Elements of Geometry for Computer Vision and Computer Graphics

Translation of Euclid's Elements by Adelardus Bathensis (1080–1152)

2021 Lecture 12

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3.4.2 Point computation

Let us assume having camera projection matrices P_1, P_2 and image points $\vec{x}_{1\beta_1}, \vec{x}_{2\beta_2}$ such that

$$\zeta_1 \, \vec{x}_{1\beta_1} = P_1 \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix} \quad \text{and} \quad \zeta_2 \, \vec{x}_{2\beta_2} = P_2 \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix} \tag{3.68}$$

We can get $\vec{X}_\delta$, and ζ_1, ζ_2 by solving the following system of (inhomogeneous) linear equations

$$\begin{bmatrix} \vec{x}_{1\beta_1} & \vec{0} & -P_1 \\ \vec{0} & \vec{x}_{2\beta_2} & -P_2 \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \vec{X}_\delta \\ 1 \end{bmatrix} = 0 \tag{3.69}$$

3.5 Calibrated relative camera pose computation

Essential matrix computation

1) 7 points → 5 points
2) Works for $X \in$ 1 plane
3) More difficult

In the previous chapter, we had first computed a multiple of the fundamental matrix from seven point correspondences and only then used camera calibration matrices to recover a multiple of the essential matrix. Here we will use the camera calibration right from the beginning to obtain a multiple of the essential matrix directly from only five image correspondences. Not only that five is smaller than seven but using the calibration right from the beginning permits all points of the scene generating the correspondences to lie in a plane.

We start from Equation 3.42 to get $\vec{x}_{1\gamma_1}$ and $\vec{x}_{2\gamma_2}$ from Equation 3.43 which are related by

$$\vec{x}_{2\beta_2}^\top K_2^{-\top} E \, K_1^{-1} \vec{x}_{1\beta_1} = 0 \tag{3.70}$$

$$\vec{x}_{2\gamma_2}^\top E \, \vec{x}_{1\gamma_1} = 0 \tag{3.71}$$

Calibrated essential matrix computation

The above equation holds true for all pairs of image points $(\vec{x}_{1\gamma_1}, \vec{x}_{2\gamma_2})$ that are in correspondence, i.e. are projections of the same point of the scene.

3.5.1 Constraints on E

Matrix $\mathtt{E}$ has rank two, and therefore there holds

$$|\mathtt{E}| = 0 \tag{3.72}$$

Known for F too → |F|=0

true.

We will now derive additional constraints on $\mathtt{E}$. Let us consider that we can write, Equation 3.48,

$$\mathtt{E} = \mathtt{R}\left[\vec{C}_{\epsilon_1}\right]_\times \tag{3.73}$$

Additional constraints valid for E (but NOT F)

Let us introduce $\vec{C}_{\epsilon_1} = \begin{bmatrix} x & y & z \end{bmatrix}^\top$ and evaluate

$$\begin{aligned}
\mathtt{E}^\top\mathtt{E} &= \left(\mathtt{R}\left[\vec{C}_{\epsilon_1}\right]_\times\right)^\top \mathtt{R}\left[\vec{C}_{\epsilon_1}\right]_\times = \left[\vec{C}_{\epsilon_1}\right]_\times^\top \mathtt{R}^\top\mathtt{R}\left[\vec{C}_{\epsilon_1}\right]_\times = \left[\vec{C}_{\epsilon_1}\right]_\times^\top \left[\vec{C}_{\epsilon_1}\right]_\times & (3.74)\\
&= \begin{bmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{bmatrix} \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} = \begin{bmatrix} z^2+y^2 & -x\,y & -x\,z \\ -x\,y & z^2+x^2 & -y\,z \\ -x\,z & -y\,z & y^2+x^2 \end{bmatrix}\\
&= \begin{bmatrix} x^2+y^2+z^2 & & \\ & x^2+y^2+z^2 & \\ & & x^2+y^2+z^2 \end{bmatrix} - \begin{bmatrix} x\,x & x\,y & x\,z \\ x\,y & y\,y & y\,z \\ x\,z & y\,z & z\,z \end{bmatrix}\\
&= \|\vec{C}_{\epsilon_1}\|^2 \mathtt{I} - \vec{C}_{\epsilon_1}\vec{C}_{\epsilon_1}^\top & (3.75)
\end{aligned}$$

Making a polynomial constraint on E

We can multiply the above expression by $\mathtt{E}$ from the left again to get an interesting equation

$$\mathtt{E}\,\mathtt{E}^\top\mathtt{E} = \mathtt{E}\left(\|\vec{C}_{\epsilon_1}\|^2\mathtt{I} - \vec{C}_{\epsilon_1}\vec{C}_{\epsilon_1}^\top\right) = \|\vec{C}_{\epsilon_1}\|^2\mathtt{E} = \frac{1}{2}\operatorname{trace}\left(\mathtt{E}^\top\mathtt{E}\right)\mathtt{E} \tag{3.76}$$

or equivalently

$$2\,\mathtt{E}\,\mathtt{E}^\top\mathtt{E} = \operatorname{trace}\left(\mathtt{E}^\top\mathtt{E}\right)\mathtt{E} \tag{3.77}$$

Demazure constraints

which provides nine equations on elements of $\mathtt{E}$.

In fact, these equations also imply $|\mathtt{E}| = 0$. Consider that Equation 3.77 implies

$$\left(2\,\mathtt{E}\,\mathtt{E}^\top - \operatorname{trace}(\mathtt{E}^\top\mathtt{E})\,\mathtt{I}\right)\,\mathtt{E} = \mathtt{0} \qquad (3.78)$$

For Equation 3.78 to hold true, either $\mathtt{E}$ can't have the full rank, i.e. $|\mathtt{E}| = 0$, or $2\,\mathtt{E}\,\mathtt{E}^\top - \operatorname{trace}(\mathtt{E}^\top\mathtt{E})\,\mathtt{I} = \mathtt{0}$. The latter case gives

$$0 = \operatorname{trace}(2\,\mathtt{E}\,\mathtt{E}^\top - \operatorname{trace}(\mathtt{E}^\top\mathtt{E})\,\mathtt{I}) = 2\operatorname{trace}(\mathtt{E}\,\mathtt{E}^\top) - 3\operatorname{trace}(\mathtt{E}^\top\mathtt{E}) \qquad (3.79)$$

Let us check the relationship between $\operatorname{trace}(\mathtt{E}^\top\mathtt{E})$ and $\operatorname{trace}(\mathtt{E}\,\mathtt{E}^\top)$ now. We write

$$\begin{aligned}\operatorname{trace}(\mathtt{E}^\top\mathtt{E}) &= (\mathtt{E}_{11}^2 + \mathtt{E}_{21}^2 + \mathtt{E}_{31}^2) + (\mathtt{E}_{12}^2 + \mathtt{E}_{22}^2 + \mathtt{E}_{32}^2) + (\mathtt{E}_{13}^2 + \mathtt{E}_{23}^2 + \mathtt{E}_{33}^2)\\ &= (\mathtt{E}_{11}^2 + \mathtt{E}_{12}^2 + \mathtt{E}_{13}^2) + (\mathtt{E}_{21}^2 + \mathtt{E}_{22}^2 + \mathtt{E}_{23}^2) + (\mathtt{E}_{31}^2 + \mathtt{E}_{32}^2 + \mathtt{E}_{33}^2)\\ &= \operatorname{trace}(\mathtt{E}\,\mathtt{E}^\top) \qquad (3.80)\end{aligned}$$

Substituting the above into Equation 3.79 gets us

$$0 = 2\operatorname{trace}(\mathtt{E}\,\mathtt{E}^\top) - 3\operatorname{trace}(\mathtt{E}^\top\mathtt{E}) = -\operatorname{trace}(\mathtt{E}^\top\mathtt{E}) \qquad (3.81)$$

Equation $2\,\mathtt{E}\,\mathtt{E}^\top - \operatorname{trace}(\mathtt{E}^\top\mathtt{E})\,\mathtt{I} = \mathtt{0}$ also implies

$$\begin{aligned}2\,\mathtt{E}\,\mathtt{E}^\top &= \operatorname{trace}(\mathtt{E}^\top\mathtt{E})\,\mathtt{I} \;\; = 0\cdot \mathtt{I} && (3.82)\\ |2\,\mathtt{E}\,\mathtt{E}^\top| &= |\operatorname{trace}(\mathtt{E}^\top\mathtt{E})\,\mathtt{I}| \;\; = 0 && (3.83)\\ 2^3|\mathtt{E}|^2 &= (\operatorname{trace}(\mathtt{E}^\top\mathtt{E}))^3 \;\; = 0 && (3.84)\\ 2^3|\mathtt{E}|^2 &= 0 && (3.85)\\ |\mathtt{E}| &= 0 && (3.86)\end{aligned}$$

Therefore, Equation 3.77 implies $|\mathtt{E}| = 0$.

Let us now look at constraints on matrix $\mathtt{G} = \tau\,\mathtt{E}$, for some non-zero real τ. We can multiply Equation 3.78 by τ^3 to get

$$\begin{aligned}\tau^3\,\left(2\,\mathtt{E}\,\mathtt{E}^\top - \operatorname{trace}(\mathtt{E}^\top\mathtt{E})\,\mathtt{I}\right)\,\mathtt{E} &= \mathtt{0} && (3.87)\\ \left(2\,(\tau\,\mathtt{E})\,(\tau\,\mathtt{E}^\top) - \operatorname{trace}((\tau\,\mathtt{E}^\top)\,(\tau\,\mathtt{E}))\,\mathtt{I}\right)\,(\tau\,\mathtt{E}) &= \mathtt{0} && (3.88)\\ \left(2\,\mathtt{G}\,\mathtt{G}^\top - \operatorname{trace}(\mathtt{G}^\top\mathtt{G})\,\mathtt{I}\right)\,\mathtt{G} &= \mathtt{0} && (3.89)\end{aligned}$$

3x3 matrix

Show that

$2\,\mathtt{E}\,\mathtt{E}^\top\mathtt{E} = \operatorname{trace}(\mathtt{E}^\top\mathtt{E})\,\mathtt{E}$ implies $|\mathtt{E}| = 0$

$\operatorname{trace}\begin{bmatrix}\operatorname{trace}(\mathtt{E}^\top\mathtt{E}) & & \\ & \operatorname{trace}(\mathtt{E}^\top\mathtt{E}) & \\ & & \operatorname{trace}(\mathtt{E}^\top\mathtt{E})\end{bmatrix} = 3\operatorname{trace}(\mathtt{E}^\top\mathtt{E})$

Clearly, rank $(\mathtt{G})$ = rank $(\tau\,\mathtt{E})$ = rank $(\mathtt{E}) = 2$.

We conclude that constraints on $\mathtt{E}$ and $\mathtt{G}$ are the same.

3.5.2 Geometrical interpretation of Equation 3.77

Demazure constraints

$$2\,\mathtt{E}\,\mathtt{E}^\top\mathtt{E} = \mathrm{trace}\,(\mathtt{E}^\top\mathtt{E})\,\mathtt{E}$$

True for every $\vec{y} \in \mathbb{R}^3$

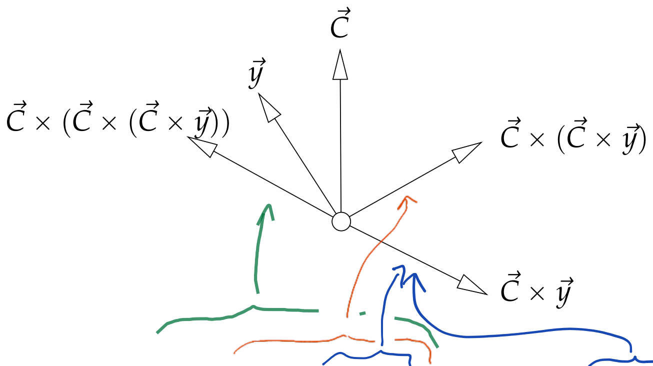

Figure 3.3: Identity $\vec{C}_{\epsilon_1}\times(\vec{C}_{\epsilon_1}\times(\vec{C}_{\epsilon_1}\times\vec{y})) = -\|\vec{C}_{\epsilon_1}\|^2(\vec{C}_{\epsilon_1}\times\vec{y})$.

Let us provide a geometrical interpretation of Equation 3.77. We will mutiply both sides of Equation 3.77 by a vector $\vec{y}\in\mathbb{R}^3$ and write

$$2\,\mathtt{E}\,\mathtt{E}^\top\mathtt{E}\,\vec{y} = \mathrm{trace}\,(\mathtt{E}^\top\mathtt{E})\,\mathtt{E}\,\vec{y} \tag{3.90}$$

$$2\,\mathtt{R}\left[\vec{C}_{\epsilon_1}\right]_\times\left[\vec{C}_{\epsilon_1}\right]_\times^\top\left[\vec{C}_{\epsilon_1}\right]_\times\vec{y} = 2\,\|\vec{C}_{\epsilon_1}\|^2\,\mathtt{R}\left[\vec{C}_{\epsilon_1}\right]_\times\vec{y} \tag{3.91}$$

$$-\mathtt{R}\left[\vec{C}_{\epsilon_1}\right]_\times\left[\vec{C}_{\epsilon_1}\right]_\times\left[\vec{C}_{\epsilon_1}\right]_\times\vec{y} = \mathtt{R}\,\|\vec{C}_{\epsilon_1}\|^2\left[\vec{C}_{\epsilon_1}\right]_\times\vec{y} \tag{3.92}$$

$$\left[\vec{C}_{\epsilon_1}\right]_\times\left[\vec{C}_{\epsilon_1}\right]_\times\left[\vec{C}_{\epsilon_1}\right]_\times\vec{y} = -\|\vec{C}_{\epsilon_1}\|^2\left[\vec{C}_{\epsilon_1}\right]_\times\vec{y} \tag{3.93}$$

$\mathtt{E} = \mathtt{R}\,[\vec{C}_{\varepsilon_1}]_\times$

$[\vec{C}_{\varepsilon_1}]_\times\,\vec{y}_{\varepsilon_1} = \vec{C}_{\varepsilon_1}\times\vec{y}_{\varepsilon_1}$

Now, we use that for every two vectors $\vec{x},\vec{y}\in\mathbb{R}^3$ there holds $[\vec{x}]_\times\,\vec{y} = \vec{x}\times\vec{y}$ true to get

$$\vec{C}_{\epsilon_1}\times(\vec{C}_{\epsilon_1}\times(\vec{C}_{\epsilon_1}\times\vec{y})) = -\|\vec{C}_{\epsilon_1}\|^2(\vec{C}_{\epsilon_1}\times\vec{y}) \tag{3.94}$$

which is a familiar identity of the vector pruduct in $\mathbb{R}^3$, Figure 3.3.

3.5.3 Characterization of $\mathtt{E}$

Let us next see that a non-zero 3×3 real matrix satisfying Equation 3.77 has rank two and can be written in the form of Equation 3.73 for some rotation $\mathtt{R}$ and some vector $\mathtt{C}_{\epsilon_1}$.

Consider a real 3×3 matrix $\mathtt{E}$ such that Equation 3.77 holds true. We will make here use of the SVD decomposition [2, p. 411] of real matrices. We can write

$$\mathtt{E} = \mathtt{U} \begin{bmatrix} a & & \\ & b & \\ & & c \end{bmatrix} \mathtt{V}^\top \tag{3.95}$$

for some real non-negative a, b, c and some orthogonal real 3×3 matrices $\mathtt{U}, \mathtt{V}$, such that $\mathtt{U}^\top \mathtt{U} = \mathtt{I}$, and $\mathtt{V}^\top \mathtt{V} = \mathtt{I}$ [2, p. 411]. One can see that $\mathtt{U}^\top \mathtt{U} = \mathtt{I}$, and $\mathtt{V}^\top \mathtt{V} = \mathtt{I}$ implies $|\mathtt{U}| = \pm 1, |\mathtt{V}| = \pm 1$.

Using Equation 3.95 we get

$$\mathtt{E}\,\mathtt{E}^\top = \mathtt{U} \begin{bmatrix} a^2 & & \\ & b^2 & \\ & & c^2 \end{bmatrix} \mathtt{U}^\top, \quad \mathtt{E}^\top \mathtt{E} = \mathtt{V} \begin{bmatrix} a^2 & & \\ & b^2 & \\ & & c^2 \end{bmatrix} \mathtt{V}^\top \tag{3.96}$$

and $\operatorname{trace}(\mathtt{E}^\top \mathtt{E}) = \operatorname{trace}(\mathtt{V}\,\mathtt{D}^2 \mathtt{V}^\top) = \operatorname{trace}(\mathtt{V}\,\mathtt{D}^2 \mathtt{V}^{-1}) = \operatorname{trace}(\mathtt{D}^2)$ since matrices $\mathtt{D}^2$ and $\mathtt{E}\,\mathtt{E}^\top$ are similar and hence their traces, which are the sums of their eigenvalues, are equal. Now, we can rewrite Equation 3.77 as

$$\left(2\,\mathtt{U} \begin{bmatrix} a^2 & & \\ & b^2 & \\ & & c^2 \end{bmatrix} \mathtt{U}^\top - (a^2 + b^2 + c^2)\,\mathtt{I}\right) \mathtt{U} \begin{bmatrix} a & & \\ & b & \\ & & c \end{bmatrix} \mathtt{V}^\top = 0 \tag{3.97}$$

$$2\,\mathtt{U} \begin{bmatrix} a^3 & & \\ & b^3 & \\ & & c^3 \end{bmatrix} \mathtt{V}^\top - (a^2 + b^2 + c^2)\,\mathtt{U} \begin{bmatrix} a & & \\ & b & \\ & & c \end{bmatrix} \mathtt{V}^\top = 0 \tag{3.98}$$

Matrices $\mathtt{U}$, $\mathtt{V}$ are regular and thus we get

$$2\begin{bmatrix} a^3 & & \\ & b^3 & \\ & & c^3 \end{bmatrix} - (a^2+b^2+c^2)\begin{bmatrix} a & & \\ & b & \\ & & c \end{bmatrix} = \mathbf{0} \tag{3.99}$$

which finally leads to the following three equations

$$a^3 - a\,b^2 - a\,c^2 = a\,(a^2 - b^2 - c^2) = 0 \tag{3.100}$$
$$b^3 - b\,a^2 - b\,c^2 = b\,(b^2 - c^2 - a^2) = 0 \tag{3.101}$$
$$c^3 - c\,a^2 - c\,b^2 = c\,(c^2 - a^2 - b^2) = 0 \tag{3.102}$$

We see that there are the following two exclusive cases:

1. If any two of a, b, c are zero, then the third one is zero too. For instance, if $a = b = 0$, then Equation 3.102 gives $c^3 = 0$. This can't happen for a non-zero $\mathtt{E}$.

2. If any two of a, b, c are non-zero, then the two non-zero are equal and the third is zero. For instance, if $a \neq 0$ and $b \neq 0$, then Equations 3.100, 3.101 imply $c^2 = 0$ and thus $a^2 = b^2$, which gives $a = b$ since a, b are non-negative, i.e. $\operatorname{rank}(\mathtt{E}) = 2$.

We thus conclude that $\mathtt{E}$ can be written as

$$\mathtt{E} = \mathtt{U}\begin{bmatrix} a & & \\ & a & \\ & & 0 \end{bmatrix}\mathtt{V}^\top = \mathtt{U}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -a & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\mathtt{V}^\top \tag{3.103}$$

$$= \mathtt{W}\left[\begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix}\right]_\times \mathtt{V}^\top = \mathtt{W}\left[\mathtt{V}^\top\mathtt{V}\begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix}\right]_\times \mathtt{V}^\top = \mathtt{W}\,\frac{(\mathtt{V}^\top)^{-\top}}{|(\mathtt{V}^\top)^{-\top}|}\left[\mathtt{V}\begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix}\right]_\times \tag{3.104}$$

$$= (\operatorname{sign}(|\mathtt{W}|))^2\,\mathtt{W}\,\mathtt{V}^\top \operatorname{sign}(|\mathtt{V}^\top|)\;[a\,\mathrm{v}_3]_\times \tag{3.105}$$

$$= \operatorname{sign}(|\mathtt{W}|)\,\mathtt{W}\,\mathtt{V}^\top \operatorname{sign}(|\mathtt{V}^\top|)\;[\operatorname{sign}(|\mathtt{W}|)\,a\,\mathrm{v}_3]_\times \tag{3.106}$$

$$= \mathtt{R}\,[\operatorname{sign}(|\mathtt{U}|)\,a\,\mathrm{v}_3]_\times \tag{3.107}$$

for some non-negative a and the third column $\mathtt{v}_3$ of $\mathtt{V}$. Parameter a is zero for $\mathtt{E} = \mathtt{0}$ and positive for rank two matrices $\mathtt{E}$. We introduced a new matrix $\mathtt{W}$ in Equation 3.104, which is the product of $\mathtt{U}$ and a rotation round the z axis. We also used $\mathtt{V}^\top \mathtt{V} = \mathtt{I}$, and finally Equation 1.51. In Equation 3.105 we used $(\operatorname{sign}(|\mathtt{W}|))^2 = 1$, $\mathtt{V}^{-\top} = \mathtt{V}$ for $\mathtt{V}^\top \mathtt{V} = \mathtt{I}$. Matrix $\mathtt{R} = \operatorname{sign}(|(\mathtt{W})|)\, \mathtt{W}\, \mathtt{V}^\top \operatorname{sign}(|\mathtt{V}^\top|)$ in Equation 3.107 is a rotation since $\operatorname{sign}(|(\mathtt{W})|)\, \mathtt{W}$ as well as $\mathtt{V}^\top \operatorname{sign}(|\mathtt{V}^\top|)$ are both rotations. Finally, we see that $\operatorname{sign}(|\mathtt{W}|) = \operatorname{sign}(|\mathtt{U}|)$.

3.5.4 Computing a non-zero multiple of $\mathtt{E}$

Let us now disscuss how to compute a non-zero multiple of matrix $\mathtt{E}$ from image matches.

3.5.4.1 Selecting equations

Every pair of image matches $(\vec{x}_{1\gamma_1}, \vec{x}_{2\gamma_2})$ provides a linear constratint on elements of $\mathtt{E}$ in the form of Equation 3.71 and matricial Equation 3.77 gives nine polynomial constraints for elements of $\mathtt{E}$.

We have already seen in Paragraph 3.2 that a non-zero multiple of $\mathtt{E}$ can be obtained from seven absolutely accurate point correspondences using the constraint $|\mathtt{E}| = 0$. The solution was obtained by solving a set of polynomial equations out of which seven were linear and the eighth one was a third order polynomial.

Let us now see how to exploit Equation 3.77 in order to compute a non-zero multiple of $\mathtt{E}$ from as few image matches as possible.

An idea might be to use Equations 3.77 instead of $|\mathtt{E}| = 0$. It would be motivated by the fact that Equations 3.77 imply equation $|\mathtt{E}| = 0$ for real 3×3 matrices $\mathtt{E}$. Unfortunately, this implication does not hold true when we allow complex numbers in $\mathtt{E}$[1], which we have to do if we want to

[1] Equation $|\mathtt{E}| = 0$ can't be generated from Equations 3.77 as their algebraic combination, i.e. $|\mathtt{E}| = 0$ is not in the ideal [12] generated by Equations 3.77. It means that there

obtain E as a solution to a polynomial system without using any additional constraints. We have to therefore use $|E| = 0$ as well.

The next question is whether we have to use all nine Equations 3.77. It can be shown similarly as above that indeed none of the equations 3.77 is in the ideal [12] generated by the others[2]. Therefore, we have to use all

might be some matrices E satisfying Equations 3.77 which do not satisfy $|E| = 0$. We know that such matrices can't be real. The proof of the above claim can be obtained by the following program in Maple [13]

```
>with(LinearAlgebra):
>with(Groebner):
>E:=¡¡e11—e12—e13¿,¡e21—e22—e23¿,¡e31—e32—e33¿¿:
>eM:=2*E.Transpose(E).E-Trace(Transpose(E).E)*E:
>eq:=expand(convert(convert(eM,Vector),list)):
>v:=indets(eq):
>mo:=tdeg(op(v)):
>G:=Basis(eq,mo):
>Reduce(Determinant(E),G,mo);
e11 e22 e33 - e11 e23 e32 + e21 e32 e13 - e21 e12 e33 + e31 e12 e23 - e31 e22 e13
```

which computes the Groebner basis G of the ideal generated by Equations 3.77 and verifies that the remainder on division of $|E|$ by G is non-zero [12].

[2] To show that none of the equations 3.77 is in the ideal generated by the others, we run the following test in Maple.

```
>with(LinearAlgebra):
>with(Groebner):
>E:=¡¡e11—e12—e13¿,¡e21—e22—e23¿,¡e31—e32—e33¿¿:
>eM:=2*E.Transpose(E).E-Trace(Transpose(E).E)*E:
>eq:=expand(convert(convert(eM,Vector),list)):
>
>ReduceEqByEqn:=proc(eq,eqn)
     local mo,G;
     mo:=tdeg(op(indets(eqn)));
     G:=Basis(eqn,mo);
     Reduce(eq,G,mo);
  end proc:
>
>for i from 1 to 9 do
```

Equations 3.77 as well as $|\mathtt{E}| = 0$. Hence we have altogether ten polynomial equations of order higher than one.

We have more equations than unknowns but they still do not fully determine E. We have to add some more equations from image matches. To see how many equations we have to add, we evaluate the Hilbert dimension [12] of the ideal generated by Equations 3.77 and $|\mathtt{E}| = 0$. We know [12] that a system of polynomial equations has a finite number of solutions if and only if the Hilbert dimension of the ideal generated by the system is zero.

The Hilbert dimension of the ideal generated by Equations 3.77 and $|\mathtt{E}| = 0$ is equal to six[3]. An extra linear equation reduces the Hilbert dimension

ReduceEqByEqn(eq[i],eq[[op({$1..9} minus {i})]]);
end;

$$e11^3 + e11\,e12^2 + e11\,e13^2 + e11\,e21^2 + 2\,e21\,e12\,e22 + 2\,e21\,e13\,e23 + e11\,e31^2 + 2\,e31\,e12\,e32 + 2\,e31\,e13\,e33 - e11\,e22^2 - e11\,e32^2 - e11\,e23^2 - e11\,e33^2$$

$$e11^2\,e21 + 2\,e11\,e12\,e22 + 2\,e11\,e13\,e23 + e21^3 + e21\,e22^2 + e21\,e23^2 + e21\,e31^2 + 2\,e31\,e22\,e32 + 2\,e31\,e23\,e33 - e21\,e12^2 - e21\,e32^2 - e21\,e13^2 - e21\,e33^2$$

$$e11^2\,e31 + 2\,e11\,e12\,e32 + 2\,e11\,e13\,e33 + e21^2\,e31 + 2\,e21\,e22\,e32 + 2\,e21\,e23\,e33 + e31^3 + e31\,e32^2 + e31\,e33^2 - e31\,e12^2 - e31\,e22^2 - e31\,e13^2 - e31\,e23^2$$

$$e12\,e11^2 + e12^3 + e12\,e13^2 + 2\,e22\,e11\,e21 + e12\,e22^2 + 2\,e22\,e13\,e23 + 2\,e32\,e11\,e31 + e12\,e32^2 + 2\,e32\,e13\,e33 - e12\,e21^2 - e12\,e31^2 - e12\,e23^2 - e12\,e33^2$$

$$2\,e12\,e11\,e21 + e12^2\,e22 + 2\,e12\,e13\,e23 + e22\,e21^2 + e22^3 + e22\,e23^2 + 2\,e32\,e21\,e31 + e22\,e32^2 + 2\,e32\,e23\,e33 - e22\,e11^2 - e22\,e31^2 - e22\,e13^2 - e22\,e33^2$$

$$2\,e12\,e11\,e31 + e12^2\,e32 + 2\,e12\,e13\,e33 + 2\,e22\,e21\,e31 + e22^2\,e32 + 2\,e22\,e23\,e33 + e32\,e31^2 + e32^3 + e32\,e33^2 - e32\,e11^2 - e32\,e21^2 - e32\,e13^2 - e32\,e23^2$$

$$e13\,e11^2 + e13\,e12^2 + e13^3 + 2\,e23\,e11\,e21 + 2\,e23\,e12\,e22 + e13\,e23^2 + 2\,e33\,e11\,e31 + 2\,e33\,e12\,e32 + e13\,e33^2 - e13\,e21^2 - e13\,e31^2 - e13\,e22^2 - e13\,e32^2$$

$$2\,e13\,e11\,e21 + 2\,e13\,e12\,e22 + e13^2\,e23 + e23\,e21^2 + e23\,e22^2 + e23^3 + 2\,e33\,e21\,e31 + 2\,e33\,e22\,e32 + e23\,e33^2 - e23\,e11^2 - e23\,e31^2 - e23\,e12^2 - e23\,e32^2$$

$$2\,e13\,e11\,e31 + 2\,e13\,e12\,e32 + e13^2\,e33 + 2\,e23\,e21\,e31 + 2\,e23\,e22\,e32 + e23^2\,e33 + e33\,e31^2 + e33\,e32^2 + e33^3 - e33\,e11^2 - e33\,e21^2 - e33\,e12^2 - e33\,e22^2$$

[3] The Hilber Dimension of the ideal is computed in Maple as follows

```
>with(LinearAlgebra):
>E:=¡¡e11—e12—e13¿,¡e21—e22—e23¿,¡e31—e32—e33¿¿:
>eM:=2*E.Transpose(E).E-Trace(Transpose(E).E)*E:
>eq:=expand(convert(convert(eM,Vector),list)):
>with(PolynomialIdeals):
```

by one [12]. Hence, five additional (independent) linear equations from image matches will reduce the Hilbert dimension of the system to one.

Since all equations 3.71, 3.77 and $|\mathtt{E}| = 0$ are homogeneous, we can't reduce the Hibert dimension below one by adding more equations 3.77 from image matches. This reflects the fact that $\mathtt{E}$ is fixed by image measurements only up to a non-zero scale.

To conclude, five independent linear equations 3.71 plus Equations 3.77 and $|\mathtt{E}| = 0$ fix $\mathtt{E}$ up to a non-zero scale.

The scale of $\mathtt{E}$ has to be fixed in a different way. For instance, one often knows that some of the elements of $\mathtt{E}$ can be set to one. By doing so, an extra independent linear equation is obtained and the Hilbert dimension is reduced to zero. Alternatively, one can ask for $\|\mathtt{E}\|^2 = 1$, which adds a second order equation. That also reduces the Hilbert dimension to zero but doubles the number of solutions for $\mathtt{E}$.

3.5.4.2 Solving the equations

We will next describe one way how to solve equations

$$\vec{x}_{i,2\gamma_2}^{\top} \mathtt{E}\, \vec{x}_{i,1\gamma_1} = 0, \quad \left(2\,\mathtt{E}\,\mathtt{E}^{\top} - \mathrm{trace}\,(\mathtt{E}^{\top}\mathtt{E})\,\mathtt{I}\right)\,\mathtt{E} = \mathtt{0}, \quad |\mathtt{E}| = 0, \quad i = 1, \ldots, 5 \tag{3.108}$$

We will present a solution based on [14], which is somewhat less efficient than [15, 16] but requires only eigenvalue computation.

>HilbertDimension(¡op(eq),Determinant(E)¿);
6

First, using Equation 1.90 from Paragraph 1.5, we can write

$$\begin{bmatrix} \vec{x}_{1,1\gamma_1}^\top \otimes \vec{x}_{1,2\gamma_2}^\top \\ \vec{x}_{2,1\gamma_1}^\top \otimes \vec{x}_{2,2\gamma_2}^\top \\ \vec{x}_{3,1\gamma_1}^\top \otimes \vec{x}_{3,2\gamma_2}^\top \\ \vec{x}_{4,1\gamma_1}^\top \otimes \vec{x}_{4,2\gamma_2}^\top \\ \vec{x}_{5,1\gamma_1}^\top \otimes \vec{x}_{5,2\gamma_2}^\top \\ \vec{a}^\top \end{bmatrix} v(\mathtt{E}) \quad = \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{3.109}$$

to obtain a 6×9 matrix of a system of linear equations on $v(\mathtt{E})$. Row $\vec{a}^\top$ can be chosen randomly to fix the scale of $v(\mathtt{E})$. There is only a negligible chance that it will be chosen in the orthogonal complement of the span of the solutions to force the solutions be trivial. If so, it can be detected and a new $\vec{a}^\top$ generated.

Assuming that the rows of the matrix of the system are linearly independent, we obtain a 3-dimensional affine space of solutions. After rearranging the particular solution, resp. the basis of the solution of the associated homogeneous system, back to 3×3 matrices $\mathtt{G}_0$, resp. $\mathtt{G}_1, \mathtt{G}_2, \mathtt{G}_3$, we will get all solutions compatible with Equation 3.109 in the form

$$\mathtt{G} = \mathtt{G}_0 + x\,\mathtt{G}_1 + y\,\mathtt{G}_2 + z\,\mathtt{G}_3 \tag{3.110}$$

for $x, y, z \in \mathbb{R}$.

Now, we can substitute $\mathtt{G}$ for $\mathtt{E}$ into the two remaining equations in 3.108. We get ten trird-order polynomial equations in three unknowns and with 20 monomials. We can write it as

$$\mathtt{M}\,\mathtt{m} = 0 \tag{3.111}$$

where $\mathtt{M}$ is a constant 10×20 matrix[4] and

$$\mathtt{m}^\top = [x^3, y\,x^2, y^2x, y^3, z\,x^2, z\,y\,x, z\,y^2, z^2x, z^2y, z^3, x^2, y\,x, y^2, z\,x, z\,y, z^2, x, y, z, 1] \tag{3.112}$$

[4]Matrix $\mathtt{M}$ can be obtained by the following Maple [13] program

is a vector of 20 monomials.

Next, we rewrite the system 3.112 as

$$(z^3 \mathtt{C}_3 + z^2 \mathtt{C}_2 + z\,\mathtt{C}_1 + \mathtt{C}_0)\,\mathtt{c} = 0 \tag{3.113}$$

with

$$\mathtt{c} = z^3 \mathtt{c}_3 + z^2 \mathtt{c}_2 + z\,\mathtt{c}_1 + \mathtt{c}_0 \tag{3.114}$$

containing 10 monomials. Matrices $\mathtt{C}_0, \ldots, \mathtt{C}_4$ are constant 10×10 matrices

$$\mathtt{C}_0 = \begin{bmatrix} \mathtt{m}_1 & \mathtt{m}_2 & \mathtt{m}_3 & \mathtt{m}_4 & \mathtt{m}_{11} & \mathtt{m}_{12} & \mathtt{m}_{13} & \mathtt{m}_{17} & \mathtt{m}_{18} & \mathtt{m}_{20} \end{bmatrix} \tag{3.115}$$

$$\mathtt{C}_1 = \begin{bmatrix} \mathtt{0} & \mathtt{0} & \mathtt{0} & \mathtt{0} & \mathtt{m}_5 & \mathtt{m}_6 & \mathtt{m}_7 & \mathtt{m}_{14} & \mathtt{m}_{15} & \mathtt{m}_{19} \end{bmatrix} \tag{3.116}$$

$$\mathtt{C}_2 = \begin{bmatrix} \mathtt{0} & \mathtt{0} & \mathtt{0} & \mathtt{0} & \mathtt{0} & \mathtt{0} & \mathtt{0} & \mathtt{m}_8 & \mathtt{m}_9 & \mathtt{m}_{16} \end{bmatrix} \tag{3.117}$$

$$\mathtt{C}_3 = \begin{bmatrix} \mathtt{0} & \mathtt{0} & \mathtt{0} & \mathtt{0} & \mathtt{0} & \mathtt{0} & \mathtt{0} & \mathtt{0} & \mathtt{0} & \mathtt{m}_{10} \end{bmatrix} \tag{3.118}$$

where $\mathtt{m}_i$ are columns of $\mathtt{M}$.

Since $\mathtt{m}$ contains all monomials in x, y, z up to degree three, we could have written similar equations as Equation 3.113 with x and y.

```
>with(LinearAlgebra):
>G0:=¡¡g011—g012—g013¿,¡g021—g022—g023¿,¡g031—g032—g033¿¿:
>G1:=¡¡g111—g112—g113¿,¡g121—g122—g123¿,¡g131—g132—g133¿¿:
>G2:=¡¡g211—g212—g213¿,¡g221—g222—g223¿,¡g231—g232—g233¿¿:
>G3:=¡¡g311—g312—g313¿,¡g321—g322—g323¿,¡g331—g332—g333¿¿:
>trc:=E-¿simplify((2*E.Transpose(E)-Trace(Transpose(E).E)*IdentityMatrix(3,3)).E):
>eq:=[op(convert(trc(G),listlist)),Determinant(G)]:
>mo:=tdeg(x,y,z);
>m:=PolyVarMonomials(eq,mo);
```

$m := [x^3, y\,x^2, y^2 x, y^3, z\,x^2, z\,y\,x, z\,y^2, z^2 x, z^2 y, z^3, x^2, y\,x, y^2, z\,x, z\,y, z^2, x, y, z, 1]$

```
>M:=PolyCoeffMatrix(eq,m,mo):
>M[1,1];
```

$2\,g122\,g112\,g121 + 2\,g133\,g113\,g131 - g123^2\,g111 - g122^2\,g111 + 2\,g132\,g112\,g131 - g132^2\,g111 + g131^2\,g111 + g112^2\,g111 + g111^3 + 2\,g123\,g113\,g121 - g133^2\,g111 + g121^2\,g111 + g113^2\,g111$

Equation 3.113 is known as a Polynomial Eigenvealue Problem (PEP) [17] of degree three. The standard solution to such a problem is to relax it into a generelized eigenvalue problem of a larger size as follows.

We can write $z^2\mathtt{c} = z\,(z\mathtt{c})$ and $z\mathtt{c} = z\,(\mathtt{c})$ altogether with Equation 3.113 in a matrix form as

$$\begin{bmatrix} \mathtt{0} & \mathtt{I} & \mathtt{0} \\ \mathtt{0} & \mathtt{0} & \mathtt{I} \\ -\mathtt{C}_0 & -\mathtt{C}_1 & -\mathtt{C}_2 \end{bmatrix} \begin{bmatrix} \mathtt{c} \\ z\mathtt{c} \\ z^2\mathtt{c} \end{bmatrix} = z \begin{bmatrix} \mathtt{I} & \mathtt{0} & \mathtt{0} \\ \mathtt{0} & \mathtt{I} & \mathtt{0} \\ \mathtt{0} & \mathtt{0} & \mathtt{C}_3 \end{bmatrix} \begin{bmatrix} \mathtt{c} \\ z\mathtt{c} \\ z^2\mathtt{c} \end{bmatrix} \tag{3.119}$$

$$\mathtt{A}\,\mathtt{v} = z\,\mathtt{B}\,\mathtt{v} \tag{3.120}$$

This is a Generelized Eigenvalue Problem (GEP) [17] of size 30×30, which can be solved for z and $\mathtt{v}$. Values of x, y can be recovered from $\mathtt{v}$ as $x = \mathtt{c}_8/\mathtt{c}_{10}$ and $x = \mathtt{c}_9/\mathtt{c}_{10}$. It provides 30 solutions in general.

When $\mathtt{C}_0$ is regular, we can pass to a standard eigenvalue problem for a non-zero z by inverting $\mathtt{A}$ and using $w = 1/z$

$$\begin{bmatrix} -\mathtt{C}_0^{-1}\mathtt{C}_1 & -\mathtt{C}_0^{-1}\mathtt{C}_2 & -\mathtt{C}_0^{-1}\mathtt{C}_3 \\ \mathtt{I} & \mathtt{0} & \mathtt{0} \\ \mathtt{0} & \mathtt{I} & \mathtt{0} \end{bmatrix} \begin{bmatrix} w^2\mathtt{c} \\ w\mathtt{c} \\ \mathtt{c} \end{bmatrix} = w \begin{bmatrix} w^2\mathtt{c} \\ w\mathtt{c} \\ \mathtt{c} \end{bmatrix} \tag{3.121}$$