Homography between images of a plane

$$\zeta\vec{x}_\gamma = \begin{bmatrix}\mathtt{R} & -\mathtt{R}\vec{C}_\delta\end{bmatrix}\begin{bmatrix}\vec{X}_\delta \\ 1\end{bmatrix}, \quad \zeta'\vec{x}'_{\gamma'} = \begin{bmatrix}\mathtt{R}' & -\mathtt{R}'\vec{C}'_\delta\end{bmatrix}\begin{bmatrix}\vec{X}_\delta \\ 1\end{bmatrix}$$

Let the plane be defined by

$$\vec{n}_\delta^\top \vec{X}_\delta = d,$$

i.e. the plane doesn't need necessarily coincide with the plane spanned by δ_1 and δ_2. Since

$$\vec{X}_\delta = \zeta\mathtt{R}^{-1}\vec{x}_\gamma + \vec{C}_\delta = \zeta\mathtt{R}^\top\vec{x}_\gamma + \vec{C}_\delta$$

then

$$d = \vec{n}_\delta^\top\vec{X}_\delta = \vec{n}_\delta^\top(\zeta\mathtt{R}^\top\vec{x}_\gamma + \vec{C}_\delta) = \zeta\vec{n}_\delta^\top\mathtt{R}^\top\vec{x}_\gamma + \vec{n}_\delta^\top\vec{C}_\delta \Rightarrow \frac{1}{\zeta} = \frac{\vec{n}_\delta^\top\mathtt{R}^\top\vec{x}_\gamma}{d - \vec{n}_\delta^\top\vec{C}_\delta}$$

We have

$$\zeta'\vec{x}'_{\gamma'} = \mathtt{R}'\vec{X}_\delta - \mathtt{R}'\vec{C}'_\delta = \mathtt{R}'(\zeta\mathtt{R}^\top\vec{x}_\gamma + \vec{C}_\delta) - \mathtt{R}'\vec{C}'_\delta = \zeta\mathtt{R}'\mathtt{R}^\top\vec{x}_\gamma + \mathtt{R}'\underbrace{(\vec{C}_\delta - \vec{C}'_\delta)}_{\vec{t}_\delta} = \zeta\mathtt{R}'\mathtt{R}^\top\vec{x}_\gamma + \mathtt{R}'\vec{t}_\delta$$

and thus

$$\frac{\zeta'}{\zeta}\vec{x}'_{\gamma'} = \mathtt{R}'\mathtt{R}^\top\vec{x}_\gamma + \mathtt{R}'\vec{t}_\delta\frac{1}{\zeta} = \mathtt{R}'\mathtt{R}^\top\vec{x}_\gamma + \mathtt{R}'\vec{t}_\delta\frac{\vec{n}_\delta^\top\mathtt{R}^\top\vec{x}_\gamma}{d - \vec{n}_\delta^\top\vec{C}_\delta}$$

$$\frac{\zeta'}{\zeta}\vec{x}'_{\gamma'} = \left(\underbrace{\mathtt{R}'\mathtt{R}^\top + \frac{\mathtt{R}'\vec{t}_\delta\vec{n}_\delta^\top\mathtt{R}^\top}{d - \vec{n}_\delta^\top\vec{C}_\delta}}_{\mathtt{H}}\right)\vec{x}_\gamma$$

Usually, in practice we attach the world coordinate system δ to the first camera in such a way that its pose becomes

$$\mathtt{R} = \mathtt{I}, \quad \vec{C}_\delta = \mathbf{0}.$$

In that case, the homography matrix takes the form

$$\mathtt{H} = \mathtt{R}' - \frac{\mathtt{R}'\vec{C}'_\delta\vec{n}_\delta^\top}{d}$$

GVG'2021 Exercise-06 EN

1. Consider two fully calibrated cameras

$$\mathtt{P} = \begin{bmatrix}\mathtt{R} & -\mathtt{R}\vec{C}_\delta\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}, \quad \mathtt{P}' = \begin{bmatrix}\mathtt{R}' & -\mathtt{R}'\vec{C}'_\delta\end{bmatrix} = \begin{bmatrix}0 & 0 & 1 & 0\\ 1 & 0 & 0 & 2\\ 0 & 1 & 0 & 1\end{bmatrix}$$

which observe the plane with normal $\vec{n}_\delta = [1, 0, 2]^\top$ and offset $d = 1$. We know that the coordinates of the projection of some 3D point lying in that plane to the first camera are $[u, v]^\top = [1, 2]^\top$. Find the coordinates $[u', v']$ of the projection of the same 3D point to the second camera considering that the calibration matrices of these cameras are

$$\mathtt{K} = \mathtt{K}' = \begin{bmatrix}1 & 0 & 3\\ 0 & 1 & 3\\ 0 & 0 & 1\end{bmatrix}$$

Answer: $[u', v']^\top = [2, 5]^\top$.

2. Points in an affine plane with affine coordinates

$$\vec{X}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \vec{X}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{X}_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \vec{X}_4 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

are mapped by a homography into image points with affine coordinates

$$\vec{u}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad \vec{u}_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \vec{u}_4 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

(a) Find a homography matrix.

(b) Find the affine coordinates of the point of the affine plane that is mapped into point $[1, 1]^\top$ in the image.