Elements of Geometry for Computer Vision and Computer Graphics

Translation of Euclid's Elements by Adelardus Bathensis (1080–1152)

2021 Lecture 11

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has a special form which corresponds to a special change of a coordinate system in the three-dimensional affine space.

3.4 Reconstruction from two calibrated views

Let us further assume that camera calibration matrices $\mathtt{K}_1, \mathtt{K}_2$ are known. Hence we can pass from $\mathtt{F}$ to $\mathtt{E}$ using Equations 3.14, 3.15 as

$$\boxed{\mathtt{E} = \mathtt{K}_2^\top \, \mathtt{F} \, \mathtt{K}_1} \tag{3.41}$$

then recover the relative pose of the cameras, set their coordinate systems and finally reconstruct points of the scene.

3.4.1 Camera computation

To simplify the setting, we will first pass from "uncalibrated" image points $\vec{x}_{1\beta_1}, \vec{x}_{2\beta_2}$ using $\mathtt{K}_1, \mathtt{K}_2$ to "calibrated"

$$\vec{x}_{1\gamma_1} = \mathtt{K}_1^{-1} \vec{x}_{1\beta_1} \quad \text{and} \quad \vec{x}_{2\gamma_2} = \mathtt{K}_2^{-1} \vec{x}_{2\beta_2} \tag{3.42}$$

and then use camera projection matrices as follows

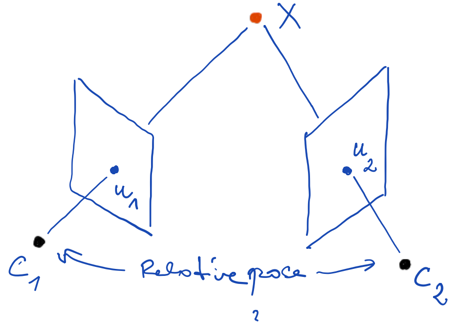
$$\zeta_1 \, \vec{x}_{1\gamma_1} = \mathtt{P}_{1\gamma_1} \overset{H^{-1}H}{} \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix} \quad \text{and} \quad \zeta_2 \, \vec{x}_{2\gamma_2} = \mathtt{P}_{2\gamma_2} \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix} \tag{3.43}$$

Matrix $\mathtt{H}$ allows us to choose the global coordinate system of the scene as (C_1, ϵ_1). Setting

$$\mathtt{H}^{-1} = \begin{bmatrix} \mathtt{R}_1^\top & \vec{C}_{1\delta} \\ \vec{0}^\top & 1 \end{bmatrix} \tag{3.44}$$

$P_{1\beta_1} = [I \mid 0]$

3D reconstruction by triangulation



$u_1, u_2 \xrightarrow{?} X$

$\begin{bmatrix} \vec{u}_{1\alpha_1} \\ \vec{u}_{2\alpha_2} \end{bmatrix} \longrightarrow \vec{X}_\delta$

$\mathbb{R}^4 \longrightarrow \mathbb{R}^3$

Calibrated

K_1, K_2 known

$\Downarrow$

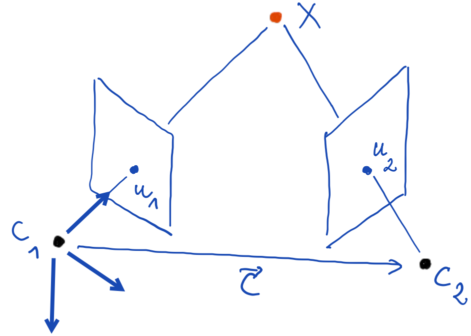
$\vec{x}_{1\beta_i} \longrightarrow \vec{x}_{1\gamma_i}$

Essential matrix

$E = R_2 [\vec{C}_{2\delta} - \vec{C}_{1\delta}]_\times R_1^T$

Fundamental matrix

$F = K_2^{-T} E K_1^{-1}$



we get from Equation 3.38

$$
\begin{aligned}
P_{1\gamma_1} &= \left[\, I \mid \vec{0} \,\right] && (3.45)\\
P_{2\gamma_2} &= \left[\, R_2\, R_1^\top \mid -R_2\,(\vec{C}_{2\delta} - \vec{C}_{1\delta}) \,\right] = \left[\, R_2\, R_1^\top \mid -R_2 R_1^\top (\vec{C}_{2\epsilon_1} - \vec{C}_{1\epsilon_1}) \,\right] && (3.46)\\
&= \left[\, R \mid -R\, \vec{C}_{\epsilon_1} \,\right] && (3.47)
\end{aligned}
$$

and the corresponding essential matrix

$$\boxed{E = R\left[\vec{C}_{\epsilon_1}\right]_\times} \qquad (3.48)$$

From image measurements, $\vec{x}_{1\gamma_1}, \vec{x}_{2\gamma_2}$, we can compute, Section 3.2, matrix

computed up to scale :
$$G = \tau\, E = \tau\, R\left[\vec{C}_{\epsilon_1}\right]_\times \qquad (3.49)$$

and hence we can get E only up to a non-zero multiple τ. Therefore, we can recover $\vec{C}_{\epsilon_1}$ only up to τ.

We will next fix τ up to its sign s_1. Consider that the *Frobenius norm* of a matrix G

$$
\begin{aligned}
\|G\|_F &= \sqrt{\sum_{i,j=1}^{3} G_{ij}^2} = \sqrt{\mathrm{trace}\,(G^\top G)} = \sqrt{\mathrm{trace}\left(\tau^2 \left[\vec{C}_{\epsilon_1}\right]_\times^\top \underbrace{R^\top R}_{I} \left[\vec{C}_{\epsilon_1}\right]_\times\right)}\\
&= \sqrt{\tau^2\,\mathrm{trace}\left(\left[\vec{C}_{\epsilon_1}\right]_\times^\top \left[\vec{C}_{\epsilon_1}\right]_\times\right)} && (3.50)\\
&= |\tau|\, \sqrt{2\,\|\vec{C}_{\epsilon_1}\|^2} = |\tau|\,\sqrt{2}\,\|\vec{C}_{\epsilon_1}\| && (3.51)
\end{aligned}
$$

$y^2+z^2+x^2+z^2+x^2+y^2 = 2\|\vec{C}_{\epsilon_1}\|^2$

trace

We have used the following identities

$$
\begin{aligned}
G^\top G &= \tau^2 \left[\vec{C}_{\epsilon_1}\right]_\times^\top R^\top R \left[\vec{C}_{\epsilon_1}\right]_\times = \tau^2 \left[\vec{C}_{\epsilon_1}\right]_\times \left[\vec{C}_{\epsilon_1}\right]_\times && (3.52)\\
&= \tau^2 \begin{bmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{bmatrix} \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} = \tau^2 \begin{bmatrix} y^2+z^2 & -x\,y & -x\,z \\ -x\,y & x^2+z^2 & = y\,z \\ -x\,z & -y\,z & x^2+y^2 \end{bmatrix}
\end{aligned}
$$

$(O,\delta) := (C_1, \varepsilon_1)$

The world coordinate system in the first camera

$$P_{1\gamma_1} = \left[R_1 \mid -R_1 \vec{C}_{1\delta}\right] \begin{bmatrix} R_1^T & \vec{C}_{1\delta} \\ 0 & 1 \end{bmatrix} = \left[\, I \mid 0 \,\right]$$

$$H^{-1} = \begin{bmatrix} R_1^\top & \vec{C}_{1\delta} \\ \vec{0}^\top & 1 \end{bmatrix}$$

$$P_{2\gamma_2} = \left[R_2 \mid -R_2 \vec{C}_{2\delta}\right] \begin{bmatrix} R_1^T & \vec{C}_{1\delta} \\ 0 & 1 \end{bmatrix} = \left[R_2 R_1^T \mid -R_2 R_1^T \vec{C}_{1\varepsilon_1}\right] = \left[R \mid -R\vec{C}_{\varepsilon_1}\right]$$

We can now construct normalized matrix $\bar{\mathtt{G}}$ as

$$\bar{\mathtt{G}} = \frac{\sqrt{2}\,\mathtt{G}}{\sqrt{\sum_{i,j=1}^{3} \mathtt{G}_{ij}^2}} = \frac{\tau}{|\tau|}\,\mathtt{R}\left[\frac{\vec{C}_{\epsilon_1}}{\|\vec{C}_{\epsilon_1}\|}\right]_\times = s_1\,\mathtt{R}\left[\vec{t}_{\epsilon_1}\right]_\times \doteq \pm E \tag{3.53}$$

with new unknown $s_1 \in \{+1,-1\}$ and $\vec{t}_{\epsilon_1}$ denoting the unit vector in the direction of the second camera center in ϵ_1 basis.

We can find vector $\vec{v}_{\epsilon_1} = s_2\,\vec{t}_{\epsilon_1}$ with new unknown $s_2 \in \{+1,-1\}$ by solving

$$\bar{\mathtt{G}}\,\vec{v}_{\epsilon_1} = 0 \quad \text{subject to} \quad \|\vec{v}_{\epsilon_1}\| = 1 \tag{3.54}$$

to get

$$\bar{\mathtt{G}} = s_1\,\mathtt{R}\left[\frac{1}{s_2}\,\vec{v}_{\epsilon_1}\right]_\times = \frac{s_1}{s_2}\,\mathtt{R}\left[\vec{v}_{\epsilon_1}\right]_\times \tag{3.55}$$

1/s

$$s\,\bar{\mathtt{G}} = \mathtt{R}\left[\vec{v}_{\epsilon_1}\right]_\times \tag{3.56}$$

$$3\times 3 \quad \begin{bmatrix} s\,\mathtt{g}_1 & s\,\mathtt{g}_2 & s\,\mathtt{g}_3 \end{bmatrix} = \mathtt{R}\begin{bmatrix} \mathtt{v}_1 & \mathtt{v}_2 & \mathtt{v}_3 \end{bmatrix} \quad 3\times 3 \tag{3.57}$$

with unknown $s \in \{+1,-1\}$, unknown rotation $\mathtt{R}$ and known matrices $\begin{bmatrix} \mathtt{g}_1 & \mathtt{g}_2 & \mathtt{g}_3 \end{bmatrix} = \bar{\mathtt{G}}$ and $\begin{bmatrix} \mathtt{v}_1 & \mathtt{v}_2 & \mathtt{v}_3 \end{bmatrix} = \left[\vec{v}_{\epsilon_1}\right]_\times$.

This is a matricial equation. Matrices $\bar{\mathtt{G}}$, $\left[\vec{v}_{\epsilon_1}\right]_\times$ are of rank two and hence do not determine $\mathtt{R}$ uniquely unless we use $\mathtt{R}^\top\mathtt{R} = \mathtt{I}$ and $|\mathtt{R}| = 1$. That leads to a set of polynomial equations. They can be solved but we will use the property of vector product, § 2, to directly construct regular matrices that will determine $\mathtt{R}$ uniquely for a fixed s.

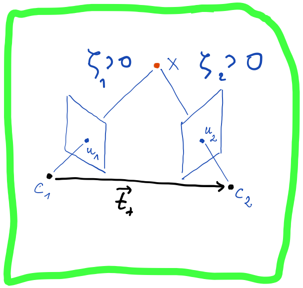
Consider that for every regular $\mathtt{A} \in \mathbb{R}^{3\times 3}$, we have, § 2,

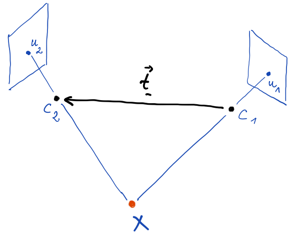
$$(\mathtt{A}\,\vec{x}_\beta) \times (\mathtt{A}\,\vec{y}_\beta) = \vec{x}_{\beta'} \times \vec{y}_{\beta'} = \frac{\mathtt{A}^{-\top}}{|\mathtt{A}^{-\top}|}\,(\vec{x}_\beta \times \vec{y}_\beta) \tag{3.58}$$

which for $\mathtt{R}$ gives

A rotation

$$(\mathtt{R}\,\vec{x}_\beta) \times (\mathtt{R}\,\vec{y}_\beta) = \mathtt{R}\,(\vec{x}_\beta \times \vec{y}_\beta) \tag{3.59}$$





rank $\bar{G} = 2$

rank $[\vec{v}_{\epsilon_1}]_\times = 2$

3x3 = 9 eqns. linear

8 unkn. × 9+7 eq.

$s\,\bar{G} = R\,[\vec{v}_{\epsilon_1}]_\times$

↓ ?

R

deg 2, deg 3

→ 6 + 1 = 7

R rotation $\Leftrightarrow$ $\underbrace{R^\top R = I,\ \det R = 1}_{\text{non-linear}}$

Using it for $i, j = 1, 2, 3$ to get

$$(s\,\mathtt{g}_i) \times (s\,\mathtt{g}_j) = (\mathtt{R}\,\mathtt{v}_i) \times (\mathtt{R}\,\mathtt{v}_j) \tag{3.60}$$

$1 = (\pm 1)^2$

$$\cancel{s}^2\,(\mathtt{g}_i \times \mathtt{g}_j) = \mathtt{R}\,(\mathtt{v}_i \times \mathtt{v}_j) \tag{3.61}$$

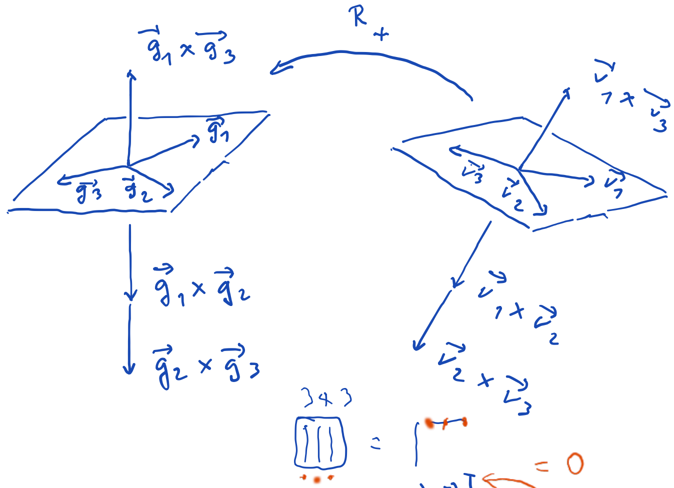
$$(\mathtt{g}_i \times \mathtt{g}_j) = \mathtt{R}\,(\mathtt{v}_i \times \mathtt{v}_j) \tag{3.62}$$

$$\left\{\begin{aligned} \vec{g}_1 \times \vec{g}_2 &= R\,(\vec{v}_1 \times \vec{v}_2) \\ \vec{g}_2 \times \vec{g}_3 &= R\,(\vec{v}_2 \times \vec{v}_3) \\ \vec{g}_1 \times \vec{g}_3 &= R\,(\vec{v}_1 \times \vec{v}_3) \end{aligned}\right.$$

i.e. three more vector equations. Notice how s disappeared in the vector product.

We see that we can write

$$\begin{bmatrix} s\,\mathtt{g}_1 & s\,\mathtt{g}_2 & s\,\mathtt{g}_3 & \mathtt{g}_1 \times \mathtt{g}_2 & \mathtt{g}_2 \times \mathtt{g}_3 & \mathtt{g}_1 \times \mathtt{g}_3 \end{bmatrix} = \\ = \mathtt{R}_s \begin{bmatrix} \mathtt{v}_1 & \mathtt{v}_2 & \mathtt{v}_3 & \mathtt{v}_1 \times \mathtt{v}_2 & \mathtt{v}_2 \times \mathtt{v}_3 & \mathtt{v}_1 \times \mathtt{v}_3 \end{bmatrix} \tag{3.63}$$



There are two solutions $\mathtt{R}_+$ for $s = +1$ and $\mathtt{R}_-$ for $s = -1$. We can next compute two solutions $\vec{t}_{+\epsilon_1} = +\vec{v}_{\epsilon_1}$ and $\vec{t}_{-\epsilon_1} = -\vec{v}_{\epsilon_1}$ and combine them together to four possible solutions

$$\mathtt{P}_{2\gamma_2++} = \mathtt{R}_+ \left[\mathtt{I} \,|\, -\vec{t}_{+\epsilon_1}\right] \tag{3.64}$$

$$\mathtt{P}_{2\gamma_2+-} = \mathtt{R}_+ \left[\mathtt{I} \,|\, -\vec{t}_{-\epsilon_1}\right] \tag{3.65}$$

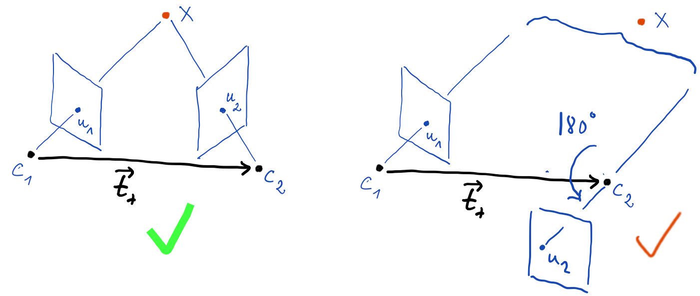
$$\mathtt{P}_{2\gamma_2-+} = \mathtt{R}_- \left[\mathtt{I} \,|\, -\vec{t}_{+\epsilon_1}\right] \tag{3.66}$$

$$\mathtt{P}_{2\gamma_2--} = \mathtt{R}_- \left[\mathtt{I} \,|\, -\vec{t}_{-\epsilon_1}\right] \tag{3.67}$$

$$E = R\,[\vec{t}]_\times = R\,\underbrace{\left(-I + 2\,\frac{\vec{t}\,\vec{t}^{\top}}{\vec{t}^{\top}\vec{t}}\right)}_{\text{rotation by } 180^\circ}[\vec{t}]_\times$$

$= 0$

The above four camera projection matrices are compatible with $\bar{\mathtt{G}}$. The one which corresponds to the actual matrix can be selected by requiring that all reconstructed points lie in front of the cameras, i.e. that the reconstructed points are all positive multiples of vectors $\vec{x}_{1\epsilon_1}$ and $\vec{x}_{2\epsilon_2}$ for all image points.



3.4.2 Point computation

Let us assume having camera projection matrices P_1, P_2 and image points $\vec{x}_{1\beta_1}, \vec{x}_{2\beta_2}$ such that

$[I|0]$ 4 cases

$$\zeta_1 \underbrace{\vec{x}_{1\beta_1}}_{\gamma_1} = \underbrace{P_1}_{\gamma_1} \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix} \quad \text{and} \quad \zeta_2 \underbrace{\vec{x}_{2\beta_2}}_{\gamma_2} = \underbrace{P_2}_{\gamma_2} \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix} \tag{3.68}$$

Projection equations

We can get $\vec{X}_\delta$, and ζ_1, ζ_2 by solving the following system of (inhomogeneous) linear equations

$$\begin{array}{c c} & \begin{array}{ccc} 1 & 1 & 4 \end{array} \\ \begin{array}{c} 3 \\ 3 \end{array} & \begin{bmatrix} \vec{x}_{1\beta_1} & \vec{0} & -P_1 \\ \vec{0} & \vec{x}_{2\beta_2} & -P_2 \end{bmatrix} \end{array} \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \vec{X}_\delta \\ 1 \end{bmatrix} = 0 \tag{3.69}$$

6×6 read out

Solve this linear system for $\zeta_1, \zeta_2, \vec{X}_\delta \in \mathbb{R}^3$

3.5 Calibrated relative camera pose computation

In the previous chapter, we had first computed a multiple of the fundamental matrix from seven point correspondences and only then used camera calibration matrices to recover a multiple of the essential matrix. Here we will use the camera calibration right from the beginning to obtain a multiple of the essential matrix directly from only five image correspondences. Not only that five is smaller than seven but using the calibration right from the beginning permits all points of the scene generating the correspondences to lie in a plane.

We start from Equation 3.42 to get $\vec{x}_{1\gamma_1}$ and $\vec{x}_{2\gamma_2}$ from Equation 3.43 which are related by

$$\vec{x}_{2\beta_2}^\top K_2^{-\top} E\, K_1^{-1} \vec{x}_{1\beta_1} = 0 \tag{3.70}$$

$$\vec{x}_{2\gamma_2}^\top E\, \vec{x}_{1\gamma_1} = 0 \tag{3.71}$$

The above equation holds true for all pairs of image points $(\vec{x}_{1\gamma_1}, \vec{x}_{2\gamma_2})$ that are in correspondence, i.e. are projections of the same point of the scene.

4 cases

$P_1 = [I\ 0]$ $P_{2++} \rightarrow X_{++}$

-||- $P_{2+-} \rightarrow X_{+-}$

-||- $P_{2-+} \rightarrow X_{-+}$

-||- $P_{2--} \rightarrow X_{--}$

$\underset{>0}{\zeta_1} \vec{x}_{1\gamma_1} = P_1 X$

$\underset{>0}{\zeta_2} \vec{x}_{2\gamma_2} = P_{2ab} X$