathtt{R}' - \frac{\mathtt{R}'\vec{C}'_\delta\vec{n}_\delta^\top}{d}\right)\vec{x}_\gamma$$

$$\vec{x}_\beta = \mathtt{K}\vec{x}_\gamma, \quad \vec{x}_\beta = \begin{bmatrix}\vec{x}_\alpha\\ 1\end{bmatrix} \quad \rightarrow \quad \vec{x}_\gamma = \mathtt{K}^{-1}\begin{bmatrix}\vec{x}_\alpha\\ 1\end{bmatrix}$$

$$\frac{\zeta'}{\zeta}\mathtt{K}'^{-1}\begin{bmatrix}\vec{x}'_{\alpha'}\\ 1\end{bmatrix} = \left(\mathtt{R}' - \frac{\mathtt{R}'\vec{C}'_\delta\vec{n}_\delta^\top}{d}\right)\mathtt{K}^{-1}\begin{bmatrix}\vec{x}_\alpha\\ 1\end{bmatrix} \quad \rightarrow \quad \frac{\zeta'}{\zeta}\begin{bmatrix}\vec{x}'_{\alpha'}\\ 1\end{bmatrix} = \mathtt{K}'\left(\mathtt{R}' - \frac{\mathtt{R}'\vec{C}'_\delta\vec{n}_\delta^\top}{d}\right)\mathtt{K}^{-1}\begin{bmatrix}\vec{x}_\alpha\\ 1\end{bmatrix}$$

$$\frac{\zeta'}{\zeta}\begin{bmatrix}\vec{x}'_{\alpha'}\\ 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 3\\ 0 & 1 & 3\\ 0 & 0 & 1\end{bmatrix}\left(\begin{bmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix} - \frac{-\begin{bmatrix}0\\ 2\\ 1\end{bmatrix}\begin{bmatrix}1 & 0 & 2\end{bmatrix}}{1}\right)\begin{bmatrix}1 & 0 & 3\\ 0 & 1 & 3\\ 0 & 0 & 1\end{bmatrix}^{-1}\begin{bmatrix}1\\ 2\\ 1\end{bmatrix} =$$

$$\begin{bmatrix}1 & 0 & 3\\ 0 & 1 & 3\\ 0 & 0 & 1\end{bmatrix}\left(\begin{bmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix} + \begin{bmatrix}0 & 0 & 0\\ 2 & 0 & 4\\ 1 & 0 & 2\end{bmatrix}\right)\begin{bmatrix}1 & 0 & -3\\ 0 & 1 & -3\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\ 2\\ 1\end{bmatrix} = \begin{bmatrix}-2\\ -5\\ -1\end{bmatrix} \quad \rightarrow \quad \begin{bmatrix}u'\\ v'\end{bmatrix} = \begin{bmatrix}2\\ 5\end{bmatrix}$$

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2. Points X are all in a single plane (let's call the plane e.g. $\mathtt{N}$). World coordinate frame (O, $\delta = (\vec{d}_1, \vec{d}_2, \vec{d}_3)$) is selected in such way that:

   - the origin $O \in \mathtt{N}$,
   - $\vec{d}_1$, $\vec{d}_2$ span $\mathtt{N} \rightarrow$ any point $\in \mathtt{N}$ can be fully defined just by linear combination of $\vec{d}_1$, $\vec{d}_2$.

   We can define a new coordinate frame $(C, \tau = (\vec{d}_1, \vec{d}_2, \vec{d}_4))$, where $\vec{d}_4 = \vec{CO}$ (vector from the camera center to the origin of world coordinate frame). All the points $\in \mathtt{N}$ represented in the coordinate frame (C, τ) have coordinates $[x, y, 1]^\top$ ($1 \cdot \vec{d}_4$ to get from C to O and then $x \cdot \vec{d}_1$ and $y \cdot \vec{d}_2$ to get along the plane).

   We get points in plane defined in $(O, (\vec{d}_1, \vec{d}_2))$

$$\vec{X}_1 = \begin{bmatrix}0\\ 0\end{bmatrix}, \; \vec{X}_2 = \begin{bmatrix}1\\ 0\end{bmatrix}, \; \vec{X}_3 = \begin{bmatrix}0\\ 1\end{bmatrix}, \; \vec{X}_4 = \begin{bmatrix}1\\ 1\end{bmatrix},$$

and their projections to image plane defined in $(o, \alpha = (\vec{b}_1, \vec{b}_2))$

$$\vec{u}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ \vec{u}_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \ \vec{u}_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \vec{u}_4 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

(a) $\mathtt{H} = ???$

(b) $\vec{x}_\alpha = [1, 1]^\top, \ X = ???$

(a) The projection of points on plane in space to the image plane of perspective camera can be described by a homography transformation.

$$\zeta \vec{x}_\beta = \zeta \begin{bmatrix} \vec{x}_\alpha \\ 1 \end{bmatrix} = \mathtt{H}\vec{X}_\tau = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \vec{X}_\tau = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} \vec{X}_{(O,(\vec{d}_1,\vec{d}_2))} \\ 1 \end{bmatrix}$$

Sidenote: Normally the transformation of points in 3D space to the image plane (projection) is described by the projection matrix $\mathtt{P}$, but thanks to the selection of world coordinate frame we can describe the projection just by a homography matrix.

$$\zeta \vec{x}_\beta = \mathtt{P}\vec{X}_\delta = \begin{bmatrix} \vec{p}_1 & \vec{p}_2 & \vec{p}_3 & \vec{p}_4 \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \vec{p}_1 & \vec{p}_2 & \vec{p}_4 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \mathtt{H}\vec{X}_\tau$$

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i = 1: $\vec{X}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \vec{u}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\zeta_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \rightarrow \begin{matrix} h_{13} = 0 \\ h_{23} = 0 \\ h_{33} = \zeta_1 \end{matrix}$$

i = 2: $\vec{X}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$$\zeta_2 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \rightarrow \begin{matrix} h_{11} = 2\zeta_2 \\ h_{21} = 0 \\ h_{31} = \zeta_2 - \zeta_1 \end{matrix}$$

i = 3: $\vec{X}_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\zeta_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \rightarrow \begin{matrix} h_{12} = 0 \\ h_{22} = \zeta_3 \\ h_{32} = \zeta_3 - \zeta_1 \end{matrix}$$

i = 4: $\vec{X}_4 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{u}_4 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$$\zeta_4 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \rightarrow \begin{matrix} h_{11} = 2\zeta_4 \rightarrow \zeta_2 = \zeta_4 \\ h_{22} = 2\zeta_4 \rightarrow \zeta_3 = 2\zeta_4 \\ h_{31} + h_{32} + h_{33} = \zeta_4 \end{matrix}$$

$$h_{31}+h_{32}+h_{33} = (\zeta_2-\zeta_1)+(\zeta_3-\zeta_1)+(\zeta_1) = (\zeta_4-\zeta_1)+(2\zeta_4-\zeta_1)+(\zeta_1) = \zeta_4 \rightarrow \zeta_1 = 2\zeta_4$$

$$\mathtt{H} = \begin{bmatrix} 2\zeta_4 & 0 & 0 \\ 0 & 2\zeta_4 & 0 \\ -\zeta_4 & 0 & 2\zeta_4 \end{bmatrix}$$

We can omit ζ_4 as $\mathtt{H} = c \cdot \mathtt{H}$ for any scalar $c \neq 0$.

$$\mathtt{H} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

(b)

$$\frac{1}{\zeta}\vec{X}_\tau = \mathtt{H}^{-1}\vec{x}_\beta$$

$$\frac{1}{\zeta}\vec{X}_\tau = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{3}{4} \end{bmatrix} \rightarrow \vec{X}_{(O,(\vec{d}_1,\vec{d}_2))} = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}$$