GVG Lab-03 Solution

Task 1. *Assume a camera with the following image projection matrix*

$$\mathtt{P}_\beta = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

Find the coordinates of the projection center.

Solution: By definition,

$$\mathtt{P}_\beta = \begin{bmatrix} \mathtt{A} & -\mathtt{A}\vec{C}_\delta \end{bmatrix}$$

where $\mathtt{A} = \mathtt{T}_{\delta\to\beta}$. Thus, the camera projection center expressed in the world coordinate system $\vec{C}_\delta$ can be retrieved from the kernel of $\mathtt{P}_\beta$, since

$$\mathtt{P}_\beta \begin{bmatrix} \vec{C}_\delta \\ 1 \end{bmatrix} = \begin{bmatrix} \mathtt{A} & -\mathtt{A}\vec{C}_\delta \end{bmatrix} \begin{bmatrix} \vec{C}_\delta \\ 1 \end{bmatrix} = \mathtt{A}\vec{C}_\delta - \mathtt{A}\vec{C}_\delta = \mathbf{0}$$

Computing the kernel of $\mathtt{P}_\beta$ means solving the system of linear equations

$$\mathtt{P}_\beta \mathbf{x} = \mathbf{0}, \quad \mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^\top \tag{1}$$

We solve it by applying Gaussian elimination method:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

From the last row ($-x_3 + x_4 = 0$) we let x_4 to be any real number and conclude that $x_3 = x_4$. Substituting it to the second row we get $x_2 = -2x_4$. Further, substituting x_2 and x_3 to the first row we obtain $x_1 = -x_2 = x_4$. Thus, the solutions to (1) are

$$S = \left\{ \begin{bmatrix} x_4 \\ -2x_4 \\ x_4 \\ x_4 \end{bmatrix} \;\middle|\; x_4 \in \mathbb{R} \right\}$$

We know that the kernel of every image projection matrix is one-dimensional, since rank of this matrix is always equal to 3. We are interested in the representative of the kernel with last coordinate equal to 1. For this we take $x_4 = 1$ and get

$$\begin{bmatrix} \vec{C}_\delta \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 1 \end{bmatrix} \in S \Rightarrow \vec{C}_\delta = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Another way to compute $\vec{C}_\delta$ would be to invert $\mathtt{A}$. For this we stack $\mathtt{A}$ and $\mathtt{I}$ together into 3×6 matrix and apply Gaussian elimination so that the left 3×3 part becomes $\mathtt{I}$. The inverse $\mathtt{A}^{-1}$ then appears as the right 3×3 block:

$$\begin{bmatrix} \mathtt{A} \mid \mathtt{I} \end{bmatrix} = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right]$$

Thus,

$$\mathtt{A}^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \Rightarrow \vec{C}_\delta = -\mathtt{A}^{-1}\mathtt{P}_{\beta_{1:3,4}} = -\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

□

Task 2. *Assume the following camera projection matrix*

$$\mathtt{P} = \begin{bmatrix} 2 & 0 & 4 & 0 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Find the principal point of the image plane.

Solution: By definition,

$$\mathtt{P} = \begin{bmatrix} \mathtt{KR} \mid -\mathtt{KR}\vec{C}_\delta \end{bmatrix}$$

The principal point is defined to be the elements k_{13} and k_{23} of the camera calibration matrix $\mathtt{K}$. We use [1, Equations (7.45) and (7.46)] to compute it. If we denote the matrix $\mathtt{KR}$ by $\mathtt{B}$ and its rows by column vectors $\mathtt{b}_i$, then we can rewrite those equations as

$$k_{13} = \mathtt{b}_1^\top \mathtt{b}_3 = \begin{bmatrix} 2 & 0 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 4,$$

$$k_{23} = \mathtt{b}_2^\top \mathtt{b}_3 = \begin{bmatrix} 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 2.$$

The principal point of the image plane equals

$$\vec{u}_{0\alpha} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

□

Task 3. *Let us assume the following image projection matrix*

$$\mathtt{P}_\beta = \begin{bmatrix} 6 & -8 & 50 & 800 \\ 16 & 12 & 40 & -1200 \\ 0 & 0 & 0.1 & 0 \end{bmatrix}$$

Find $\mathtt{K}$, $\mathtt{R}$, $\vec{C}_\delta$, f.

Solution: We first compute the focal length f of the given camera according to [1, Equation (7.25)]:

$$\left\| \mathtt{P}_{\beta_{3,1:3}} \right\| = \frac{1}{f} \Rightarrow f = \frac{1}{\left\| \mathtt{P}_{\beta_{3,1:3}} \right\|} = \frac{1}{0.1} = 10$$

According to [1, Equation (7.24)]:

$$\mathtt{P}_{\beta_{1:3,1:3}} = \frac{1}{f}\mathtt{KR} \Rightarrow \mathtt{KR} = f \cdot \mathtt{P}_{\beta_{1:3,1:3}} = \begin{bmatrix} 60 & -80 & 500 \\ 160 & 120 & 400 \\ 0 & 0 & 1 \end{bmatrix}$$

Using [1, Equations (7.45)-(7.49)] we decompose $\mathtt{B} = \mathtt{KR}$ into $\mathtt{K}$ and $\mathtt{R}$:

$$k_{23} = \mathtt{b}_2^\top \mathtt{b}_3 = \begin{bmatrix} 160 & 120 & 400 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 400,$$

$$k_{13} = \mathtt{b}_1^\top \mathtt{b}_3 = \begin{bmatrix} 60 & -80 & 500 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 500,$$

$$k_{22}^2 + 400^2 = \mathtt{b}_2^\top \mathtt{b}_2 = \begin{bmatrix} 160 & 120 & 400 \end{bmatrix} \begin{bmatrix} 160 \\ 120 \\ 400 \end{bmatrix} = 200000 \Rightarrow k_{22} = \sqrt{200000 - 160000} = \sqrt{40000} = 200,$$

$$k_{12} \cdot 200 + 500 \cdot 400 = \mathtt{b}_1^\top \mathtt{b}_2 = \begin{bmatrix} 60 & -80 & 500 \end{bmatrix} \begin{bmatrix} 160 \\ 120 \\ 400 \end{bmatrix} = 200000 \Rightarrow k_{12} = \frac{200000 - 200000}{200} = 0,$$

$$k_{11}^2 + 0^2 + 500^2 = \mathtt{b}_1^\top \mathtt{b}_1 = \begin{bmatrix} 60 & -80 & 500 \end{bmatrix} \begin{bmatrix} 60 \\ -80 \\ 500 \end{bmatrix} = 260000 \Rightarrow k_{11} = \sqrt{260000 - 250000} = \sqrt{10000} = 100.$$

Thus,

$$\mathtt{K} = \begin{bmatrix} 100 & 0 & 500 \\ 0 & 200 & 400 \\ 0 & 0 & 1 \end{bmatrix}$$

The rotation matrix $\mathtt{R}$ of the camera can be computed as

$$\mathtt{R} = \mathtt{K}^{-1}\mathtt{B} = \begin{bmatrix} 0.01 & 0 & -5 \\ 0 & 0.005 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 60 & -80 & 500 \\ 160 & 120 & 400 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.6 & -0.8 & 0 \\ 0.8 & 0.6 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The camera projection center $\vec{C}_\delta$ can be obtained as

$$\vec{C}_\delta = -\mathtt{P}_{\beta_{1:3,1:3}}^{-1} \mathtt{P}_{\beta_{1:3,4}} = - \begin{bmatrix} 6 & -8 & 50 \\ 16 & 12 & 40 \\ 0 & 0 & 0.1 \end{bmatrix}^{-1} \begin{bmatrix} 800 \\ -1200 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 100 \\ 0 \end{bmatrix}$$

□

Task 4. *Write the transition matrix from basis δ to basis γ, coordinates of the camera projection center in the camera cartesian coordinate system $C_{(C,\gamma)}$ and $\vec{C}_\gamma$ of the camera from the previous example.*

Solution: According to [1, Equation (7.6)], the transition matrix $\mathtt{T}_{\delta\to\gamma}$ equals

$$\mathtt{T}_{\delta\to\gamma} = \frac{1}{f}\mathtt{R} = \frac{1}{10} \begin{bmatrix} 0.6 & -0.8 & 0 \\ 0.8 & 0.6 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.06 & -0.08 & 0 \\ 0.08 & 0.06 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}$$

Also,

$$C_{(C,\gamma)} = \overrightarrow{CC}_\gamma = \vec{0}_\gamma = \mathbf{0},$$

$$\vec{C}_\gamma = \mathtt{T}_{\delta\to\gamma}\vec{C}_\delta = \begin{bmatrix} 0.06 & -0.08 & 0 \\ 0.08 & 0.06 & 0 \\ 0 & 0 & 0.1 \end{bmatrix} \begin{bmatrix} 0 \\ 100 \\ 0 \end{bmatrix} = \begin{bmatrix} -8 \\ 6 \\ 0 \end{bmatrix}$$

□

References

[1] Tomas Pajdla, *Elements of geometry for computer vision*, `https://cw.fel.cvut.cz/wiki/_media/courses/gvg/pajdla-gvg-lecture-2021.pdf`.